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Seventy-Five Years of Global Analysis around the Forced Pendulum Equation

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Es dürfte nützlich sein, zuweilen die allgemeinen Forschritte,
welche die Analysis in den letzen Jahren gemacht hat,
an einem bestimmten Beispiel zu prüfen.(...)
Wir wollen die Differentialgleichung des Herrn Duffing
in der ursprünglichen Form hier etwa genauer untersuchen.
G. Hamel, 1922

Description of the set P of f for which the equation $u'' + \sin u = f(t)$
has a T-periodic solution seems to remain a terra incognita.
S. Fučik, 1979

Abstract. We survey the recent progress made in the study of harmonic, subharmonic and other solutions of the forced pendulum equation

$$u'' + cu' + a \sin u = h(t)$$

when the forcing term h is periodic, almost periodic or bounded. The results depend upon various methods of nonlinear functional analysis, critical point theory and dynamical systems.

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1 Introduction

Seventy-five year ago, in a paper published in 1922 in the special issue of the *Mathematische Annalen* dedicated to Hilbert's sixtieth birthday anniversary [86], Hamel, one of his former students, has provided the first general existence results for the periodic solutions of the periodically forced pendulum equation

$$y'' + a \sin y = b \sin t. \tag{1.1}$$

This equation had been the central topics of a monograph published four years earlier by Duffing [55], who had restricted his study to the approximate determination of the periodic solutions for the following approximation of equation (1.1)

$$y'' + ay - cy^3 = b \sin t,$$

which still bears his name today.

Hamel's paper starts by an existence result for a 2π-periodic solution of equation (1.1) by using the *direct method of the calculus of variations* made rigorous by Hilbert at the beginning of the century. He shows indeed that the *action integral*

$$A(y) := \int_0^{2\pi} \left(\frac{y'^2(t)}{2} + a \cos y(t) + b y(t) \sin t \right) dt$$

has a minimum on the space of 2π-periodic functions of class C^1. His argument extends easily to the more general case where $b \sin t$ is replaced by any 2π-periodic function with mean value zero, a fact rediscovered independently, in the easier framework of Sobolev spaces, some sixty years later [179,47], and rapidly followed by the proof of the existence of a *second* 2π-periodic solution [124] through the use of more sophisticated tools of *critical point theory.* In the second section of [86], Hamel uses the *Ritz method* to find a first approximation of the amplitude of the periodic solution found in the previous section. In Section 4, Hamel observes that the symmetries of the equation imply that any solution of equation (1.1) such that

$$y(0) = y(\pi) = 0 \tag{1.2}$$

can be extended as an *odd* 2π-periodic solution. He then reduces the problem (1.1)–(1.2) to the integral equation

$$y(t) = -a \int_0^{2\pi} K(t,\tau) \sin y(\tau)\, d\tau - b \sin t := F(y)(t), \tag{1.3}$$

where $K(t,\tau)$ is the Green function of its linear part, and shows that the corresponding *method of successive approximations*

$$y_{n+1} = F(y_n), \quad y_0(t) = -b \sin t,$$

converges. His argument is equivalent to showing the existence of a sufficiently large integer m, for which the m^{th} iterate F^m of F is a contraction in the space of $C[0,\pi]$. Observe that this is published the very same year where Banach publishes his version of the *contraction mapping theorem*! Notice also that this part of Hamel's paper will inspire Hammerstein's famous researches on *nonlinear integral equations.* Hamel uses not only the equivalent integral equation for existence and uniqueness conclusions, but also to obtain approximations to the solutions. For this, Hamel relies upon Schmidt's version of the *Lyapunov-Schmidt's method.*

This short description clearly shows that Hamel's paper anticipates or uses several of the fundamental methods of nonlinear analysis, and that the opening sentence of his paper, recalled in exergue of this work, is fully justified.

The survey papers [110,111] describe the contributions to the forced pendulum equation in the fifty-five years following Hamel's work. An important role in renewing the interest to the forced pendulum equation was played in the late seventies by Fučik, who wrote, in the Introduction of Chapter 26 of his monograph [72] : *Finally we shall present here one attempt to obtain the existence of a T-periodic solution of the mathematical pendulum equation*

$$-u''(x) + \sin u(x) = f(x). \tag{1.4}$$

The result is not final since the necessary and sufficient condition obtained for T-periodic solvability of (1.4) *is not useful.* After describing very partial results in this direction and mentioning extensions personally communicated by Dancer, Fučik concluded with the sentence mentioned in exergue of this paper.

Motivated by Fučik's remarks, but unaware of the existence of Hamel's paper, Castro [38], Dancer [47] and Willem [179], reintroduced in the early eighties the use of variational methods in the study of the forced pendulum. The time was ripe for the obtention, more than sixty years after the first one, of a *second periodic solution,* using a version of the *mountain pass lemma* [124]. The survey papers [110,111,114,117,118,121,183], as well as to the monographs [112,126,40,79,98,151] provide a description of the state of the art till the early nineties for the global results on the existence and multiplicity of periodic solutions.

At the same time, the forced pendulum equation also became a paradigm for the *theory of chaos,* and appeared in the description of *Josephson type junctions.* According to Baker and Gollub [13]: *Now 400 years after Galileo's initial work, the pendulum has again become an object of research as a chaotic system.* We shall not develope this viewpoint here and refer to a nice survey of Chenciner [42] and to the papers or monographs [13,22,25,26,29,49,54,80,81,82,87,88,89,94,95] [96,97,100,101,104,127,152,160,168,150,170] and their references.

Despite its fundamental role in the development of the qualitative theory of nonlinear differential equations and its applications to engineering, we shall not discuss here the special case of the *pendulum equation with a constant torque*

$$y'' + cy' + a \sin y = b,$$

initiated by Tricomi [175,176] and widely developed since (see e.g. [10,15,85,105] [154,153,163,164] and their references).

Moreover, to keep the size of the paper reasonable and make more easy the comparaison between results obtained through different methods, we shall only state the theorems for the special case of the standard forced pendulum equation

$$y'' + cy' + a \sin y = h(t).$$

Most of the assertions remain valid if $a \sin y$ is replaced by an arbitrary continuous function $g(y)$ which is S-periodic for some $S > 0$ and of mean value zero.

There are even recent results which depend upon the fact that $a \sin y$ is replaced by a S-periodic function whose Fourier series contains higher harmonics [93]. Also, some conclusions survive when the friction term cy' is replaced by a more general one of Liénard type $f(y)y'$ or of Rayleigh type $f(y')$ (see e.g. [124,83]).

For the same reason, we shall not describe the possible generalizations to *systems of equations of the pendulum-type,* and in particular to the equations of the *forced multiple pendulum,* and to *higher order pendulum-type equations.* The reader can consult the original papers [125,36,53,66,149,64,119,40,62,171,173] [174,56,57,128,63,58,67,21]. Some of those results are related to the famous solution by Conley and Zehnder [44] of a *conjecture of Arnold* in symplectic geometry. See also, for example, [45,189,190,61,106]. We shall not describe the results dealing with *symmetric forcing terms* $h(t)$, which have been recently considered in [161,162,155,16,136]. Also we shall leave aside the existence of forced oscillations for the *spherical pendulum* (which depend upon methods of a quite different nature, and have been the object of a sequence of papers by Furi, Pera and Spadini [73,74,75,76,77]), for some pendulum-type equations describing the *libration of satellites* (see [17,99,113,116,147,84]), and for delay-differential equations of the pendulum type like the *sunflower equation* [37].

Let us mention also that the corresponding problem for the case of *Dirichlet boundary conditions,* namely

$$y'' + y + a \sin y = h(t), \quad y(0) = 0 = y(\pi),$$

and its analog for partial differential equations, has been the object, since the pioneering paper of Ward [178], of a number of studies based upon various methods. See [169,107,156,157,158,159,46,11,31,32,33,34]. This problem has both deep analogies and strong differences with the periodic boundary value problem for the forced pendulum.

Finally, let us warn the reader that, despite of its substantial size, the given bibliography is undoubtly far to be complete, but its size is sufficient to show how stimulating has been the study of the forced pendulum equation in the recent development of nonlinear and global analysis, and of the theory of dynamical systems.

2 Periodic forcing

2.1 The problems

We consider the (possibly dissipative) *periodically forced pendulum equation*

$$y'' + cy' + a \sin y = h(t), \tag{2.1}$$

where, without loss of generality, $c \geq 0$, $a > 0$, and h is T-periodic, for some period $T > 0$, and corresponding frequency $\omega := \frac{2\pi}{T}$. For the simplicity of exposition, we shall assume that h is continuous. Most results hold under weaker regularity conditions.

A *T-periodic* solution of equation (2.1) is a solution $y : \mathbb{R} \to \mathbb{R}$ such that $y(t+T) = y(t)$ for all $t \in \mathbb{R}$. By integrating equation (2.1) over $[0, T]$, we immediately see that a *necessary condition* for the existence of a T-periodic solution to equation (2.1) is that

$$\left| \frac{1}{T} \int_0^T h(t)\, dt \right| \leq a.$$

The main questions which can be raised about the T-periodic problem for equation (2.1) are the following ones:

1. Determine the nature and the properties of the set

   $$\mathcal{R} = \mathcal{R}(c, a, T)$$

   of T-periodic forcings h such that equation (2.1) has at least one T-periodic solution, i.e. the *range* of the nonlinear operator

   $$\frac{d^2}{dt^2} + c\frac{d}{dt} + a \sin(\cdot)$$

   over the space of T-periodic functions of class C^2.
2. For $h \in \mathcal{R}$, discuss the *multiplicity* of the T-periodic solutions.
3. For $h \in \mathcal{R}$, discuss the *stability* of the T-periodic solutions.
4. Discuss the existence of *other solutions* and *properties of the set of all solutions.*

Concerning the multiplicity, it is clear that if y is a T-periodic solution of equation (2.1), then the same is true for $y + 2k\pi$, $k \in \mathbb{Z}$. Consequently, we shall say that y_1 and y_2 are *distinct T-periodic solutions of* (2.1) if they do not differ by a multiple of 2π.

In the sequel of the paper, we shall use the following notations.

$$L^p_T = \{h \in L^p_{loc}(\mathbb{R}) : h(t+T) = h(t) \text{ for a.e. } t \in \mathbb{R}\}$$

$$C_T = \{h \in C(\mathbb{R}) : h(t+T) = h(t) \text{ for all } t \in \mathbb{R}\}$$

$$H^1_T = \{h \in AC_{loc}(\mathbb{R}) : h' \in L^2\}$$

$$\|h\|_p = \left(\frac{1}{T} \int_0^T |h(t)|^p\, dt \right)^{1/p}, \quad \|h\|_\infty = \max_{t \in [0,T]} |h(t)|,$$

$$\|h\|_{H^1} = \left(\|h\|_2^2 + \|h'\|_2^2 \right)^{1/2}$$

$$\overline{h} = \frac{1}{T} \int_0^T h(t)\, dt, \quad \widetilde{h}(t) = h(t) - \overline{h} \qquad \left(\int_0^T \widetilde{h}(t)\, dt = 0 \right)$$

$$\widetilde{L^p_T} = \{h \in L^p_T : \overline{h} = 0\}, \quad \widetilde{C_T} = \{h \in C_T : \overline{h} = 0\}$$

Consequently,

$$L^p_T = \mathbb{R} \oplus \widetilde{L^p_T}, \quad C_T = \mathbb{R} \oplus \widetilde{C_T},$$

with the corresponding decomposition $y = \overline{y} + \widetilde{y}$.

We shall also use an interesting *equivalent formulation* of the problem of T-periodic solutions for equation (2.1).

Lemma 1. *If $\widetilde{H}(t) = \widetilde{H}_{c,T}(t)$ denotes the unique T-periodic solution in $\widetilde{C_T}$ of*

$$y'' + cy' = \widetilde{h},$$

then $y(t)$ is a T-periodic solution of equation (2.1) if and only $x(t) = y(t) - \widetilde{H}(t)$ is a T-periodic solution of equation

$$x'' + cx' + a\sin(x + \widetilde{H}(t)) = \overline{h}. \tag{2.2}$$

2.2 The methods

Various methods have been used in the study of the T-periodic solutions of equation (2.1) or (2.2). For the reader's convenience, we shall give a brief survey of the ones directly involved in the results described in this survey.

2.2.1 Poincaré's method

Let $y(t;u)$ be the solution of equation (2.1) such that

$$y(0,u) = u_1, \; y'(0;u) = u_2,$$

and let

$$P : \mathbb{R}^2 \to \mathbb{R}^2, u \mapsto [y(T;u), y'(T;u)].$$

Then $y(t;u)$ is a T-periodic solution of equation (2.1) if and only if u is a fixed point of P. P is called the *Poincaré's operator.*

If $c = 0$, P is area-preserving, and one can then use various *twist theorems.* Take polar coordinates (r,θ) in the plane, and denote by A the annulus $[a,b] \times S^1$. A first useful result is *Poincaré-Birkhoff's twist theorem* [148,23].

Lemma 2. *Every area-preserving homeomorphism $\phi : A \to A$ with lift*

$$(r,\theta) \mapsto (f(r,\theta), \theta + g(r,\theta)), \tag{2.3}$$

rotating the two boundaries in opposite directions, i.e. such that

$$g(a,\theta)g(b,\theta) < 0, \theta \in \mathbb{R},$$

possesses at least two fixed points in the interior .

A second one is *Moser's twist theorem* [129].

Lemma 3. *Let* $l \geq 5$, $\alpha \in C^5(\mathbb{R})$ *be such that* $|\alpha'(r)| \geq \nu > 0$ *for all* $r \in [a,b]$, *and let* $\varepsilon > 0$. *Then there exists* $\delta = \delta(\varepsilon, l, \alpha) > 0$ *such that any area-preserving mapping* (2.3) *of* A *into* $\mathbb{R}^2$ *with* $f, g \in C^l$ *such that*

$$|f - r|_{C^l} + |g - \alpha|_{C^l} < \nu\delta,$$

possesses an invariant curve of the form

$$r = c + u(\xi),\ \theta = \xi + v(\xi),$$

in A, *where* u, v *are of class* C^1, 2π*-periodic, such that* $|u|_{C^1} + |v|_{C^1} < \varepsilon$, *and* $c \in]a,b[$ *is constant. Moreover, the induced mapping on this curve is given by* $\xi \to \xi + \omega$, *where* ω *is incommensurable with* 2π, *and satisfies infinitely many conditions*

$$\left|\frac{\omega}{2\pi} - \frac{p}{q}\right| \geq \gamma q^{-\tau},$$

with some positive γ, τ, *for all integers* $q > 0$, *and* p. *In fact, each choice of* ω *in the range of* α *satisfying the above Diophantine inequalities gives rise to such an invariant curve.*

Call $\phi : A \to A$ a *monotone twist homeomorphism* if it preserves orientation, preserves boundary components of A and if for a lift $F(r,\theta) = (f(r,\theta), g(r,\theta))$, the function $g(\cdot, \theta)$ is a strictly monotone function for each θ. For definiteness, we assume this function to be strictly increasing. Let $F^j(r,\theta) = (r_j, \theta_j)$, and

$$\alpha_r(\phi) = \lim_{j\to\infty} \frac{\theta_j}{j}$$

be its *rotation number*. The *twist interval* of ϕ is the interval $[\alpha_a(\phi), \alpha_b(\phi)]$. It is defined up to an integral translation. If $\phi^q(z) = z$, then $F^q(r,\theta) = T^p(r,\theta)$, for some integer p determined up to a multiple of q, and $T(r,\theta) = (r, \theta + 2\pi)$. $\frac{p}{q}$ is called the *rotation number* of z. One calls such a point $z = (r,\theta)$ a *Birkhoff point of type (p,q)* if there exists a sequence $(r_n, \theta_n)_{n\in\mathbb{Z}}$ such that $(r_0, \theta_0) = (r,\theta)$, $\theta_{n+1} > \theta_n,\ (n \in \mathbb{N})$, $(r_{n+q}, \theta_{n+q}) = (r_{n+q}, \theta_n + 2\pi)$, $(r_{n+q}, \theta_{n+q}) = F(r_n, \theta_n)$.

One then has the *Birkhoff's twist theorem* [24].

Lemma 4. *Let* $\phi : A \to A$ *be an area-preserving monotone twist homeomorphism and*

$$\frac{p}{q} \in [\alpha_a(\phi), \alpha_b(\phi)]$$

be a rational number with p, q *relatively prime. Then there exist two Birkhoff periodic orbits of type* (p,q) *for* ϕ.

A *Mather set* of rotation number α for F is a closed invariant set for F with representation $u = u(\theta), v = v(\theta)$ where u is monotone increasing, $u - Id$ and v are 2π-periodic (not necessarily continuous!), and $u(\theta+\alpha) = \phi_1(u,v),\ v(\theta+\alpha) = \phi_2(u,v)$.

The following result is the *Aubry-Mather's twist theorem* [12,109].

Lemma 5. *Let $\phi : A \to A$ be an area-preserving monotone twist homeomorphism and let $\alpha \in [\alpha_a(\phi), \alpha_b(\phi)]$. Then there exists an invariant Mather set Γ_α with rotation number α. Furthermore, Γ_α is a subset of a closed curve $y = w(x)$ where w is 2π-periodic and Lipschitz continuous, i.e. $v(\theta) = w(u(\theta)$. For rational $\alpha = \frac{p}{q}$, this theorem provides orbits (r_j, θ_j) satisfying $\theta_{j+q} = \theta_j + 2p\pi$, $r_{j+q} = r_0$ for $j \in \mathbb{Z}$.*

For some surveys on the Aubry-Mather's twist theorem, see [14,43,92].

2.2.2 Lyapunov-Schmidt's method

The *Lyapunov-Schmidt's method* (see e.g. [78]) is based upon the following elementary fact.

Lemma 6. *$y = \overline{y} + \widetilde{y}$ is a T-periodic solution of equation (2.1) if and only if it is a solution of the system*

$$\widetilde{y}'' + c\widetilde{y}' + a\sin(\overline{y} + \widetilde{y}) = \overline{a\sin(\overline{y} + \widetilde{y})} + \widetilde{h}(t), \quad \overline{a\sin(\overline{y} + \widetilde{y})} = \overline{h} \tag{2.4}$$

In the classical Lyapunov-Schmidt's method, the first equation in (2.4) is solved with respect to $\widetilde{y}$ for fixed $\overline{y}$ (using a fixed point or implicit function theorem, or critical point theory) and this solution is introduced in the second equation, which then becomes the (one-dimensional) *bifurcation equation*. One can also study directly the equivalent system (2.4) by *degree theory* or *critical point theory*.

2.2.3 Upper and lower solutions

The *method of upper and lower solutions* for the periodic solutions of equation (5) (see e.g. [112]) consists in the following statement.

Lemma 7. *If α and β are of class C^2, T-periodic and such that, for all $t \in \mathbb{R}$,*
i) $\alpha(t) \leq \beta(t)$
ii) $\alpha''(t) + c\alpha'(t) + a\sin\alpha(t) \geq h(t) \geq \beta''(t) + c\beta'(t) + a\sin\beta(t)$,
then (2.1) has at least one T-periodic solution y such that $\alpha(t) \leq y(t) \leq \beta(t)$.

The reader will easily state the analogous statement for the periodic solutions of (2.2).

2.2.4 Critical point theory

The starting point of the use of a *variational method* or of *critical point theory* to the periodic solutions of the forced pendulum equation without dissipation is the following classical observation.

Lemma 8. *y is a T-periodic solution of*

$$y'' + a\sin y = h(t) \tag{2.5}$$

if and only if y is a critical point of the action functional

$$A_h : H^1_T \to \mathbb{R}, y \mapsto \int_0^T \left(\frac{y'^2(t)}{2} + a\cos y(t) + h(t)y(t) \right) dt. \tag{2.6}$$

Various tools of critical point theory like *minimization, mountain pass lemma, Lyusternik-Schnirelmann theory, Morse theory* (see e.g. [124]) can be applied to (2.5) or to its equivalent form (2.2). Notice that a semi-variational method has been used in [1] to study the dissipative forced pendulum.

2.3 Results valid for all c, a, T, h

Rewrite equation (2.1) as

$$y'' + cy' + a\sin y = \overline{h} + \widetilde{h}(t) \tag{2.7}$$

The following results are now classical and can be found in [124,68,112]. Their proof uses Lyapunov-Schmidt's argument, topological degree, upper and lower solutions. Some of them can already been found in [47] and some have been reobtained in [91].

Theorem 1. *For each $\widetilde{h} \in L^1_T$, there exists*

$$m_{\widetilde{h}} = m_{\widetilde{h}}(c,a,T) \leq M_{\widetilde{h}} = M_{\widetilde{h}}(c,a,T)$$

such that the following hold.

1. $-a \leq m_{\widetilde{h}} \leq M_{\widetilde{h}} \leq a$ *et* $-a = m_0 < M_0 = a$.
2. $m_{\widetilde{h}_k} \to m_{\widetilde{h}}$ *and* $M_{\widetilde{h}_k} \to M_{\widetilde{h}}$ *if* $\widetilde{H}_k \to \widetilde{H}$ *uniformly on* $\mathbb{R}$.
3. *Equation (2.7) has at least one T-periodic solution if and only if* $\overline{h} \in [m_{\widetilde{h}}, M_{\widetilde{h}}]$.
4. *Equation (2.7) has at least two distinct T-periodic solutions if* $\overline{h} \in]m_{\widetilde{h}}, M_{\widetilde{h}}[$.
5. *If* $m_{\widetilde{h}} = M_{\widetilde{h}}$*, equation (2.7) has, for each $\xi \in \mathbb{R}$, at least one T-periodic solution y with $\overline{y} = \xi$.*

In particular, $\mathcal{R}(c,a,T)$ is closed and

$$\mathcal{R}(c,a,T) = \bigcup_{\widetilde{h} \in \widetilde{C_T}} [m_{\widetilde{h}}, M_{\widetilde{h}}] \times \{\widetilde{h}\} \subset [-a,a] \times \widetilde{C_T}.$$

2.4 Open problems and partial solutions.

Some important questions are left open by the results of Theorem 1, and are only partially solved.

2.4.1 Find an explicit element in $[m_{\widetilde{h}}, M_{\widetilde{h}}]$

Theorem 2. *When* $c = 0$, *then* $0 \in [m_{\widetilde{h}}, M_{\widetilde{h}}]$.

This is shown by proving the existence of a global minimum for the action functional A_h ([86,179,180,47]). The reason of the success of the minimization method is that $A_h(y + 2\pi) = A_h(y)$ if and only if $\overline{h} = 0$. This property together with the coercivity of A_h with respect to $\widetilde{y}$ allows easily to obtain a bounded minimizing sequence for A_h. The periodicity property of A_h when $\overline{h} = 0$ allows also the use of a Lusternik-Schnirelmann type argument to prove directly that A_h has two distinct critical points (see [119,40,149]). Another proof of this fact has been given in [71] using a generalized Poincaré-Birkhoff theorem. No proof based upon degree theory is known at this day.

Theorem 3. *When* $\frac{c}{T} > \frac{1}{\pi\sqrt{3}}\|\widetilde{h}\|_2$, *then* $0 \in \,]m_{\widetilde{h}}, M_{\widetilde{h}}[$.

This is proved by topological degree arguments ([124]).

The question was then raised to know if $0 \in [m_{\widetilde{h}}, M_{\widetilde{h}}]$ for each $c > 0$. A *negative answer* was first given by a counterexample of Ortega [140], recently improved by another one of Alonso [2] showing that *for each* $c > 0$, *there exists* $T_0 = T_0(a, c)$ *such that for each* $T > T_0$, $0 \notin [m_{\widetilde{h}}, M_{\widetilde{h}}]$. The idea of Alonso's counterexample consists in constructing a forcing term close to a piecewise constant function $h(t)$ taking a large positive value p in the interval $[0, \tau]$ and a small negative value $-q$ in the interval $[\tau, T]$, where $p\tau - q(T - \tau) = 0$.

2.4.2 Prove or disprove the existence of some $\widetilde{h}$ such that $m_{\widetilde{h}} = M_{\widetilde{h}}$

This problem remains open. Here is some known partial information.

Theorem 4. *The set* $\{\widetilde{h} \in \widetilde{C_T} : m_{\widetilde{h}} < M_{\widetilde{h}}\}$ *is open and dense.*

This has been proved using various arguments [124,112,108], and in particular a *generalized Sard-Smale's theorem.* Thus, *generically,* $[m_{\widetilde{h}}, M_{\widetilde{h}}]$ is a non degenerate interval.

Theorem 5. *For* $c = 0$,

$$\{\widetilde{h} \in \widetilde{C_T} : \lim_{|\lambda| \to \infty} m(\lambda\widetilde{h}) = \lim_{|\lambda| \to \infty} M(\lambda\widetilde{h}) = 0\}$$

contains an open and dense subset of $\widetilde{C_T}$.

This has been proved by Kannan and Ortega [91], who also gave an example showing that this set is not open. The proof makes use of some Riemann-Lebesgue lemma and asymptotic analysis techniques.

2.5 The conservative case $c = 0$

We shall now concentrate on some results which hold for the conservative case $c = 0$. Recall the a *regular value* for a continuously differentiable mapping f between two smooth Banach manifolds is the image by f of a point c such that f'_c is onto.

Theorem 6. *The set $\mathcal{G}$ of regular values for $y'' + a\sin y$ is open and dense in $\widetilde{C_T}$, and, for every $g \in \mathcal{G}$, there exists $\varepsilon > 0$ such that, if $\|h - g\|_\infty \le \varepsilon$, then equation (2.5) has a T-periodic solution.*

This has been proved [108] using a generalized Sard-Smale lemma.

Recently, using techniques of critical point theory (a suitable *minimax method*), Serra, Tarallo and Terracini [167] have introduced a new condition in order that $m_{\widetilde{h}} < M_{\widetilde{h}}$.

Theorem 7. *If $\overline{h} = 0$, and if $c_0 = \inf_{H^1_T} A_h$, then $m_{\widetilde{h}} < M_{\widetilde{h}}$ if and only if the following condition*

$$(K_0) \qquad \mathcal{K}(\xi) := \{y \in H^1_T : A_h(y) = c_0,\ \overline{y} = \xi\} = \emptyset \text{ for some } \xi \in \mathbb{R}$$

holds. Moreover, if (K_0) does not hold, then, for each $\xi \in \mathbb{R}$, $\mathcal{K}(\xi) = \{y_\xi\}$, with $\xi \to y_\xi$ continuous and $y_{\xi_1}(t) < y_{\xi_2}(t)$ for all $t \in \mathbb{R}$ whenever $\xi_1 < \xi_2$, and equation (2.5) has no other periodic solutions.

In a subsequent paper [166], Serra and Tarallo have introduced a new *reduction method of Lyapunov-Schmidt's type*, which sheds some light on some of the unsolved problems for the conservative forced pendulum equation.

Theorem 8. *For each $\xi \in \mathbb{R}$, let*

$$\varphi_h(\xi) := \min_{\overline{y}=\xi} A_h(y), \qquad M_h(\xi) = \{y \in H^1_T : \overline{y} = \xi,\ A_h(y) = \varphi_h(\overline{y})\},$$

and let

$$M_h = \bigcup_{\xi\in\mathbb{R}} M_h(\xi) = \{u \in H^1_T : A_h(u) = \varphi_h(\overline{u})\}.$$

Then the following results hold.

1. *φ_h is defined and locally Lipschitz continuous on $\mathbb{R}$.*
2. *$M_h(\xi) \neq \emptyset$ and compact for each $\xi \in \mathbb{R}$ and $M_h : \mathbb{R} \to 2^{H^1_T}$ upper semi-continuous.*
3. *If $y \in M$ and $\overline{y}$ is a local minimum for φ_h, then y is a local minimum for A_h.*
4. *φ_h is differentiable at ξ if and only if $y \mapsto \int_0^T (a\sin y(t) - h(t))\,dt$ is constant on $M_h(\xi)$.*
5. *If φ_h has a critical point, then A_h has a critical point.*
6. *If φ_h is not strictly monotone, then A_h has a critical point.*

It is interesting to compare this approach to the classical method of Lyapunov-Schmidt. In this case, one proves (by critical point theory if $c = 0$ and Schauder's fixed point theorem in all cases) that, for each $\xi \in \mathbb{R}$, the set

$$K_h(\xi) = \{y \in C_T : \overline{y} = \xi \text{ and } \widetilde{y} \text{ solves the first equation in (2.4)}\}$$

is not empty, and then the problem is reduced to find the elements of the set $K_h = \bigcup_{\xi \in \mathbb{R}} K_h(\xi)$ such that $\overline{a \sin y} = \overline{h}$. In the Serra-Tarallo's approach, on each *slice* $\xi + \widetilde{H_T^1}$ of H_T^1, one considers only the elements of $K_h(\xi)$ which minimize the restriction of A_h on this slice, which provides the subset $M_h(\xi) \subset K_h(\xi)$, and then, instead of trying to solve the second equation of (2.4) on this set, one concentrates on the reduced functional φ_h and relates its critical points to those of A_h. Hence the spirit is more variational than in the earlier approaches combining a Lyapunov-Schmidt argument with some variational method, in that the emphasis, at each step, remains on the functional instead of on its gradient. Because the minimization is made on each slice on the function space, one can imitate the type of humor which has led from the name *Klein-Gordon equation* for $u_{tt} - \Delta u + u = 0$ to the name *Sine-Gordon equation* for $u_{tt} - \Delta u + \sin u = 0$, and call the Serra-Tarallo's approach a *Lyapunov-Schnitt's method.*

Notice that one of the main features of this approach is that, in contrast to most other ones, it applies when $a \sin y$ is replaced by a more general almost periodic function.

2.6 The case where $c = 0$ and $a < \omega^2$

In the conservative case, more precise results can be obtained when the following condition

$$a < \omega^2 \tag{2.8}$$

holds.

Using global analysis and *singularity theory,* Donati [51] has proved the following result about the multiplicity of solutions.

Theorem 9. *If (2.8) holds and $\overline{h} \in [m_{\widetilde{h}}, M_{\widetilde{h}}]$, then equation (2.5) has at most finitely many distinct T-periodic solutions when $[m_{\widetilde{h}}, M_{\widetilde{h}}] \neq \{0\}$. Otherwise, equation (2.5) has an analytic unbounded curve of solutions.*

Serra and Tarallo [166] have used their *Lyapunov-Schnitt's method* to obtain more precise information.

Theorem 10. *Assume that (2.8) holds. Then*

1. *If φ_h is constant, then $M_h(\xi) = \{y_\xi\}$, and if y is a periodic solution of equation (2.5), then $\overline{h} = 0$ and $y = y_\xi$ for some $\xi \in \mathbb{R}$.*
2. *φ_h is not constant if and only if there exists $\varepsilon_0 > 0$ such that equation (2.5) has at least one T-periodic solution for each $|\overline{h}| < \varepsilon_0$.*

3. *$[m_{\widetilde{h}}, M_{\widetilde{h}}] = \{0\}$ if and only if φ_h is constant.*
4. *$\{\widetilde{h} \in \widetilde{C_T} : \varphi_{\widetilde{h}}$ is not constant$\}$ is open and dense in $\widetilde{C_T}$.*
5. *If φ_h is constant and $\overline{h} \neq 0$, then equation (2.5) has no bounded solution.*

The same approach has also been used by Calanchi and Tarallo [30] to show the following result.

Theorem 11. *There exists $K = K(a,T) > 0$ such that if $\|h\|_2 < K$, each critical point of A_h over H^1_T is a local minimum or a point of mountain pass type.*

2.7 Stability of the T-periodic solutions

2.7.1 The dissipative case $c > 0$

By imposing some restrictions upon c, a, and T, it is possible to obtain on one hand *exact multiplicity results* for the T-periodic solutions, and, on the other hand, informations upon their *Lyapunov stability.* The pioneering work in the first direction is due to Tarantello [172] (using a Lyapunov-Schmidt approach) and, in the second direction, to Ortega [141,142,143] (using some relations between stability and the Brouwer degree of Poincaré's operator). A recent paper of Čepička, Drábek and Jenšiková [39] provides the sharpest known conditions.

Theorem 12. *If*

$$c > 0, \quad a < \max\left\{\frac{c^2}{4} + \omega^2, \omega\sqrt{c^2+\omega^2}\right\}$$

then equation (2.7) has :

1. *exactly one T-periodic solution if either $\overline{h} = m_{\widetilde{h}}$ or $\overline{h} = M_{\widetilde{h}}$.*
2. *exactly two T-periodic solutions if $\overline{h} \in]m_{\widetilde{h}}, M_{\widetilde{h}}[$.*

If

$$c > 0, \quad a < \max\left\{\frac{c^2+\omega^2}{4}, \frac{\omega}{2}\sqrt{c^2+\frac{\omega^2}{4}}\right\},$$

then the conclusions (1.-2.) remain true and the periodic solution obtained in (1.) is unstable while one solution obtained in (2.) is asymptotically stable and the other unstable.

The proof of the exact multiplicity results in Theorem 5 is based upon the Lyapunov-Schmidt's reduction method together with the *real analytic version of the implicit function theorem* to analyze the bifurcation equation. The uniqueness in the solution of the first equation in (2.4) is deduced from some preliminary assertions on the T-periodic solutions of linear equations of the type

$$y'' + cy' + g(t)y = 0,$$

with g T-periodic. The stability conclusion is obtained in the same way as in Ortega's papers.

2.7.2 The conservative case $c = 0$

The difficulty in analyzing the stability in the conservative case is that asymptotic stability can no more be expected. In a recent paper, Dancer and Ortega [48] have proved the following proposition.

Lemma 9. *A stable isolated fixed point of an orientation preserving local homeomorphism on $\mathbb{R}^2$ has fixed point index equal to one.*

The proof of this result depends upon a variant of Brouwer's lemma on translation arcs. One of the given applications is the following result.

Lemma 10. *If y is an isolated T-periodic solution of the second order equation, with continuous right-hand member T-periodic with respect to t,*

$$y'' = \frac{\partial V}{\partial y}(t, y), \tag{2.9}$$

and y reaches a local minimum on H_T^1 of the action functional

$$f(y) = \int_0^T \left(\frac{y'^2(t)}{2} + V(t, y(t)) \right) dt,$$

then y is unstable.

This result is proved by showing first, through a result of Amann on the computation of degree of gradient mappings and a relatedness principle of Krasnosel'skii-Zabreiko, that the index of y is equal to minus one. The result then follows from the previous one.

An immediate consequence for the pendulum equation is the following one.

Theorem 13. *If $\overline{h} = 0$, and if a T-periodic solution minimizing $A_{\widetilde{h}}$ is isolated, then it is unstable.*

One can then raise the question to known if the above results still hold without the assumption that the T-periodic solution is isolated. Ortega [145] has proved the following interesting result.

Lemma 11. *If $D \subset \mathbb{R}$ is a domain and $F : D \subset \mathbb{R}^2 \to \mathbb{R}^2$ is real analytical and not the identity on D, its Jacobian is equal to 1 on D, and if p is a stable fixed point of F, then p is isolated in the fixed points set of F.*

The delicate proof of this result uses Brouwer's plane translation theorem.

As an application, the following unstability result is proved in [145].

Lemma 12. *If V is T-periodic with respect to t and real analytic, and y is a T-periodic solution of equation (2.9) such that y reaches a local minimal of f on H_T^1, then y is unstable.*

An immediate consequence for the forced pendulum equation is the following one.

Theorem 14. *If h is analytical and $\overline{h} = 0$, then, given $N \in \mathbb{Z}$, the number of T-periodic solutions of equation (2.5) that are stable and geometrically different is finite.*

2.8 Existence of more than two T-periodic solutions

In [50], Donati proved that *given $a > 0$ and $T > 0$, there exists some $h^* \in C^T$ with $\overline{h^*} = 0$ and a neighborhoud V of h^* such that for each $h \in V$, equation (2.5) has at least four distinct T-periodic solutions.* The proof is based upon a *classification of singularities of the nonlinear Fredholm operator* $\frac{d^2}{dt^2} + a\sin(\cdot)$. Applying to (2.2) a classical *perturbation method* as used for example by Loud for Duffing's equation, Ortega [146] has recently improved this result by replacing 4 by any even number.

Theorem 15. *Given $a > 0$ and an integer $N \geq 1$, there exists $h^* \in C^T$ satisfying $\overline{h^*} = 0$ and such that equation (2.5) with h replaced by h^* has at least $2N$ distinct T-periodic solutions. In addition, there exists $\delta > 0$ such that if h satisfies $\overline{h} = 0$ and $\|h - h^*\|_{L^1} < \delta$, then the conclusion also holds for equation (2.5).*

The idea of the proof consists in considering the equation

$$y'' + a\sin\left(y + P_0(t)\right) = 0, \tag{2.10}$$

where

$$P_0(t) = 2\pi\left(\frac{t}{T} - \left[\frac{t}{T}\right]\right),$$

which has a continuum $(y_c)_{c\in\mathbb{R}}$ of T-periodic solutions, and in considering a perturbation of equation (2.10)

$$y'' + a\sin\left(y + P_0(t) + \Psi(t,\varepsilon)\right) = 0, \tag{2.11}$$

with conditions upon Ψ insuring that $P_0(t) + \Psi(t,\varepsilon)$ is smooth and that one has at least 2N periodic simultaneous bifurcations for $\varepsilon = 0$.

To motivate a further multiplicity result of perturbation type, let us recall that for the undamped free pendulum equation

$$y'' + a\sin y = 0 \tag{2.12}$$

it is known that the period $T(A)$ of the periodic solutions of (2.12) as a function of their amplitude $A > 0$ is an increasing function such that

$$\lim_{A\to 0+} T(A) = \frac{2\pi}{\sqrt{a}}, \quad \lim_{A\to\pi-} T(A) = +\infty.$$

Consequently, given any positive integer N, then, if $a > \frac{4\pi^2 N^2}{T^2}$, equation (2.12) has a closed orbit with least period $\frac{T}{k}$ for each $k = 1, 2, \ldots, N$. Using a perturbation argument and W. Ding's generalization of the Poincaré-Birkhoff fixed point theorem for area-preserving twist mappings of an annulus, Fonda and Zanolin [65] have proved the following result for the forced case.

Theorem 16. *Given any positive integer N, there exists a constant $a_0 > 0$ such that, for any $a \geq a_0$, equation (2.5) has at least N periodic solutions with minimal period T, which can be chosen to have exactly $2j$ simple crossings with 0 in the interval $[0, T[$, with $j = 1, 2, \ldots, N$.*

2.9 Subharmonic solutions in the conservative case $c = 0$

Let us first recall that, if $k \geq 2$ is an integer, a *subharmonic solution* of order k of (2.1) is a periodic solution of equation (2.1) with minimal period kT. The first existence results for the subharmonic solutions of equation (2.5) with $\overline{h} = 0$ have been obtained by Fonda and Willem [64] (see also Offin [137] for a close result based upon an index theory for periodic extremals and a variant of the mountain pass lemma).

Theorem 17. *Suppose that the T-periodic solutions of equation (2.5) are isolated and that every T-periodic solution of equation (2.5) having Morse index equal to zero is nondegenerate. Then there exists $k_0 \geq 2$ such that, for every prime integer $k \geq k_0$, there is a periodic solution of equation (2.5) with minimal period kT. If moreover the kT-periodic solutions of equation (2.5) are nondegenerate for $k = 1$ and every prime integer k, then there exists a $k_0 \geq 3$ such that, for every prime integer $k \geq k_0$, there are two periodic solutions of equation (2.5) with minimal period kT.*

To prove this result, Fonda and Willem consider the critical points of the functional

$$A_{h,k} = \int_0^{kT} \left(\frac{y'^2(t)}{2} + a \cos y(t) + h(t) y(t) \right) dt,$$

over the Sobolev space H^1_{kT}. Then, by assumption and an easy reasoning, $A_{h,1} = A_h$ has a finite number of critical points $y_0, y_1, \ldots, y_n$, which, of course, are also critical points of $A_{h,k}$ for any $k \geq 2$. The first ingredient of the proof consists in showing the existence of some integer k_0 such that, for $k \geq k_0$ and $0 \leq i \leq n$, either the Morse index $J(y_i, kT, 1)$ of y_i is equal to 0 and y_i is nondegenerate, or $J(y_i, kT, 1) \geq 2$. This is done using an iteration formula for the Morse index due to Bott. Now, let $k \geq k_0$ be a prime number, so that the critical points of $A_{h,k}$ have minimal period T or kT. Assuming by contradiction that $y_0, \ldots, y_n$ are the only critical points of $A_{h,k}$, one is led to a contradiction in the Morse inequalities of *Morse theory* (see e.g. [126]) applied to $A_{h,k}$. The proof of the second part of Theorem 17 is similar.

Combining the Fonda-Willem's theorem with the generic results of [108], one gets the *generic existence of subharmonic solutions.*

Theorem 18. *There exists an open dense subset $\mathcal{G}$ of $\widetilde{C_T}$ such that for every $h \in \mathcal{G}$, there exists a $k_0 \geq 2$ such that, for every prime integer $k \geq k_0$, equation (2.5) has a periodic solution with minimal period kT.*

As shown in [167], the *Lyapunov-Schnitt's reduction method* also provides some information about subharmonic solutions, by relating their existence to the properties of φ_h.

Theorem 19. *Equation (2.5) with $\overline{h} = 0$ has subharmonics of infinitely many distinct levels if and only if φ_h is not constant. If $c_0^T := \min_{H_T^1} A_h$ is isolated in the set of critical levels of A_h, then equation (2.5) with $\overline{h} = 0$ admits subharmonics of arbitrary large minimal period if and only if φ_h is not constant. Finally, the isolatedness assumption in the previous statement can be dropped if $a < \omega^2$.*

Finally, the Fonda-Zanolin multiplicity result [65] has a counterpart for subharmonic solutions, proved using the same technique.

Theorem 20. *Given any two positive integers M, N, there exists a constant $a_0 > 0$ such that, for any $a \geq a_0$, equation (2.5) has, for each $k = 1, 2, \ldots, M$, at least N periodic solutions with minimal period kT.*

2.10 Rotating solutions in the conservative case $c = 0$

Besides periodic solutions, the free pendulum has also *rotating solutions* which are the sum of a linear function of t and of a periodic term. Under some conditions, the conservative forced pendulum (2.5) can also admit such solutions. Most of the results in this case are obtained via combination of Poincaré's method and some theorem for twist mappings.

The following results have been proved by Levi [103] using Moser's twist theorem. The basic idea is that, for large velocities $x = y'$, the forced pendulum equation has solutions which are close to those of the integrable system $y'' = 0$.

Theorem 21. *For any $\omega \in]0, 2\pi[$ satisfying, for some $c_0 > 0$ and $\mu > 0$, the set of inequalities*

$$\left| \frac{\omega}{2\pi} - \frac{m}{n} \right| > \frac{c_0}{n^{2+\mu}},$$

for all $m, n \in \mathbb{Z}$ with $n \neq 0$, there exists an integer $k_0 = k_0(c_0, \mu)$ such that the Poincaré's mapping associated to (2.5) possesses, for all integers k with $|k| \geq k_0$, a countable set of invariant curves $y = f_{\frac{\omega}{2\pi}+k}(x) \equiv f_{\frac{\omega}{2\pi}+k}(x+1)$. For any real number α, equation (2.5) has a Birkhoff orbit with that rotation number. For any rational $\alpha = \frac{p}{q}$ there exists at least two solutions satisfying $y(t+qT) = y(t)+2p\pi$.

A similar result was proved independently by Moser [130], using a variational method which can be traced to Percival and Mather (see [131]).

Theorem 22. *If $\overline{h} = 0$, then, for some sufficiently large irrational α (satisfying a Diophantine condition), equation (2.5) has solutions of the form $y(t) = U(t, \alpha t)$ such that $U(t, \theta) - \theta$ is continuous, T-periodic in t and 2π-periodic in θ, and $\partial_\theta U > 0$.*

Physically, the above result means that there exists a motion with any average angular velocity (see also Dovbysh [52]).

The following result of You [185] is also proved using Moser's twist theorem.

Theorem 23. *Equation* (2.5) *admits an infinite number of invariant tori, and thus an infinite number of almost periodic solutions, when* $\overline{h} = 0$, *and no invariant torus when* $\overline{h} \neq 0$.

In the case of an analytic h, Ortega's approach described in Section 2.7.2 provides some information about the number and stability of rotating solutions [145].

Theorem 24. *If h is analytic and $\overline{h} = 0$, then, given $N \in \mathbb{Z}$, the number of stable and distinct T-periodic solutions with winding number N (i.e. solution such that $y(t+T) = y(t) + 2N\pi$) of equation* (2.5) *is finite.*

Finally, the change of variable $y(t) = k\omega t + v(t)$, and the use of direct methods of the calculus of variations to the transformed equation allows a very simple proof of the following special case of Theorem 13 [121].

Theorem 25. *For each $a > 0$, $T > 0$, $k \in \mathbb{Z} \setminus \{0\}$, and each h with $\overline{h} = 0$, equation* (2.5) *has at least one solution of the form $y(t) = k\omega t + v(t)$ with v T-periodic.*

2.11 Lagrange stability

2.11.1 The conservative case $c = 0$

Equation (2.5) is called *Lagrange stable* if any solution of (2.1) is bounded over $\mathbb{R}$ in the phase cylinder $\{(y \bmod 2\pi, y')\}$. Physically, this means that any solution of (2.1) has angular velocity bounded over $\mathbb{R}$.

The problem of the Lagrange stability of equation (2.5) was raised by Moser in the Introduction of [129]. Its positive solution is a consequence of the results of Levi, Moser and You described in the previous section.

Theorem 26. *If $\overline{h} = 0$, then for any sufficiently large $N > 0$, there exists $M = M(N)$ such that any solution $y(t)$ of equation* (2.5) *with $|y'(0)| \leq M$ satisfies $|y'(t)| \leq N$ for all $t \in \mathbb{R}$.*

As shown by You [185], the conditions that the mean value of h is zero is necessary and sufficient for the Lagrange stability.

Theorem 27. *If $a > 0$, then equation* (2.5) *is Lagrange stable if and only if* $\overline{h} = 0$.

2.11.2 The dissipative case $c > 0$

Some results for Lagrange stability in the dissipative case have been obtained by Andres [4,5] and Andres-Staněk [9] using Lyapunov function techniques. See also [6,7,8] for further discussions and problems.

Here, *Lagrange stability* of (2.1) has to be understood as the boundedness *over* $\mathbb{R}_+$ of any solution in the phase cylinder $\{y \bmod 2\pi, y'\}$. Physically, that means that any solution of (2.1) has angular velocity bounded in the future.

Theorem 28. *The equation* (2.1) *is Lagrange stable provided* $\overline{h} = 0$ *and*

$$c > \frac{(a + \|h\|_\infty)\left\{\|H\|_\infty + \left[\|H\|_\infty^2 + 4(2a + \pi(a + \|h\|_\infty))\right]^{1/2}\right\}}{2(2a + \pi(c + \|h\|_\infty))},$$

where $H(t) = \int_0^t h(s)\,ds$.

3 Bounded or almost periodic forcing

3.1 Bounded forcing

Using a version of the *method of upper and lower solutions* for solutions bounded over $\mathbb{R}$ going back to to Opial [138] (see also [122]), one can prove the following result, which is the one dimensional case of a result for elliptic partial differential equations due to Fournier, Szulkin et Willem [69]. Consider the dissipative forced pendulum-type equation

$$y'' + cy' + a\sin y = h(t), \tag{3.1}$$

where $a > 0$, $c \geq 0$, and $h : \mathbb{R} \to \mathbb{R}$ is continuous and bounded.

Theorem 29. *If* $c \geq 0$ *and if* $h : \mathbb{R} \to \mathbb{R}$ *continuous is such that*

$$-a \leq h(t) \leq a, \tag{3.2}$$

for all $t \in \mathbb{R}$, *then equation* (3.1) *has at least one solution* y *such that*

$$\frac{\pi}{2} \leq y(t) \leq \frac{3\pi}{2}$$

for all $t \in \mathbb{R}$. *If condition* (3.2) *is restricted to*

$$\|h\|_\infty < a, \tag{3.3}$$

then there exists $\varepsilon > 0$ *such that equation* (3.1) *has a unique solution* y *such that*

$$\frac{\pi}{2} + \varepsilon \leq y(t) \leq \frac{3\pi}{2} - \varepsilon \tag{3.4}$$

for all $t \in \mathbb{R}$.

Using the equivalent formulation for the forced pendulum problem together with some results of Ortega on bounded solutions of second order linear equations [144] (see also [122]), one can prove in a similar way the following existence and uniqueness theorem [123].

Theorem 30. *If $c > 0$, $h = h^* + h^{**}$ where h^{**} is bounded and h^* has a bounded primitive over $\mathbb{R}$, and if inequalities*

$$osc_{\mathbb{R}} H_c^* \leq \pi,$$

and

$$\|h^{**}\|_\infty \leq a \cos\left(\frac{osc_{\mathbb{R}} H_c^*}{2}\right),$$

hold, where H_c^* is the unique bounded solution of $y'' + cy' = h^*(t)$, then equation (3.1) has at least one solution y such that

$$\frac{\pi}{2} + H_c^*(t) \leq y(t) \leq \frac{3\pi}{2} + H_c^*(t),$$

for all $t \in \mathbb{R}$. If the inequalities above are strenghtened to

$$osc_{\mathbb{R}} H_c^* < \frac{\pi}{2}, \tag{3.5}$$

$$\|h^{**}\|_\infty \leq \frac{a\sqrt{2}}{2}\left[\sin\left(\frac{osc_{\mathbb{R}} H_c^*}{2}\right) + \cos\left(\frac{osc_{\mathbb{R}} H_c^*}{2}\right)\right], \tag{3.6}$$

then there exists $\varepsilon > 0$ such that equation (3.1) has a unique solution y satisfying the inequality

$$\frac{\pi}{2} + \varepsilon \leq y(t) \leq \frac{3\pi}{2} - \varepsilon, \tag{3.7}$$

for all $t \in \mathbb{R}$. When $c = 0$, the above results hold if $h^{**} = 0$, $h = h^*$ has a second primitive H^1 bounded over $\mathbb{R}$ and H_c^* is replaced by H^1 in (3.5).

3.2 Particular almost periodic forcings

3.2.1 A class of almost periodic functions

The following classes of almost periodic functions was introduced by Belley, Fournier and Saadi Drissi [19,20,21]. Given a countable set $\Gamma \subset \mathbb{R}$, symmetric with respect to the origin, put

$$C_\Gamma = \left(\sum_{\lambda \in \Gamma \setminus \{0\}} \frac{1}{\lambda^2}\right)^{1/2}.$$

Let $P_\Gamma(\mathbb{R})$ denote the class of all (real-valued) trigonometric polynomials $p(t) = \sum_{\lambda \in \Gamma} \alpha_\lambda e^{i\lambda t}$ where all but finitely many of the coefficients α_λ vanish, and $\alpha_{-\lambda}$

is the complex conjugate of α_λ. On $P_\Gamma(\mathbb{R})$ one can put the uniform norm $\|\cdot\|_\infty$ and the norm $\|\cdot\|_2$ associated with the inner product

$$\langle p, q\rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T p(t)q(t)\,dt.$$

Let $AP_\Gamma(\mathbb{R})$ and $B^2_\Gamma(\mathbb{R})$ denote the completion of $P_\Gamma(\mathbb{R})$ with respect to the norms $\|\cdot\|_\infty$ and $\|\cdot\|_2$ respectively. The operation $\langle\cdot,\cdot\rangle$ can be extended to $B^2_{\mathbb{R}}(\mathbb{R})$ by defining it to be the inner product on $B^2_{\mathbb{R}}(\mathbb{R})$ associated with the norm $\|\cdot\|_2$.

For any $x \in AP_{\mathbb{R}}(\mathbb{R})$, define $\widehat{x} : \mathbb{R} \to \mathbb{C}$ by

$$\widehat{x}(\lambda) = \langle x(t), e^{-i\lambda t}\rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)e^{-i\lambda}\,dt.$$

This notation can be extended to $x \in B^2_{\mathbb{R}}(\mathbb{R})$ by $\widehat{x}(\lambda) = \lim_{n\to\infty} \widehat{p_n}(\lambda)$ for any sequence $\{p_n\}$ in $P_{\mathbb{R}}(\mathbb{R})$ such that $\|p_n - x\|_2 \to 0$. For any subset X of $B^2_\Gamma(\mathbb{R})$, let $\widetilde{X} : \{x \in X : \widehat{x}(0) = 0.\}$ One often writes

$$\overline{x} = \widehat{x}(0) = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,dt,$$

and $\widetilde{x}(t) = x(t) - \overline{x}$.

If $x \in B^2_\Gamma(\mathbb{R})$ and $y \in \widetilde{B}^2_\Gamma(\mathbb{R})$ are such that

$$\langle x, p'\rangle = -\langle y, p\rangle$$

for all $p \in P_\Gamma(\mathbb{R})$, then y is said to be the *weak derivative* of x. Note that y is necessarily unique in $\widetilde{B}^2_\Gamma(\mathbb{R})$, and we write $y = x'$.

3.2.2 The results

Consider first the dissipative forced pendulum-type equation

$$y'' + cy' + a\sin y = h(t), \tag{3.8}$$

where $a > 0$, $c \geq 0$, and h almost periodic.

The following result is due to Belley-Fournier-Saadi Drissi [20], and proved using a Lyapunov-Schmidt's argument modeled on that of [62].

Theorem 31. *Let $e \in B^2_\Gamma(\mathbb{R})$ be fixed and assume that the following conditions hold.*

1. $C_\Gamma < +\infty$.
2. $c > 0$ *and* $a < \frac{c}{C_\Gamma}$.
3. $\beta := C_\Gamma(C_\Gamma^{-2} + c^2)^{-1/2} a \leq \delta(\widetilde{e})$, *where*

$$\delta(\widetilde{e}) = \left[\left(\overline{\cos \widetilde{E}}\right)^2 + \left(\overline{\sin \widetilde{E}}\right)^2\right]^{1/2},$$

and $\widetilde{E}(t)$ is the unique weak almost periodic solution of equation $y'' + cy' = \widetilde{e}(t)$.

4. $|\overline{e}| \leq (\delta(\widetilde{e}) - \beta)A$.

Then there exists some $\widetilde{\mathcal{Y}}$ *in the orthogonal supplement of* $B^2(\Gamma)$ *in* $B^2(\mathbb{R})$ *such that equation* (3.8) *with* $h = e + \widetilde{\mathcal{Y}}$ *has at least one weak almost periodic solution* $y \in AP_\Gamma(\mathbb{R})$ *such that* $y' \in \widetilde{AP}_\Gamma(\mathbb{R})$ *and* $y'' \in \widetilde{B}^2_\Gamma(\mathbb{R})$.

Consider now the conservative forced pendulum equation

$$y'' + a \sin y = h(t), \tag{3.9}$$

where $a > 0$, and h almost periodic.

The following result was proved by Belley-Fournier-Saadi Drissi [19] and Belley-Fournier-Hayes [18] using a Lyapunov-Schmidt's argument modeled on that of [62].

Theorem 32. *If* $C_\Gamma < \infty$, *then given* $\xi \in \mathbb{R}$, *and* $\widetilde{e} \in \widetilde{B}^2_\Gamma(\mathbb{R})$, *there exists a function* $\gamma \in B^2_{\mathbb{R}}(\mathbb{R}) \ominus \widetilde{B}^2_\Gamma(\mathbb{R})$ *such that the equation*

$$z'' + a \sin(\xi + z) = \gamma(t) + \widetilde{e}(t),$$

holds in $\widetilde{B}^2_\Gamma(\mathbb{R})$ *for some* $z \in \widetilde{AP_\Gamma}(\mathbb{R})$ *for which the weak derivative* $z' \in \widetilde{B}^2_\Gamma(\mathbb{R})$ *exists and admits a weak derivative* $z'' \in \widetilde{B}^2_\Gamma(\mathbb{R})$. *Furthermore, if* $a < C_\Gamma^{-2}$, *this solution* z *is unique.*

3.3 General almost periodic forcing

Combining some results on the existence and uniqueness of bounded solutions over $\mathbb{R}$ with Amerio's criterion on the existence of almost periodic solutions (see e.g. [60]), Fink [59] has given in 1968 some partial extension of the *method of upper and lower solutions* to almost periodic solutions. A special case of his results is the following proposition.

Lemma 13. *Let* $c \in \mathbb{R}$, $g \in C^1(\mathbb{R}, \mathbb{R})$ *and* h *continuous and almost periodic. Assume that there exist* $a < b$ *and* $\lambda \in \mathbb{R}$ *such that* $g'(x) > 0$ *for all* $x \in [a, b]$, *and*

$$g(a) + h(t) \leq 0 \leq g(b) + h(t)$$

for all $t \in \mathbb{R}$. *Then equation*

$$y'' + cy' = g(y) + h(t)$$

has a unique almost periodic solution y *such that* $a \leq y(t) \leq b$ *for all* $t \in \mathbb{R}$.

This result implies the following existence theorem, also proved independently by Fournier-Szulkin-Willem [69] as a special case of a more general result for elliptic partial differential equations.

Theorem 33. *For each* $c \geq 0$ *and each* $h \in AP(\mathbb{R})$ *such that* $\|h\|_\infty < a$, *equation* (3.8) *has a unique solution* $y \in AP(\mathbb{R})$ *such that* $\pi/2 < y(t) < 3\pi/2$.

Indeed, the condition upon $\|h\|_\infty$ implies the existence of $\varepsilon > 0$ such that $a = \frac{\pi}{2} + \varepsilon$ and $b = \frac{3\pi}{2} - \varepsilon$ satisfy the conditions of Lemma 13. The result with $c = 0$ generalizes an earlier approximate solvability result of Blot [27] for equation (3.9), based upon variational techniques and convex analysis, which provides the existence for a dense subset of forcing functions h only.

Similar arguments applied to the equivalent formulation of the forced pendulum equation provide the following existence theorem [123].

Theorem 34. *If $c > 0$, $h = h^* + h^{**}$ where h^{**} is almost periodic and h^* has an almost periodic primitive, and if conditions (3.5) and (3.6) are satisfied, then there exists $\varepsilon > 0$ such that equation (3.8) has a unique almost periodic solution verifying inequality (3.7). If $c = 0$, and $h \in C$ has an almost periodic second primitive H^1 satisfying (3.5) with H_c^* replaced by H^1, then the same conclusion holds.*

This result when $c = 0$ generalizes an earlier approximate solvability result of Blot [28] for equation (3.9), based upon variational techniques and convex analysis, which gives existence for a dense subset of forcing functions h only.

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