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Irregular boundary value problems for ordinary differential equations

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Abstract. Birkhoff-irregular boundary value problems for quadratic ordinary differential pencils of the second order have been considered. The spectral parameter may appear in a boundary condition, the equation contains an abstract linear operator while the boundary conditions contain internal points of an interval and a linear functional. Isomorphism and coerciveness with a defect are proved for such problems. Two-fold completeness of root functions of corresponding spectral problems is also established. As an application of the obtained results, an initial boundary value problem for second order parabolic equations is considered, and the well-posedness and completeness of the elementary solutions are proved. These and some other results have been published in [1] .

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1 Isomorphism and Two-fold Completeness

Consider a principally (because of the addition of B, T, and x_{ji}) boundary value problem for ordinary differential equations

$$\begin{aligned} L(\lambda)u := \lambda^2 u(x) + \lambda\left[a_1 u'(x) + b_1 u(x)\right] + \left[a_2 u''(x) + b_2 u'(x)\right] + \\ + Bu|_x = f(x), \quad x \in (0,1), \quad (1) \end{aligned}$$

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This is an overview article.

$$\begin{cases} L_1(\lambda)u := \lambda\Big[\alpha_{11}u(0) + \beta_{11}u(1) + \sum_{i=1}^{N_0}\eta_{0i}u(x_{0i})\Big] + \\ \qquad + [\alpha_{10}u'(0) + \beta_{10}u'(1) + \gamma_{10}u(0) + \delta_{10}u(1)] + \\ \qquad + \sum_{i=1}^{N_1}\eta_{1i}u'(x_{1i}) + \sum_{i=1}^{N_2}\eta_{2i}u(x_{2i}) + Tu = f_1, \\ L_2u := [\alpha_{20}u(0) + \beta_{20}u(1)] + \sum_{i=1}^{N_3}\eta_{3i}u(x_{3i}) = f_2, \end{cases} \tag{2}$$

where a_k, b_k, $\alpha_{\nu k}$, $\beta_{\nu k}$, γ_{10}, δ_{10}, η_{ji}, f_ν are complex numbers, $f(x)$ is a given function; $x_{ji} \in (0,1)$; B is a linear operator in $L_q(0,1)$ and T is a linear functional in $L_q(0,1)$, a real $q \in (1,\infty)$.

We assume that $a_2 \neq 0$ and denote by

$$\omega_1 := \frac{-a_1 + (a_1^2 - 4a_2)^{\frac{1}{2}}}{2a_2}, \quad \omega_2 := \frac{-a_1 - (a_1^2 - 4a_2)^{\frac{1}{2}}}{2a_2}$$

the roots of the equation

$$a_2\omega^2 + a_1\omega + 1 = 0,$$

where $z^{\frac{1}{2}} := |z|^{\frac{1}{2}}e^{i\frac{\arg z}{2}}$, $-\pi < \arg z \leq \pi$. We also assume that $\arg\omega_1 \neq \arg\omega_2$. Further,

$$\underline{\omega} := \min\{\arg\omega_1, \arg\omega_2 + \pi\},$$
$$\overline{\omega} := \max\{\arg\omega_1, \arg\omega_2 + \pi\},$$

and values $\arg\omega_j$ are chosen up to a multiple of 2π, so that $\overline{\omega} - \underline{\omega} < \pi$.

Introduce now the following notations:

$$\theta_0(\omega_1,\omega_2) := \begin{vmatrix} \alpha_{11} + \alpha_{10}\omega_1 & \beta_{11} + \beta_{10}\omega_2 \\ \alpha_{20} & \beta_{20} \end{vmatrix},$$

$$\theta_1(\omega_1,\omega_2) := \begin{vmatrix} -\alpha_{10}\frac{b_1+b_2\omega_1}{(a_1^2-4a_2)^{\frac{1}{2}}} + \gamma_{10} & \beta_{10}\frac{b_1+b_2\omega_2}{(a_1^2-4a_2)^{\frac{1}{2}}} + \delta_{10} \\ \alpha_{20} & \beta_{20} \end{vmatrix}.$$

Definition 1. Problem (1)—(2) is called *regular (regular with defect 1) with respect to the numbers* ω_1, ω_2 if:

(1) $a_2 \neq 0$, $\arg\omega_1 \neq \arg\omega_2$;

(2) $x_{ji} \in (0,1)$, for some real $q \in (1,\infty)$ the operator B from $W_q^1(0,1)$ into $L_q(0,1)$ is compact and the functional T is continuous in $L_q(0,1)$;

(3) $\theta_0(\omega_1,\omega_2) \neq 0 \quad \big(\theta_0(\omega_1,\omega_2) = 0,\ \theta_1(\omega_1,\omega_2) \neq 0\big)$.

Theorem 2. *Let problem* (1)—(2) *be regular with defect 1 with respect to the numbers* $\omega_1,\ \omega_2$.

Then for any $\varepsilon > 0$ *there exists* $R_\varepsilon > 0$ *such that for all complex numbers* λ *that satisfy* $|\lambda| > R_\varepsilon$ *and lying inside the angle*

$$\frac{\pi}{2} - \underline{\omega} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \overline{\omega} - \varepsilon,$$

the operator

$$\mathbb{L}(\lambda):\ u \to \mathbb{L}(\lambda)u := (L(\lambda)u, L_1(\lambda)u, L_2u)$$

from $W_q^2(0,1)$ *onto* $L_q(0,1) \dotplus \mathbb{C}^2$ *is an isomorphism and for these* λ *the following estimate holds for a solution* $u(x)$ *of problem* (1)—(2)

$$\sum_{k=0}^{2} |\lambda|^{1-k} \|u\|_{W_q^k(0,1)} \le C(\varepsilon)\Bigg(\|f\|_{L_q(0,1)} + \sum_{\nu=1}^{2} |\lambda|^{\nu-\frac{1}{q}} |f_\nu| \Bigg).$$

Theorem 3. *Let* $|\alpha_{\nu 0}|+|\beta_{\nu 0}| \ne 0,\ \nu = 1,2$ *and let homogeneous problem* (1)—(2) *be regular with defect 1 with respect to the numbers* $\omega_1,\ \omega_2$ *and regular, or regular with defect 1 with respect to the numbers* $\omega_2,\ \omega_1$, *for* $q = 2$.

Then the spectrum of homogeneous problem (1)—(2) *is discrete and a system of its root functions (eigenfunctions and associated functions) is two-fold complete in the space*

$$H := \{v \mid v := (v_1, v_2) \in W_2^1(0,1) \oplus L_2(0,1),\ L_2 v_1 = 0\}.$$

2 Well-posedness and Completeness of Elementary Solutions

Consider, in $[0,T] \times [0,1]$, the following initial boundary value problem

$$u_t(t,x) + [a u_{xx}(t,x) + b u_x(t,x)] + Bu(t,\cdot)|_x = f(t,x), \tag{3}$$

$$\begin{cases} L_1 u := [\alpha_{10} u'_x(t,0) + \beta_{10} u'_x(t,1)] + [\gamma_{10} u(t,0) + \delta_{10} u(t,1)] + \\ \qquad + \sum_{i=1}^{N_1} \eta_{1i} u'_x(t,x_{1i}) + \sum_{i=1}^{N_2} \eta_{2i} u(t,x_{2i}) + Qu(t,\cdot) = 0, \\ L_2 u := [\alpha_{20} u(t,0) + \beta_{20} u(t,1)] + \sum_{i=1}^{N_3} \eta_{3i} u(t,x_{3i}) = 0, \end{cases} \tag{4}$$

$$u(0,x) = u_0(x). \tag{5}$$

Let E, E_1, and E_2 be Banach spaces. Introduce two Banach spaces

$$C_\mu((0,T],E) := \{f \mid f \in C((0,T],E), \|f\| = \sup_{t\in(0,T]} \|t^\mu f(t)\| < \infty\},\ \mu \ge 0,$$

$$C_\mu^\gamma((0,T],E) := \{f \mid f \in C((0,T],E),\ \|f\| = \sup_{t\in(0,T]} \|t^\mu f(t)\| + \\ + \sup_{0<t<t+h\le T} \|f(t+h)-f(t)\| h^{-\gamma} t^\mu < \infty\},\ \mu \ge 0, \gamma \in (0,1],$$

and a linear space (in the case $E_1 \subset E_2$)

$$C^1((0,T],E_1,E_2) := \{f \mid f \in C((0,T],E_1) \cap C^1((0,T],E_2)\},$$

where $C((0,T],E)$ and $C^1((0,T],E)$ are spaces of continuous and continuously differentiable, respectively, vector-functions from $(0,T]$ into E.

Theorem 4. *Let the following conditions be satisfied:*

(1) $a \ne 0,\ |\arg a| > \frac{\pi}{2},\ \alpha_{10}\beta_{20} + \alpha_{20}\beta_{10} = 0,\ \gamma_{10}\beta_{20} - \delta_{10}\alpha_{20} - \frac{b}{a}\alpha_{10}\beta_{20} \ne 0$;

(2) *the operator* B *from* $W_q^1(0,1)$ *into* $L_q(0,1)$ *is compact;*

(3) *the functional* Q *is continuous in* $L_q(0,1)$.

(4) $f \in C_\mu^\gamma((0,T],L_q(0,1))$ *for some* $\gamma \in (\frac{1}{2},1]$ *and* $\mu \in [0,\frac{1}{2})$;

(5) $u_0 \in W_q^2((0,1), L_\nu u = 0,\ \nu = 1,2)$.

Then problem (3)—(5) *has a unique solution*

$$u \in C([0,T],L_q(0,1)) \cap C^1((0,T],W_q^2(0,1),L_q(0,1))$$

and for the solution the following estimates hold

$$\|u(t,\cdot)\|_{L_q(0,1)} \le C(\|u_0\|_{W_q^2(0,1)} + \|f\|_{C_\mu((0,t],L_q(0,1))}),$$

and

$$\|u(t,\cdot)\|_{W_q^2(0,1)} + \|u'(t,\cdot)\|_{L_q(0,1)} \le Ct^{-1}(\|u_0\|_{W_q^2(0,1)} + \|f\|_{C_\mu^\gamma((0,t],L_q(0,1))}),$$

for $t \in (0,T]$.

Consider now a spectral problem coresponding to the homogeneus (3), (4):

$$\lambda u(x) + [au''(x) + bu'(x)] + Bu|_x = 0,\ x \in (0,1), \tag{6}$$

$$\begin{cases} L_1 u := [\alpha_{10}u'(0) + \beta_{10}u'(1)] + [\gamma_{10}u(0) + \delta_{10}u(1)] + \\ \qquad + \sum_{i=1}^{N_1} \eta_{1i}u'(x_{1i}) + \sum_{i=1}^{N_2} \eta_{2i}u(x_{2i}) + Qu = 0, \\ L_2 u := [\alpha_{20}u(0) + \beta_{20}u(1)] + \sum_{i=1}^{N_3} \eta_{3i}u(x_{3i}) = 0, \end{cases} \tag{7}$$

A function of the form

$$u_j(t,x) = \mathrm{e}^{\lambda_j t}\left(\frac{t^{k_j}}{k_j!}u_{j0}(x) + \frac{t^{k_j-1}}{(k_j-1)!}u_{j1}(x) + \cdots + u_{jk_j}(x)\right) \tag{8}$$

becomes an elementary solution of the homogeneus (3), (4) if and only if a system of the functions $u_{j0}(x), u_{j1}(x), \ldots, u_{jk_j}(x)$ is a chain of root functions of problem (6)—(7), corresponding to the eigenvalue λ_j.

Theorem 5. *Let conditions of the previous theorem be satisfied with $q = 2$.*
Then problem (3)—(5) (with $f(t,x) \equiv 0$) has a unique solution

$$u \in C([0,T], L_2(0,1)) \cap C^1((0,T], W_2^2(0,1), L_2(0,1))$$

and there exist numbers c_{jn} such that

$$\lim_{n\to\infty} \sup_{t\in(0,T]} \|u(t,\cdot) - \sum_{j=1}^{n} c_{jn} u_j(t,\cdot)\|_{L_2(0,1)} = 0,$$

$$\lim_{n\to\infty} \sup_{t\in(0,T]} t(\|u_t'(t,\cdot) - \sum_{j=1}^{n} c_{jn} u_{jt}'(t,\cdot)\|_{L_2(0,1)} + \\ + \|u(t,\cdot) - \sum_{j=1}^{n} c_{jn} u_j(t,\cdot)\|_{W_2^2(0,1)}) = 0,$$

where $u(t,x)$ is a solution to problem (3)—(5) (with $f(t,x) \equiv 0$) and $u_j(t,x)$ is an elementary solution (8) of homogeneus problem (3)—(4).

References

1. Yakubov Ya., *Irregular Boundary Value Problems for Ordinary Differential Equations*, Analysis, **18** 1998, 359–402.