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The Abstract Cauchy Problem in Plasticity

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Abstract. The boundary-value problem of plasticity is formulated as the evolution variational problem (EVP) over the parameter of external loading for the displacement in the framework of the small deformations theory. The questions of the mathematical correctness of the plasticity EVP are discussed. The general existence and uniqueness theorem is formulated. The main necessary and sufficient condition has the simplest algebraic form and does not coincide with the classic Drucker's hypothesis and similar thermodynamical postulates. By means of finite element approximation the plasticity EVP transforms into the Cauchy problem for a non-linear system of ordinary differential equations unsolved regarding derivative. Moreover, this system can be stiff. Therefore, for the numerical solution the implicit Euler scheme with the decomposition method of adaptive block relaxation (ABR) is used. The numerical results show that, for finding the displacement and the time of calculation, the ABR method has advantages over the standard method.

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1 Introduction

The solution of plasticity boundary-value problems (BVPs) is of particular interest in both theory and practice. At present there are many models of plasticity in the framework of the small deformations theory [1,2,3]. Adequacy and the field of application of every model must be found only by correlation between experimental data and solutions of appropriate BVPs. Therefore, the analysis of mathematical correctness and the treatment of numerical methods for these problems is very important [4,5,6,7].

In this paper the plasticity BVP is formulated as the evolution variational problem (EVP) (i.e. as the abstract Cauchy problem in the weak form) for the displacement in the Hilbert space [8]. For this reason the parameter of external loading in the interval $[0, 1]$ is used. The general existence and uniqueness theorem for the plasticity EVP is formulated. The proof of this theorem is based on

This is the final form of the paper.

the monotonous operators theory and the theory of the abstract Cauchy problem in the Hilbert space [7]. The main necessary and sufficient condition has the simplest algebraic form and does not coincide with the classic Drucker's hypothesis and similar thermodynamical postulates [2,3,9,10]. This condition is the general criterion of mathematical correctness for plasticity models. Its independence is illustrated for the plasticity model of linear isotropic-kinematic hardening with ideal Bauschinger's effect, dilatation and internal friction [11,12].

For the numerical solution of the plasticity EVP the standard spatial piecewise linear finite element approximation is used [13]. For some models the appropriate finite dimensional Cauchy problem can be stiff [14,15]. The main cause of this phenomenon consists of the following: the global shear stiffness matrix has lines with significantly different factors (it is badly determined). Moreover, for real plasticity models both initial continuum and discrete Cauchy problems are principally unsolved regarding derivative [1,2,12]. Therefore, for the numerical solution the implicit Euler scheme with the decomposition method of adaptive block relaxation (ABR) is used [4,5,6,7]. The main idea of this method consists of iterative improvement of zones with "proportional" deformation by special decomposition of domain (variables), and separate calculation in these zones (on these variables). The global convergence theorem for the ABR method is formulated. The proof of this theorem is based on the monotonous operators theory [4,5].

The numerical results show that, for finding the displacement and the time of calculation, the ABR method has advantages over the standard method.

2 Evolution formulation of the plasticity BVP

Let a homogeneous rigid body in the undeformed reference configuration occupy a domain $\Omega \subset R^3$ with boundary Γ. In the deformed configuration each point $x \in \overline{\Omega}$ moves into a position $x+u(x) \in R^3$, where $u : \overline{\Omega} \to R^3$ is the displacement. In the framework of the small deformations theory the strain Cauchy tensor $\varepsilon = \varepsilon(u) = \frac{1}{2}\left(\partial^i u_j + \partial^j u_i\right) : \Omega \to S^3$ is used as the measure of deformation, where $\partial^i = \partial/\partial x_i$; $i,j = 1,2,3$. The symbol S^3 denotes the subspace of real symmetrical 3×3 matrices.

In the mathematical theory of plasticity the isotropic material is described by the constitutive relation for speeds [1,2,3,7,10,12]

$$\begin{gathered}\dot{\sigma}_{ij} = S_{ij}\left(\varepsilon, \dot{\varepsilon}\right) = C_{ijkm}\left(\dot{\varepsilon}_{km} - \dot{P}_{km}(\varepsilon, \dot{\varepsilon})\right), \\ C_{ijkm} = 2\mu\,\delta_{ik}\delta_{jm} + \left(k_0 - \tfrac{2}{3}\mu\right)\delta_{ij}\delta_{km},\end{gathered} \tag{1}$$

where $\sigma : \Omega \to S^3$ is the Cauchy stress tensor, $P : \Omega \to S^3$ is the plastic part of the Cauchy strain tensor, C_{ijkm} are the components of elasticity acoustic tensor [2,3], $\mu > 0$ and $k_0 > 0$ are the shear and bulk moduli, respectively; δ_{ij} is the Kronecker symbol, the above point is d/dt and $t \in [0,1]$ is the parameter of external loading. Here and in what follows we use the rule of summing over repeated indices and the designation $|A| = |A_{km}| = (A_{ij}A_{ij})^{1/2}$ for modulus of matrix A.

We consider the following boundary-value problem. The quasi-static influences acting on the body are: a mass force with density f in Ω, a surface force with density F on a portion Γ^2 of the boundary, and a surface displacement u_γ on a portion Γ^1 of the boundary is also given. Here $\Gamma^1 \cup \Gamma^2 = \Gamma$, $\Gamma^1 \cap \Gamma^2 = \emptyset$ and area$(\Gamma^1) > 0$.

According to the evolution description [8] the external influences, internal displacement and stress tensor are taken as continuous and piecewise smooth abstract functions acting from interval $[0, 1]$ to appropriate Banach spaces, supposing that $\Gamma^1 = \mathrm{const}(t)$ and $f = 0$, $F = 0$, $u_\gamma = 0$ for $t = 0$.

The plasticity BVP is formulated as the evolution variational problem (EVP) (i.e. as the abstract Cauchy problem in the weak form): the sought displacement corresponds to the abstract function $u^*(t) = u^0(t) + u(t)$, where the piecewise smooth abstract function $u^0(t)$ with $u^0(0) = 0$ corresponds to the surface displacement u_γ, and unknown abstract function $u : [0, 1] \to V^0$ must satisfy the initial condition $u(0) = 0$ and the differential equation for every $v \in V^0$ and almost every $t \in (0, 1)$

$$\begin{gathered} \int\limits_{\Omega} S_{ij}\left(\varepsilon(u^0 + u), \varepsilon(\dot{u}^0 + \dot{u})\right) \partial^j v_i \, dx = L(t, v), \\ L(t, v) = \int\limits_{\Omega} \dot{f}_i(t) v_i \, dx + \int\limits_{\Gamma^2} \dot{F}_i(t) v_i \, d\gamma. \end{gathered} \tag{2}$$

Here $V^0 = \{v : \Omega \to R^3;\, v(x) = 0,\, x \in \Gamma^1\}$ — is the set of kinematically admissible variations of the displacement. For real plasticity models this equation is principally unsolved regarding $\dot{u}$ [2,7,12].

Concerning the constitutive relation S, the domain Ω and the functions f, F, u_γ we make the following hypotheses:

(H1) Matrix function $S(A, B)$ is the continuous and strongly monotonous in B, i.e. there exists a constant $m_0 > 0$ such that for every $A, B^1, B^2 \in S^3$ the following estimate is true

$$\left(S_{ij}(A, B^1) - S_{ij}(A, B^2)\right)\left(B^1_{ij} - B^2_{ij}\right) \geq m_0 \left|B^1 - B^2\right|^2.$$

(H2) Matrix function $S(A, B)$ is the Lipschitz continuous in A, i.e. there exists a scalar function $M_0 : S^3 \to (0, +\infty)$ such that for every $A^1, A^2, B \in S^3$ the following estimate is true

$$\left|S(A^1, B) - S(A^2, B)\right| \leq M_0(B) \left|A^1 - A^2\right|.$$

(H3) Matrix function $S(A, B)$ has the growth in A and B no above linear, i.e. there exists a constant $M_1 > 0$ such that for every $A, B \in S^3$ the following estimate is true

$$|S(A, B)| \leq M_1 \left(|A| + |B|\right).$$

(H4) Ω is a connected bounded domain in R^3 with a Lipschitz boundary Γ.

(H5) $f \in C^{0,1}\left([0,1], L^{6/5}(\Omega, R^3)\right)$.

(H6) $F \in C^{0,1}\left([0,1], L^{4/3}(\Gamma^2, R^3)\right)$.

(H7) $u_\gamma \in C^{0,1}\left([0,1], L^{2}(\Gamma^1, R^3)\right)$.

We define the set of kinematically admissible variations of the displacement in the following way:

$$V^0 = \left\{ v \in H^1 : \; v(x) = 0, \; x \in \Gamma^1 \right\},$$

where $H^1 := W^{1,2}(\Omega, R^3)$ is the Hilbert space.

Theorem 1 (was proved in [7]). *In the framework of the hypotheses* (H1)–(H7) *the following statements are true:*

(i) *The unique strict solution of the EVP* (2) *exists, i.e. the absolutely continuous function* $u \in C^{0,1}([0,1], V^0)$, $u(0) = 0$ *with the strong derivative* $\dot{u}$, *satisfying the equation* (2) *for a.e.* $t \in (0,1)$.

(ii) *The map* $(f, F, u_\gamma) \mapsto u$ *is continuous.*

Remark 2. For the constitutive relation S the main condition (H1) is necessary and sufficient. It is the general criterion of mathematical correctness for plasticity models. This question is in detail discussed in [7]. Therefore, we rewrite this condition for the matrix function $\dot{P}(\varepsilon, \dot{\varepsilon})$, usually used in the modern theory of plasticity [2,7,12].

(H1) Matrix function $\dot{P}(A,B)$ is continuous in B and satisfies the following estimate for every $A, B^1, B^2 \in S^3$

$$C_{ijkm} \left(\dot{P}_{km}(A, B^1) - \dot{P}_{km}(A, B^2) \right) \left(B^1_{ij} - B^2_{ij} \right) < 2\,\mu \left| B^1 - B^2 \right|^2 . \quad (3)$$

This condition does not coincide with the Lipschitz condition of the matrix function $\dot{P}(A,B)$ over second matrix argument. It is easily to get convinced that the Lipschitz condition is stronger than the condition (3). In the following section we show that this condition is independent and does not coincide with the classic Drucker's hypothesis based on the thermodynamical postulates [2,3,9,10].

3 Example of analysis of plasticity models

The independence of the main necessary and sufficient condition (3) of mathematical correctness of plasticity EVP (2) we illustrate for the generalized model of plasticity with linear isotropic-kinematic hardening, ideal Bauschinger's effect, dilatation and internal friction [12]

$$
\begin{gathered}
\dot{P}_{km} = (1 + h_0 + 3\lambda\Lambda)^{-1} H\left(\rho_e - \varepsilon_* + \lambda\,\mathrm{tr}(\varepsilon - P)\right) \times \\
\times H\left(\cos\gamma + \lambda\dot{\varepsilon}_e^{-1}\mathrm{tr}(\dot{\varepsilon})\right)\rho_e^{-2}(\rho_{km} + \Lambda\rho_e\delta_{km})(\rho_{pq} + \lambda\rho_e\delta_{pq})\dot{\varepsilon}_{pq}, \\
\rho_{km} = \varepsilon_{km}^D - (1 + h_0)P_{km}^D, \qquad \cos\gamma = (\rho_e\dot{\varepsilon}_e)^{-1}\rho_{ij}\dot{\varepsilon}_{ij}^D, \\
\rho_e = \left|\rho_{ij}\right|, \qquad \dot{\varepsilon}_e = \left|\dot{\varepsilon}_{ij}^D\right|, \\
H(q) = 0 \quad for \quad q < 0 \qquad and \qquad H(q) = 1 \quad for \quad q \geq 0,
\end{gathered} \tag{4}
$$

where h_0 is the parameter of plastic hardening, $\lambda \geq 0$ and $\Lambda \geq 0$ are the parameters of dilatation and internal friction, respectively; $\varepsilon_* \geq 0$ is the limit of elastic strain, $A_{ij}^D = A_{ij} - \frac{1}{3}\mathrm{tr}(A)\delta_{ij}$ are the components of deviatoric part and $\mathrm{tr}(A) = \delta_{ij}A_{ij}$ is the trace (first invariant) of matrix A.

For $\lambda = \Lambda = 0$ model (4) equals the classic model of plasticity with linear isotropic-kinematic hardening and ideal Bauschinger's effect [1,2,3]. In this case $\mathrm{tr}(P) = 0$ and the constitutive relation (4) is associated with the Mises yield surface $\rho_e - \varepsilon_* = 0$ [2,3].

For $\lambda = \Lambda \neq 0$ the constitutive relation (4) is associated with the yield surface for strain $\rho_e - \varepsilon_* + \lambda\,\mathrm{tr}(\varepsilon - P) = 0$. This surface for $h_0 = 0$ corresponds to the Mises-Schleiher yield surface for stress $\left|\sigma^D\right| + c^{-1}\lambda\,\mathrm{tr}(\sigma) - 2\mu\,\varepsilon_* = 0$, where $c = 3k_0/(2\mu)$ [11]. For $\lambda \neq \Lambda$ the constitutive relation (4) is non-associated with some yield surface. In both cases $\mathrm{tr}(P) \neq 0$ what is a well known experimental phenomenon of dilatation [11,12].

Let matrices $A, B^1, B^2 \in S^3$ be arbitrary. Then from condition (3) for model (4) we have

$$
\begin{aligned}
C_{ijkm}\left(\dot{P}_{km}(A, B^1) - \dot{P}_{km}(A, B^2)\right)\left(B_{ij}^1 - B_{ij}^2\right) \leq \\
\leq (1 + h_0 + 3\lambda\Lambda)^{-1}\left[\,\left|B^1 - B^2\right| + \lambda\,\mathrm{tr}\left(B^1 - B^2\right)\right] \times \\
\times \left[\,2\mu\left|B^1 - B^2\right| + 3k_0\Lambda\,\mathrm{tr}\left(B^1 - B^2\right)\right] \leq \\
\leq 2\mu\,\Psi(\lambda, \Lambda, h_0)\left|B^1 - B^2\right|^2,
\end{aligned}
$$

where

$$
\Psi = \frac{(1 + \sqrt{3}\,\lambda)(1 + \sqrt{3}\,c\Lambda)}{1 + h_0 + 3\lambda\Lambda}.
$$

The constant $c = (1 + \nu)/(1 - 2\nu) \geq 1$, because for real materials the Poisson ratio $0 \leq \nu < 1/2$ [1,2,3]. Therefore, for parameters $\lambda, \Lambda \geq 0$ the condition (3) is true ($\Psi < 1$) only for the positive parameter of plastic hardening, satisfying the following estimate

$$
h_0 > 3(c - 1)\lambda\Lambda + \sqrt{3}\,(\lambda + c\Lambda) \geq 0. \tag{5}
$$

If this condition is disturbed then the effects of bifurcation and internal instability exist in the plasticity EVP (2) [9,12].

The classic Drucker's hypothesis $\dot{\sigma}_{ij}\dot{P}_{ij} \geq 0$ is the only necessary condition for the *uniqueness* of solution of EVP (2). For the model (4) it has the following form

$$h_0 \geq 3\,(c\Lambda - \lambda)\Lambda. \tag{6}$$

For example, if $\Lambda = 0$, $\lambda > 0$ then the condition (5) is carried out only for $h_0 > \sqrt{3}\,\lambda > 0$, but the condition (6) is fulfilled for $h_0 \geq 0$. This simple example proves that the classic Drucker's hypothesis, based on thermodynamical postulates [2,3,9,10], does not provide the *existence* of solution of the plasticity EVP (2).

4 Computational method

For the numerical solution of the plasticity EVP (2) the standard spatial piecewise linear finite element approximation is here used: $\Omega_h = \cup T_h$, $\Gamma_h = \partial\Omega_h$ and $\mathrm{vol}(\Omega\backslash\Omega_h) \to 0$, $\mathrm{area}(\Gamma\backslash\Gamma_h) \to 0$ for $h \to 0$ regularity, where T_h is the simplest simplex and h is the step of approximation [13].

For the displacement the following piecewise linear approximation is used

$$u_h(t,x) = U^\gamma(t)\Phi_\gamma(x) \qquad (\gamma = 1, 2, \ldots, m),$$

where $U^\gamma \in R^3$ is the displacement in the node x^γ, $\Phi_\gamma : \Omega_h \to R$ is the standard continuous piecewise linear function such that $\Phi_\gamma(x^\alpha) = \delta_{\alpha\gamma}$ $(\alpha, \gamma = 1, 2, \ldots, m)$, m is the number of nodes. In this case the subspace $V^0 \subset H^1$ is approximated by the subspace $V_h^0 \subset R^{3m}$

$$V_h^0 = \left\{U \in R^{3m} : \, U^\alpha = 0, \, x^\alpha \in \Gamma_h^1\right\}.$$

The plasticity EVP (2) is approximated by the Cauchy problem for nonlinear system of ordinary differential equations: vector function $U : [0,1] \to V_h^0$ must satisfy the initial condition $U(0) = 0$ and the following differential equation for almost every $t \in (0,1)$

$$A_{pq}(U, \dot{U})\dot{U}_q = B_p, \tag{7}$$

where U is the global vector of free nodal displacements, A is the global shear stiffness matrix and in the end B is the global vector of nodal speeds of influences; $p, q = 1, 2, \ldots, 3m$. Here $U_p = U_i^\gamma$ with index $p = 3(\gamma - 1) + i$. Due to the properties of the real plasticity models this equation is principally unsolved regarding $\dot{U}$ in the explicit form.

For some plasticity models the differential system (7) can be stiff. The main cause of this phenomenon consists of the following: matrix A has lines with significantly different factors (it is badly determined) for the small parameter of plastic hardening $h_0 \ll 1$ [4,5,6,7].

Example 3. Let the bounded rigid body $\Omega \subset R^3$ with the regular boundary Γ consist of incompressible material describing by the model (4) with parameters $\lambda = \Lambda = 0$. The body is fastened on a portion Γ^1 of the boundary (i.e. $u_\gamma \equiv 0$)

and deformed by the external forces. In this case the set of kinematically admissible displacements is

$$V^0_{\mathrm{div}} = \left\{u \in V^0 :\ \mathrm{div}(u(x)) = 0,\ x \in \Omega\right\}.$$

We use the following approximation for unknown displacement

$$u_N(t,x) = Y_r(t) w^r(x) \qquad (r = 1,\dots,N),$$

where $\{w^r\}_{r=1}^N \subset V^0_{\mathrm{div}}$ are the basic functions.

In this case the plasticity EVP (2) is approximated by the Cauchy problem for nonlinear system of ordinary differential equations: vector function $Y : [0,1] \to R^N$ must satisfy the initial condition $Y(0) = 0$ and the following differential equation for almost every $t \in (0,1)$

$$A_{qr}(Y,\dot Y)\dot Y_r = B_q \qquad (q,r = 1,2,\dots,N), \tag{8}$$

where

$$A_{qr}(Y,\dot Y) = \int_\Omega \Psi_{qr}(Y,\dot Y)|\varepsilon(w^q)|\,|\varepsilon(w^r)|\,dx,$$

$$\Psi_{qr} = \cos\gamma_{qr} - (1-\psi)H(\rho_e - \varepsilon_*)H(\cos\gamma)\cos\gamma_q\cos\gamma_r,$$

$$B_q = (2\mu)^{-1}L(t,w^q).$$

Here and in what follows the summing over indices q,r,s does not used, $\rho_e = \rho_e(u_N)$, $\gamma = \gamma(u_N,\dot u_N)$ from (4), the parameter $\psi = h_0/(1+h_0)$ and

$$\cos\gamma_s = \left(\rho_e\,|\varepsilon(w^s)|\right)^{-1}\rho_{ij}\varepsilon_{ij}(w^s) \quad (s = q,r),$$

$$\cos\gamma_{qr} = \left(|\varepsilon(w^q)|\,|\varepsilon(w^r)|\right)^{-1}\varepsilon_{ij}(w^q)\varepsilon_{ij}(w^r).$$

Due to the properties of the finite element approximation the matrix A is symmetrical and has the largest elements on the main diagonal

$$A_{qq}(Y,\dot Y) = \int_\Omega \Psi_{qq}(Y,\dot Y)|\varepsilon(w^q)|^2\,dx,$$

$$\Psi_{qq} = \left[1 - (1-\psi)H(\rho_e - \varepsilon_*)H(\cos\gamma)\cos^2\gamma_q\right].$$

If the solution of problem (8) has the zone of active deformation with a small curvature trajectory ($\gamma \sim 0$) then for basic functions w^q with $\cos\gamma_q \approx 1$ the factors $\Psi_{qq} \approx \psi$. In the zone of passive deformation, or for a large curvature trajectory ($\gamma \sim \pi/2$), or for basic functions w^r with $\cos\gamma_r \approx 0$ the factors $\Psi_{rr} \approx 1$.

It is easily seen that for the small parameter of plastic hardening ($h_0 \ll 1$) the global shear stiffness matrix A has lines with significantly different factors (it is badly determined). As a result, the following estimate was proved in [4,6]

$$\mathrm{cond}(A) := \frac{\nu_{\max}}{\nu_{\min}} \ge C\,N^2 h_0^{-1} \gg 1,$$

where $\mathrm{cond}(A)$ is the condition number of matrix A; $\nu_{\max}$ and $\nu_{\min}$ are the largest and smallest eigenvalues of the matrix A, respectively, and $C = \mathrm{const}(N,h_0)$.

According to standard technique [14,15] for the solution of the unsolved regarding derivative and stiff problem (7) the implicit Euler scheme is used

$$
\begin{gathered}
A_{pq}(U^k + \tau V, V)V_q = B_p^{k+1}, \qquad V \in V_h^0, \\
U^{k+1} = U^k + \tau V, \qquad U^0 = 0,
\end{gathered} \tag{9}
$$

where index k corresponds to the parameter $t_k = k\tau$, $k = 0, 1, \ldots, K-1$; $\tau = 1/K$ and $K \gg 1$. Here and in what follows the summing over index k does not used.

For the numerical solution of algebraic system (9) for every $k = 0, 1, \ldots, K-1$ the decomposition method of adaptive block relaxation (ABR) is used. This method disregards the condition number of the matrix A and has the following description [4,5,6,7].

Step 1. As the initial approach the explicit solution is used (here O is the zero vector)

$$
Y_q^{(0)} = A_{pq}^{-1}(U^k, O)B_p^{k+1}.
$$

Step 2. Due to the properties of the finite element approximation the matrix A has the largest elements on the main diagonal. Therefore, by the current approach $Y^{(m)}$ variables are separated on quick and slow ones by the proximity criterion of appropriate diagonal elements of the matrix $A^{(m)} = A\left(U^k + \tau Y^{(m)}, Y^{(m)}\right)$

$$
\begin{gathered}
I_s^{(m)} = \left\{ p = 1, 2, \ldots, N : \Delta^{(s-1)/L} \le A_{pp}^{(m)}/d^{(m)} < \Delta^{s/L} \right\}, \\
I_L^{(m)} = \{1, 2, \ldots, N\} \setminus \bigcup_{s=1}^{L-1} I_s^{(m)},
\end{gathered}
$$

where $s = 1, 2, \ldots, L-1$; $\Delta = D^{(m)}/d^{(m)}$; $D^{(m)}$ and $d^{(m)}$ are the largest and smallest diagonal elements of the matrix $A^{(m)}$, respectively, $L = \mathrm{int}(\omega \lg \Delta) + 1$ is the number of blocks ($1 \le L \le N$), $\omega \ge 0$ is the decomposition parameter.

Step 3. The block version of the Seidel method is used [16]. In practice one step of this method is enough (here the summing over index s does not used)

$$
\begin{gathered}
Y^{(m+1)} = \left\{ w^1 \oplus w^2 \oplus \cdots \oplus w^L \right\}^T, \\
w_i^s = [\Lambda^{ss}]_{ij}^{-1} \left(\Xi_j^s - \sum_{t=1}^{s-1} \Lambda_{jr}^{st} w_r^t - \sum_{t=s+1}^{L} \Lambda_{jr}^{st} v_r^t \right), \\
\Lambda^{st} = \left\{ A_{pq}^{(m)} : p \in I_s^{(m)}, q \in I_t^{(m)} \right\}, \\
\Xi^s = \left\{ B_p^{k+1} : p \in I_s^{(m)} \right\}, \quad v^t = \left\{ Y_q^{(m)} : q \in I_t^{(m)} \right\}.
\end{gathered}
$$

It is easily seen that the ABR method practically disregards the condition number of the matrix $A^{(m)}$ because

$$
\mathrm{cond}\,(\Lambda^{ss}) \sim \mathrm{cond}^{1/L}\left(A^{(m)}\right) \ll \mathrm{cond}\left(A^{(m)}\right)
$$

for every $s = 1, 2, \dots, L$ even if $L = 2$.

By the new approach $Y^{(m+1)}$, variables are separated on quick and slow ones too, etc.

Step 4. For termination of the iteration process the following condition is used

$$\left| A_{pq}^{(m)} Y_q^{(m)} - B_p^{k+1} \right| < \xi, \tag{10}$$

where ξ is the prescribed precision.

Theorem 4. *In the framework of the hypotheses* (H1)–(H7) *the following statements are true:*

(i) *The solutions of systems* (7) *and* (9) *exist.*
(ii) *The ABR method converges:* $\lim_{m\to\infty} Y^{(m)} = V$.

Proof. According to the properties of the finite element approximation for the constitutive relation satisfying the conditions (H1)–(H3) the vector function $\{A_{pq}(U + \tau Y, Y)Y_q\} : R^{3m} \to R^{3m}$ is strongly monotonous in Y for every $U \in R^{3m}$ and $\tau \in [0, 1]$ [8,13]. Therefore, according to the classic results of the theory of ordinary differential equations [15] and algebra [16] the statements (i) and (ii) are true.

Remark 5. In the computational mathematics the Schwarz decomposition methods are well known [17]. But they are used only for linear BVPs without the main idea of *adaptiveness* (see References in [17]).

5 Numerical results

The numerical analysis was realized on series of BVPs with model (4) for the axisymmetrical kinematic deformation of long round tube fastened on the internal radius $\rho = a$. The complicated plane deformation was given by different regimes of the displacement on the external radius $\rho = b$ [4,6,7]: (here the summing over indices φ and ρ does not used)

$$u_\varphi^0(t) = C_\varphi Z_\varphi(t), \qquad u_\rho^0(t) = C_\rho Z_\rho(t)$$

where $t \in [0, 1]$, $C_\varphi = \varepsilon_* b(1 - a^2/b^2)$ and $C_\rho = \frac{\sqrt{3}}{2} C_\varphi$ are the maximum external displacements for which the clearly elastic deformation is realized in the framework of the classic model of plasticity (i.e. for the model (4) with parameters $\lambda = \Lambda = 0$) [6].

In the computer experiments the following data were used: $a = 10$, $b = 20$ (mm), $k_0 = 10^5$, $\mu = 7.5 \cdot 10^4$ (MPa), $\varepsilon_* = 5 \cdot 10^{-3}$, $h_0 = 0.001$ and $\lambda = \Lambda = 0$ in the model (4). The radius $[a, b]$ was discretized by 50 segments and the standard piecewise linear approximation was used for unknown functions $u_\varphi(t, \rho)$ and $u_\rho(t, \rho)$ such that $u_\varphi \equiv 0$, $u_\rho \equiv 0$ for $\rho = a$ and $u_\varphi \equiv u_\varphi^0(t)$, $u_\rho \equiv u_\rho^0(t)$ for $\rho = b$.

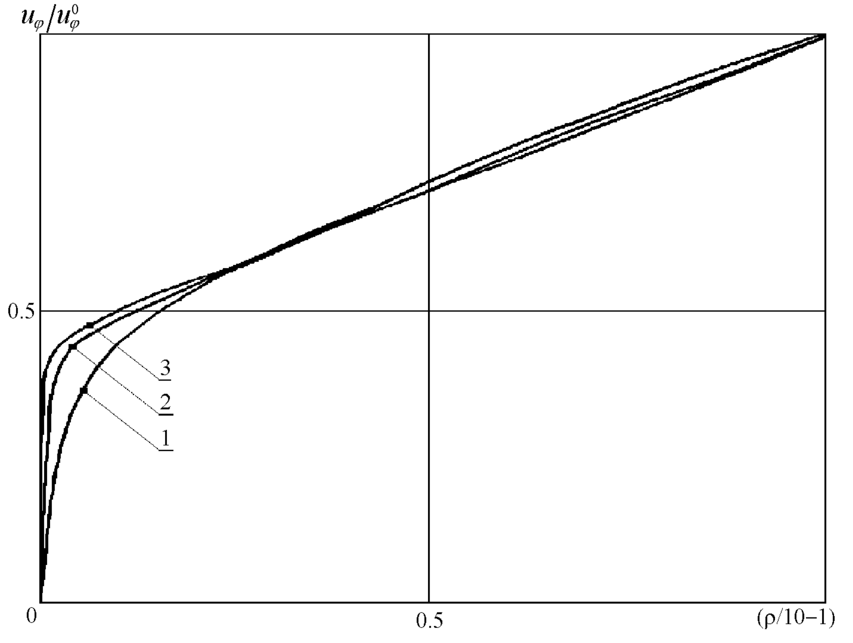

Fig. 1. The tangential displacement in the end of the simplest radial regime

The ABR method with the decomposition parameter $\omega = 0.5$ was compared with the standard method of simple iterations which equals the ABR method with parameter $\omega = 0$.

For the simplest radial regime of clear twisting $Z_\varphi(t) = 10t$, $Z_\rho(t) \equiv 0$ in the implicit Euler scheme (9) $K = 100$ steps over the parameter of loading were used. In figure 1 the following solutions in the end of process are shown: curves 1 and 2 correspond to the standard method with the single $\xi = 10^{-3}$ and double $\xi = 10^{-5}$ precision in the criterion (10), respectively; curve 3 corresponds to the ABR method with the single precision. The last numerical solution (curve 3) practically equals the analytical solution which was built in [4,5].

For the complicated cyclic regime $Z_\varphi(t) = 10\sin(4\pi t)$, $Z_\rho(t) = 10\sin(2\pi t)$ in the scheme (9) $K = 800$ steps over parameter $t \in [0, 1]$ were used. In figure 2 the following solutions in the end of process are shown: curves 1 and 2 correspond to the standard method with the single and double precision, respectively; curve 3 corresponds to the ABR method with the single precision.

In all experiments the time of calculation with single precision was approximately equal for both methods; whereas with double precision, the time of calculation was longer for the standard method than for the ABR method.

It is easily seen that, for finding the displacement and the time of calculation, the ABR method has advantages over the standard method.

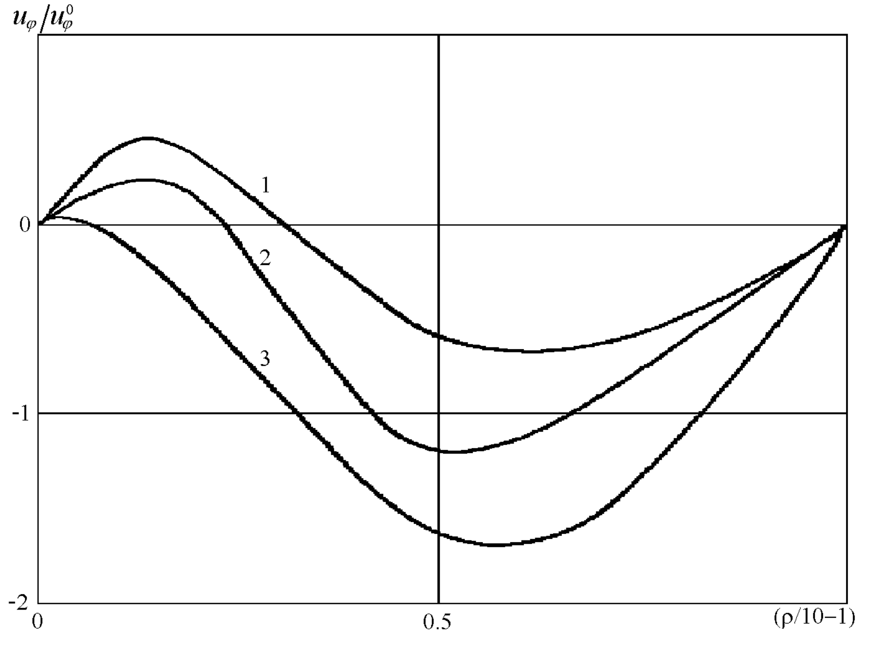

Fig. 2. The tangential displacement in the end of the cyclic regime

6 Conclusion

The questions of mathematical correctness and effective numerical solution for the plasticity BVP have been discussed. By using the evolution variational method: 1) the general algebraic criterion of mathematical correctness for plasticity models has been constructed; 2) the effective qualitative FE analysis has been realized. As a result, an original implicit adaptive strategy has been presented for the numerical simulation of practically important plastic and similar effects in the Mechanics of Solids.

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