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Solutions of a singular Cauchy problem for a nonlinear system of differential equations

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Abstract. The solvability of the singular Cauchy problem for the system of nonlinear differential equations

$$g(x)y' = A(x)\alpha(y) - \omega(x),$$

$$y(0^+) = 0$$

is investigated.

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Let us consider the system of nonlinear differential equations

$$g(x)y' = A(x)\alpha(y) - \omega(x) \tag{1}$$

and initial Cauchy problem

$$y(0^+) = 0. \tag{2}$$

Here $y = (y_1, \dots, y_n)^T$ is the vector of unknown functions; $\alpha(y) = (\alpha_1(y_1), \dots, \alpha_n (y_n))^T$ is a nonlinearity vector with entries α_i, $i = 1, \dots, n$; $A(x)$ is $n \times n$ matrix with elements $a_{ij}(x)$, $i, j = 1 \dots, n$; $\omega(x) = (\omega_1(x), \dots, \omega_n(x))^T$ and $g(x) =$

This is the preliminary version of the paper.

$\text{diag}(g_1(x), \ldots, g_n(x))$ is a diagonal matrix with diagonal entries indicated. The symbol I_s indicate an interval of the form $(0, s]$ with a fixed $s > 0$. The system (1) is considered under the following main assumptions:

(C$_1$) $g_i \in C(I_{x_0}, \mathbb{R}^+),\ i = 1, \ldots, n$ with $\mathbb{R}^+ = (0, \infty)$;
(C$_2$) $\alpha \in C^1(I_{y_0}, \mathbb{R}^n), \alpha(y) \gg 0$ on $I_{y_0}, \alpha'(y) \gg 0$ on I_{y_0} and $\alpha(0^+) = 0$;
(C$_3$) $\omega \in C^1(I_{x_0}, \mathbb{R}^n)$;
(C$_4$) $a_{ij} \in C^1(I_{x_0}, \mathbb{R}),\ a_{ii}(x) \neq 0,\ i, j = 1, \ldots, n$ and $\det A(x) \neq 0$ on I_{x_0};
(C$_5$) $\alpha_i(y) \leq M\alpha_i'(y), i = 1, \ldots, n$ on I_{y_0} with a constant $M \in \mathbb{R}^+$;
(C$_6$) $\Omega(x) \equiv A^{-1}(x)\omega(x) \gg 0, \Omega'(x) \gg 0$ on I_{x_0} and $\Omega(0^+) = 0$.

The problem (1), (2) is a singular problem if assumptions (C$_1$)–(C$_6$) hold and if, in additional, $g_i(0^+) = 0$ for at least one $i \in \{1, \ldots, n\}$. The latter condition is implicitly contained in the assumptions of Theorem 2.

Definition 1. A function $y = y(x) \in C^1(I_{x^*}, \mathbb{R}^n)$ with $0 < x^* \leq x_0$ is said to be a *solution* of the problem (1), (2) on interval I_{x^*} if y satisfies (1) on I_{x^*} and $y(0^+) = 0$.

Theorem 2. *Suppose that conditions $(C_1) - (C_6)$ are satisfied. Let*
A) *for* $i = 1, \ldots, p \leq n$:

$$\omega_i(x) < 0,\ \omega_i'(x) < 0,\ x \in I_{x_0}, \tag{3}$$

$$a_{ij}(x) \geq 0,\ j \neq i,\ j = 1, \ldots, n \text{ and } a_{ij}'(x) \geq 0,\ j = 1, \ldots, n,\ x \in I_{x_0}, \tag{4}$$

and

$$\omega_i(\delta x) > \omega_i(x) + \delta M g_i(x) \frac{\Omega_i'(\delta x)}{\Omega_i(\delta x)}, \quad x \in I_{x_0}$$

for a constant $\delta \in (0, 1)$;
B) *for* $i = p + 1, \ldots, n$:

$$\omega_i(x) > 0,\ \omega_i'(x) > 0,\ \ x \in I_{x_0}, \tag{5}$$

$$a_{ij}(x) \leq 0,\ j \neq i,\ j = 1, \ldots, n \text{ and } a_{ij}'(x) \leq 0,\ j = 1, \ldots, n,\ x \in I_{x_0}, \tag{6}$$

and

$$\omega_i(Kx) > \omega_i(x) + KMg_i(x) \frac{\Omega_i'(Kx)}{\Omega_i(Kx)}\,,\ x \in I_{x_0}$$

for a constant $K > 1$. Then there exists $(n - p)$-parametric family of solutions of the problem (1), (2), having positive coordinates, on an interval $I_{x^*} \subseteq I_{x^{**}}$ with $x^{**} \leq \min\{x_0 K^{-1}, y_0\}$.

Consequence. If Theorem 2 holds then there exist $(n-p)$-parametric family of solutions $y = y^*(x)$ of the problem (1), (2) each of which satisfies on interval I_{x^*} the inequalities

$$\varphi(\delta x) \ll y^*(x) \ll \varphi(Kx).$$

Consider the linear system

$$g(x)y' = A(x)y - \omega(x). \tag{7}$$

Theorem 3 (Linear case). *Suppose that conditions* $(C_1), (C_3), (C_4), (C_6)$, (3) – (6) *are satisfied. Let, moreover,*

$$\omega_i(\delta x) > \omega_i(x) + \delta g_i(x)\Omega_i'(\delta x), \quad x \in I_{x_0},\ i = 1, \ldots, p \leq n$$

for a constant $\delta \in (0, 1)$ *and*

$$\omega_i(Kx) > \omega_i(x) + Kg_i(x)\Omega_i'(Kx)\,,\ x \in I_{x_0},\ i = p+1, \ldots, n$$

for a constant $K > 1$. *Then there exists* $(n-p)$*-parametric family of solutions* $y = y^*(x)$ *of the problem* (7), (2), *having positive coordinates on an interval* $I_{x^*} \subset I_{x_0}$, *each of which satisfies here the inequalities*

$$\Omega(\delta x) \ll y^*(x) \ll \Omega(Kx).$$

Example 4. Let us consider a linear singular problem of the type (7), (2):

$$\begin{aligned} x^2\, y_1' &= -5y_1 + y_2 + y_3 + x + x^2, \\ x^2\, y_2' &= y_1 - 5y_2 + y_3 + x + x^2, \\ x^2\, y_3' &= -2y_1 - 3y_2 + 2y_3 - x + 3x^2, \end{aligned}$$

$$y_1(0^+) = y_2(0^+) = y_3(0^+) = 0.$$

This problem has (by Theorem 3) one-parametric family of positive solutions. Really, the general solution of system considered is expressed by means of relations

$$\begin{aligned} y_1 &= x + 11C_1 \exp(6/x) + C_2 \exp(3/x) + C_3 \exp(-1/x), \\ y_2 &= x - 10C_1 \exp(6/x) + C_2 \exp(3/x) + C_3 \exp(-1/x), \\ y_3 &= 3x - C_1 \exp(6/x) + C_2 \exp(3/x) + 5C_3 \exp(-1/x) \end{aligned}$$

with arbitrary constants C_1, C_2 and C_3. By Theorem 2 there exist one-parametric family of positive solutions of nonlinear problem

$$\begin{aligned} x^3\, y_1' &= -5y_1^2 + y_2^5 + y_3^3 + x + x^2, \\ x^4\, y_2' &= y_1^2 - 5y_2^5 + y_3^3 + x + x^2, \\ x^5\, y_3' &= -2y_1^2 - 3y_2^5 + 2y_3^3 - x + 3x^2, \end{aligned}$$

$$y_1(0^+) = y_2(0^+) = y_3(0^+) = 0.$$

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