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Boundary stabilization of the Schrödinger equation in almost star-shaped domain

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Abstract. The question of uniformly stabilizing the solution of the Schrödinger equation $y_t - i\Delta y = 0$ in $\Omega \times (0, \infty)$ (Ω is a bounded domain of R^n) subject to boundary conditions $y = 0$ on $\Gamma_0 \times (0, \infty)$ and $\dfrac{\partial y}{\partial \nu} = F(y, y_t)$ on $\Gamma_1 \times (0, \infty)$,$(\Gamma_0, \Gamma_1)$ being a partition of the boundary, is studied. We shall show that if $\{\Omega, \Gamma_0, \Gamma_1\}$ is almost star-shaped, then a suitable choice of F leads to exponential energy decay. Moreover exponential decay rate estimates will be obtained. The approach adopted is based on multipliers technique.

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1 Introduction

Let Ω be an open bounded domain in R^n with sufficiently smooth boundary Γ. Assume that Γ consists of two parts Γ_0 and $\Gamma_1$satisfying

$$\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset \tag{1}$$

We set $Q = \Omega \times (0, +\infty), \Sigma_0 = \Gamma_0 \times (0, +\infty), \Sigma_1 = \Gamma_1 \times (0, +\infty)$. Let a and l be two nonnegative functions of class C^1such that

$$\Gamma_0 \neq \emptyset \text{ or } a \neq 0 \tag{2}$$

This is the preliminary version of the paper.

Consider the problem

$$y_t - i\Delta y = 0 \qquad \text{in } Q \tag{3}$$
$$y(x,0) = y_0 \qquad \text{in } \Omega \tag{4}$$
$$y = 0 \qquad \text{on } \Sigma_0 \tag{5}$$
$$\frac{\partial y}{\partial \nu} + ay + ly_t = 0 \qquad \text{on } \Sigma_1 \tag{6}$$

where $y_t = \dfrac{dy}{dt}$ and ν is the unit normal of Γ pointing towards the exterior of Ω. The natural space for initial data is

$$V = \left\{\varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_0\right\}$$

When Γ_0 has non-empty interior in Γ, by Poincaré's inequality, we have

$$\|\varphi\|_{L^2(\Omega)} \leq \beta \|\nabla\varphi\|_{(L^2(\Omega))^n}, \qquad \forall \varphi \in V \tag{7}$$

In view of this inequality, we shall consider in V the norm induced by the inner product

$$(\varphi, \psi)_V = \Re \int_\Omega \nabla\varphi . \nabla\overline{\psi} d\Omega$$

Associated with each solution of (3)–(6) is its total energy at time t;

$$E(t) = \int_\Omega |\nabla y|^2 d\Omega + \int_{\Gamma_1} a |y|^2 d\Gamma$$

A simple calculation shows, at least formally, that

$$\tfrac{d}{dt} E(t) = -\int_{\Gamma_1} l |y_t|^2 d\Gamma \leq 0$$

hence $E(t)$ is nonincreasing.

The question we are interested in is the following: under what conditions can we establish the exponential decay of the energy and if possible obtain explicit decay rate estimates. An affirmative answer to the above question has been given by Machtyngier and Zuazua [3] under the following assumptions:

$(H1)$- $\{\Omega, \Gamma_0, \Gamma_1\}$ is "star-complemented-star-shaped", scss for short (see [1]). This means that there exists a point $x_0 \in R^n$ such that

-$(x - x_0).\nu(x) \leq 0$ on Γ_0 (Γ_0 is star-complemented with respect to x_0)

-$(x - x_0).\nu(x) \geq 0$ on Γ_1 (Γ_1 is star-shaped with respect to x_0)

$(H2)$- $a \equiv 0$ and $l = (x - x_0).\nu(x)$

The aim of this paper is to extend the result of Machtyngier and Zuazua, in two ways: first by replacing the scss domains by a larger class of domains known as almost star shaped domains, second by replacing the boundary feedback $(H2)$ by a more general one with $a \neq 0$.

The rest of the paper is organized as follows. In Section 2, we recall the notion of almost star-shaped domains. In Section 3, we state and prove the boundary stabilization theorem.

2 Almost star-shaped domains

Definition 1. $\{\Omega, \Gamma_0, \Gamma_1\}$ is an almost star-shaped domain if there exists $\varphi \in C^2(\overline{\Omega})$ such that

$$\Delta\varphi = 1 \qquad \text{in } \Omega \tag{8}$$

$$\lambda_1(\varphi) = Inf\,\{\lambda_1(x), x \in \Omega\} > 0 \tag{9}$$

$$\frac{\partial\varphi}{\partial\nu} \leq 0 \qquad \text{on } \Gamma_0 \tag{10}$$

$$\frac{\partial\varphi}{\partial\nu} \geq 0 \qquad \text{on } \Gamma_1 \tag{11}$$

where $\lambda_1(x)$ is the smallest eigenvalue of the real symmetric squared matrix $D^2\varphi(x)$.

The simplest example is the case where $\{\Omega, \Gamma_0, \Gamma_1\}$ is a scss domain. The function φ is then given by

$$\varphi(x) = \tfrac{1}{2n}\,|x - x_0|^2$$

Remark 2. We refer to [4] and [5] for other examples and further details.

3 The boundary stabilization theorem

We first state the following existence and regularity theorem for the system (3)–(6)

Theorem 3. *Assume* (1) *and* (2).
1- Given $y_0 \in V$, *the problem* (3)–(6) *has a unique weak solution*

$$y \in C((0, +\infty), V) \cap C^1((0, +\infty), L^2(\Omega))$$

2- If y_0 *satisfies the stronger conditions*

$$y_0 \in H^2(\Omega) \cap V \tag{12}$$

$$\frac{\partial y_0}{\partial\nu} + ay_0 + il\Delta y_0 = 0 \;\; on\; \Gamma_1 \tag{13}$$

Then the solution y has the stronger regularity property

$$y \in L^{\infty}((0,+\infty),H^2(\Omega)), y_t \in L^{\infty}((0,+\infty),V)$$

To prove this theorem, semigroups approach can be adopted (see [2]). Our main result is as follows.

Theorem 4. *Let $\{\Omega,\Gamma_0,\Gamma_1\}$ be an almost star shaped-domain, and choose*

$$a = \frac{1}{8\,\|\nabla\varphi\|_{\infty}^2}\frac{\partial\varphi}{\partial\nu} \qquad \text{and} \;\; l = \sqrt{\frac{2}{3}}\frac{\partial\varphi}{\partial\nu} \tag{14}$$

Then for every $y_0 \in V$, the energy corresponding to the weak solution of (3)–(6) *satisfies the estimate*

$$\forall t \geq 0, \qquad E(t) \leq E(0)e^{1-\omega t} \qquad \text{with } \omega = \frac{\lambda_1(\varphi)}{(\sqrt{6}\,\|\nabla\varphi\|_{\infty}+\beta)\,\|\nabla\varphi\|_{\infty}}$$

This theorem will be only proved for smooth initial data. The general case follows by a standard density argument. To proceed the following preliminary results are needed.

Lemma 5. *Given y_0 verifying* (12)–(13). *Then the strong solution of* (3)–(6) *satisfies*

$$\forall 0 \leq S \leq T < +\infty, \qquad E(S)-E(T) = \int_{\Sigma_1} l\,|y_t|^2\,d\Sigma \tag{15}$$

Proof. We multiply both sides of (3) by $\overline{y_t}$ and we integrate by parts over Ω We obtain

$$\begin{aligned} 0 &= \int_{\Omega}(y_t - i\Delta y)\overline{y_t}d\Omega \\ &= \int_{\Omega}(|y_t|^2 - i\overline{y_t}\Delta y)d\Omega \\ &= \int_{\Omega}|y_t|^2\,d\Omega + i\int_{\Omega}\nabla y.\nabla\overline{y_t}d\Omega + i\int_{\Gamma_1}(ay\overline{y_t} + l\,|y_t|^2)d\Sigma \end{aligned}$$

It follows that

$$\Re(\int_{\Omega}\nabla y.\nabla\overline{y_t}d\Omega + \int_{\Gamma_1}ay\overline{y_t}d\Sigma) = -\int_{\Gamma_1} l\,|y_t|^2\,d\Sigma$$

But the left-hand side of the above equality is precisely $\frac{d}{dt}E(t)$. Hence the desired result.

Lemma 6. *Given $0 \leq S < T$. Then the following identity holds true for any strong solution of (3)–(6):*

$$-2\int_{\Sigma_1}(ay+ly_t)\nabla\varphi.\nabla\overline{y}d\Sigma - \Re\int_{\Sigma_1}(|\nabla y|^2+iy\overline{y_t})\frac{\partial\varphi}{\partial\nu}d\Sigma - \Re\int_{\Sigma_1}(a\overline{y}+l\overline{y_t})yd\Sigma + \int_{\Sigma_0}\left|\frac{\partial y}{\partial\nu}\right|\frac{\partial\varphi}{\partial\nu}d\Sigma = \Im X + 2\Re\int_Q(D^2\varphi\nabla\overline{y}).\nabla ydQ \tag{16}$$

where

$$X = \left[\int_\Omega y\nabla\varphi.\nabla\overline{y}d\Omega\right]_S^T$$

Proof. (i)- We multiply both sides of (3) by $\nabla\varphi.\nabla\overline{y}$ and integrate over Q. We obtain the following identity (see the Appendix):

$$2\Re\int_\Sigma\frac{\partial\overline{y}}{\partial\nu}\nabla\varphi.\nabla yd\Sigma - \int_\Sigma|\nabla y|^2\frac{\partial\varphi}{\partial\nu} + \Im\int_\Sigma y\overline{y_t}\frac{\partial\varphi}{\partial\nu}d\Sigma + \Re\int_\Sigma\frac{\partial\overline{y}}{\partial\nu}y\Delta\varphi d\Sigma = \Im X + 2\Re\int_Q(D^2\varphi\nabla\overline{y})\nabla ydQ \tag{17}$$

(ii)- We now use the boundary conditions (5) and (6). Thus

$$\text{On } \Gamma_0 : y = y_t = 0; |\nabla y| = \left|\frac{\partial y}{\partial\nu}\right|; \nabla\varphi.\nabla y = (\nabla\varphi.\nu)\frac{\partial y}{\partial\nu} \tag{18}$$

Therefore using (6) and (18) in the left-hand side of(17), we find the sought- after identity for y satisfying (3)–(6).

Lemma 7. *Assume that a and l are defined by (14). Then for any initial data verifying (12) and (13), we have:*

$$\int_{\Sigma_0}(\frac{\partial y}{\partial\nu})^2\frac{\partial\varphi}{\partial\nu} - 2\Re\int_{\Sigma_1}(ay+ly_t)\nabla\varphi.\nabla\overline{y}d\Sigma - \int_{\Sigma_1}(|\nabla y|^2+iy\overline{y_t})\frac{\partial\varphi}{\partial\nu}d\Sigma - \Re\int_{\Sigma_1}(a\overline{y}+l\overline{y_t})yd\Sigma \leq 4\sqrt{6}\,\|\nabla\varphi\|_\infty^2(E(S)-E(T)) - \frac{1}{4}\int_{\Sigma_1}a|y|^2d\Sigma \tag{19}$$

Proof. Set $\alpha = \dfrac{1}{8\|\nabla\varphi\|_\infty^2}$ and $\lambda = \sqrt{\frac{2}{3}}$

Then

$$
\begin{aligned}
&-2\Re(ay+ly_t)\nabla\varphi.\nabla\overline{y}-|\nabla y|^2\frac{\partial\varphi}{\partial\nu}-\Re(iy\overline{y_t}\frac{\partial\varphi}{\partial\nu})-a|y|^2-\Re(ly\overline{y_t})\\
\le\ &\frac{\partial\varphi}{\partial\nu}\left[2\|\nabla\varphi\|_\infty^2(\alpha^2|y|^2+\lambda^2|y_t|^2)+|\nabla y|^2\right]-\frac{\partial\varphi}{\partial\nu}|\nabla y|^2+\\
&\frac{\lambda^2+1}{2\alpha}\frac{\partial\varphi}{\partial\nu}|y_t|^2+\frac{\alpha}{2}\frac{\partial\varphi}{\partial\nu}|y|^2\le\frac{\partial\varphi}{\partial\nu}\left[2\|\nabla\varphi\|_\infty^2\lambda^2+\frac{\lambda^2+1}{2\alpha}\right]|y_t|^2+\\
&\frac{\partial\varphi}{\partial\nu}\left[2\|\nabla\varphi\|_\infty^2\alpha-\frac{1}{2}\right]\alpha|y|^2
\end{aligned}
$$

The estimate (19) follows now from the particular choice of the coefficients α and λ given by (14), from the identity (15) and (8)–(11).

Lemma 8.

$$|X|\le 2\beta E(S)$$

Proof. Using (7), we have

$$
\begin{aligned}
\left|\int_\Omega y\nabla\varphi.\nabla\overline{y}d\Omega\right| &\le \|y(t)\|_{L^2(\Omega)}\|\nabla\varphi.\nabla\overline{y}(t)\|_{L^2(\Omega)}\\
&\le\beta\|\nabla\varphi\|_\infty\|y(t)\|_V^2
\end{aligned}
$$

Thus

$$|X|\le 2\beta\|\nabla\varphi\|_\infty E(S)$$

4 Proof of Theorem 4

Applying Lemmas 7 and 8, we deduce from the identity (16), the inequality

$$2\Re\int_Q(D^2\varphi\nabla\overline{y}).\nabla y dQ\le(4\sqrt{6}\|\nabla\varphi\|_\infty^2+2\beta\|\nabla\varphi\|_\infty)E(S)+\frac{1}{4}\int_{\Sigma_1}a|y|^2d\Sigma$$

Using (9), we get

$$2\lambda_1(\varphi)\int_S^T E(t)dt\le(4\sqrt{6}\|\nabla\varphi\|_\infty^2+2\beta\|\nabla\varphi\|_\infty)E(S)$$

Letting $T\to+\infty$, we obtain for every fixed $S\in R_+$, the estimate

$$\int_S^{+\infty}E(t)dt\le\frac{1}{\omega}E(S)$$

where

$$\omega = \frac{\lambda_1(\varphi)}{(\sqrt{6}\,\|\nabla\varphi\|_\infty + \beta)\,\|\nabla\varphi\|_\infty}$$

The conclusion of the theorem follows by applying a Gronwall-type inequality as in [2].

5 Appendix Proof of (17)

We multiply both sides of (3) by $\nabla\varphi.\nabla\overline{y}$ and integrate by parts over Q.

Term $y_t\nabla\varphi.\nabla\overline{y}$

Integrating by parts in t and using the identity

$$\int_\Omega h.\nabla\psi d\Omega = \int_\Gamma h\psi.\nu d\Gamma - \int_\Omega \psi\nabla h d\Omega \tag{A1}$$

we obtain

$$\int_Q y_t\nabla\varphi.\nabla\overline{y}dQ = \left[\int_Q y_t\nabla\varphi.\nabla\overline{y}d\Omega\right]_S^T - \int_\Sigma y\overline{y_t}\nabla\varphi.\nu d\Sigma - \\ i\int_Q \Delta\overline{y}\nabla\varphi.\nabla y dQ - i\int_Q y\Delta\overline{y}\Delta\varphi dQ \tag{A2}$$

Adding $-i\int_Q \Delta y\nabla\varphi.\nabla\overline{y}dQ$ to both sides of (A2) yields

$$2\Re\int_Q \Delta\overline{y}\nabla\varphi.\nabla y dQ = \Im X - \Im\int_\Sigma y\overline{y_t}\nabla\varphi.\nu d\Sigma - \Re\int_Q y\Delta\overline{y}\Delta\varphi dQ \tag{A3}$$

Term $\int_Q y\Delta\overline{y}\Delta\varphi dQ$

Using Green's first theorem and the identity (20), we find

$$\int_Q y\Delta\overline{y}\Delta\varphi dQ = \int_\Sigma \frac{\partial\overline{y}}{\partial\nu}y\Delta\varphi d\Sigma - \int_Q |\nabla y|^2\Delta\varphi dQ - \int_Q y\nabla\overline{y}.\nabla(\Delta\varphi)dQ \tag{A4}$$

Term $\int_Q \Delta\overline{y}\nabla\varphi.\nabla y dQ$

We use Green's first theorem and the identity (20). We obtain

$$\int_Q \Delta\overline{y}\nabla\varphi.\nabla y dQ = \int_\Sigma \frac{\partial\overline{y}}{\partial\nu}\nabla\varphi.\nabla y d\Sigma - \int_Q (D^2\varphi\nabla\overline{y}).\nabla y dQ - \\ \frac{1}{2}\int_\Sigma |\nabla y|^2\nabla\varphi.\nu d\Sigma + \frac{1}{2}\int_Q |\nabla y|^2\Delta\varphi dQ \tag{A5}$$

Inserting (A4) and (A5) into (A3), results in (17).

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