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Geothermal Flow in Porous Media

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Abstract. This contribution deals with numerical modelling of the geothermal flow of groundwater in the vicinity of sources of thermal energy. The problem is described by a set of two partial differential equations — the heat transport equation with convection and the equation for pressure. These equations are coupled together in terms of dependency of density of the fluid on temperature. As the density is assumed to be dependent on temperature only, the equation for pressure is of the elliptic type, even in the non-stationary case. The resulting system of equations is thus of parabolic-elliptic type. A suitable numerical scheme for approximation of solution to this system is proposed and it is tested on several numerical experiments which are presented in the conclusion.

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1 Introduction

Let Ω is a bounded domain in $\mathbb{R}^2$ representing a vertical cut through the soil which is fully saturated by water. In this domain, we require fulfilment of mass balance condition :

$$n\frac{\partial \rho}{\partial t} + \frac{\partial \rho q_x}{\partial x} + \frac{\partial \rho q_z}{\partial z} = Q. \tag{1}$$

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This is an overview article summarizing degree thesis [5].

Here, n denotes porosity, ρ is the density of the fluid, q_x and q_z are components of the Darcy velocity vector $\vec{q}$, and finally, Q is a source/sinks term. Equation (1) will be reffered as mass balance equation in the sequel. This equation was derived in [5] under assumption that the porosity is given function of spatial variables only — i.e. the soil is incompressible.

In the equation (1), the dependency of $\vec{q}$ on pressure p is explicitly given by the Darcy law:

$$\vec{q} = -\frac{k}{\mu}(\nabla p - \rho\vec{g}), \tag{2}$$

where μ denotes the coefficient of dynamical viscosity of the fluid and $\vec{g}$ is a vector of gravity acceleration. The quantity k is the permeability of the porous medium which is assumed to be isotropic. In the whole text, we will suppose that the coordinate system (Oxz) is oriented so that the vector of gravitational acceleration points in the direction of negative part of z-axis. So $\vec{g} = (0, g)$, where $g = -9.81\ m.s^{-2}$. Under these assumptions the Darcy law can be written in components as

$$q_x = -\frac{k}{\mu}\frac{\partial p}{\partial x}, \qquad q_z = -\frac{k}{\mu}\left(\frac{\partial p}{\partial z} - \rho g\right). \tag{3}$$

Substituting the Darcy law to the mass balance equation, we get single equation

$$-\frac{\partial}{\partial x}\left(\rho\frac{k}{\mu}\frac{\partial p}{\partial x}\right) - \frac{\partial}{\partial z}\left(\rho\frac{k}{\mu}\frac{\partial p}{\partial z} - \frac{k}{\mu}\rho^2 g\right) = Q - n\frac{\partial \rho}{\partial t}, \tag{4}$$

which will be denoted as equation for pressure in the whole text. Moreover, we add the equation of heat transport which is taken from [2]

$$\overline{\rho c}\frac{\partial T}{\partial t} + \rho c\vec{q}\cdot\nabla T = \nabla\cdot(\lambda\nabla T). \tag{5}$$

In this equation, T is the unknown temperature, c is the heat capacity per unit mass of the fluid and λ denotes the homogenized coefficient of heat conductivity. We suppose that the heat conductivity is scalar (i.e. the medium is isotropic), however, the extension for anisotropic case is possible. We use notation $\overline{\rho c} = n\rho c + (1-n)\rho_s c_s$, where ρ_s and c_s denote the density and the heat capacity per unit mass of the soil, respectively. The equations are coupled by the dependency of the density of the fluid on temperature

$$\rho(T) = \frac{\rho_0}{1 + \beta_1(T - T_0) + \beta_2(T - T_0)^2}, \tag{6}$$

in which the coefficients $\rho_0 \equiv \rho(T_0), \beta_1, \beta_2$, are given constants. Additionally, the density could depend on pressure, but this case is not considered here.

As we assumed that neither density nor porosity depend on pressure, the equation for pressure (4) is an elliptic partial differential equation with respect to pressure. Therefore, we prescribe boundary conditions of the Dirichlet, Neumann or

Newton type, i.e.

$$p(x,z,t) = f_1(x,z,t) \qquad \forall t \text{ and } \forall (x,z) \in S_1. \tag{7}$$

$$(\rho q_x n_x + \rho q_z n_z)(x,z,t) = f_2(x,z,t) \qquad \forall t \text{ and } \forall (x,z) \in S_2. \tag{8}$$

$$(\rho q_x n_x + \rho q_z n_z)(x,z,t) = \beta (p - p_{out})(x,z,t) \qquad \forall t \text{ and } \forall (x,z) \in S_3, \tag{9}$$

Here p_{out} denotes pressure at the outer side of boundary, β is the coefficient of proportionality and the $\vec{n} = (n_x, n_z)$ denotes the unit vector of outer normal. The heat transport equation (5) is parabolic partial differential equation and therefore we supply one initial condition and the boundary conditions of the Dirichlet, Neumann or Newton type, i.e.

$$T(x,z,0) = T^0(x,z) \qquad \forall (x,z) \in \Omega, \tag{10}$$

$$T(x,z,t) = f_4(x,z,t) \qquad \forall t \text{ and } \forall (x,z) \in S_4. \tag{11}$$

$$\left(-\lambda \frac{\partial T}{\partial \vec{n}}\right)(x,z,t) = f_5(x,z,t) \qquad \forall t \text{ and } \forall (x,z) \in S_5. \tag{12}$$

$$\left(-\lambda \frac{\partial T}{\partial \vec{n}}\right)(x,z,t) = \gamma (T - T_{out})(x,z,t) \qquad \forall t \text{ and } \forall (x,z) \in S_6, \tag{13}$$

where T_{out} is the outer temperature and $\gamma > 0$ is the heat transfer coefficient. The problem will be correctly formulated if the S_1, S_2, S_3 and S_4, S_5, S_6 are two (generally different) decompositions of boundary $\partial\Omega$.

2 Weak formulation

In the sequel, we will suppose that Ω is a bounded domain with the Lipschitz boundary. Let us introduce the space $V_T = \{f \in C^\infty\left(\overline{\Omega}\right) : f|_{S_4} = 0\}$ and the enthalpy by the following definition

$$H(T) = \int_0^T \overline{\rho c}(\tau)\; d\tau. \tag{14}$$

Moreover, let us denote the time interval $(0, \Theta)$ as I. At this moment, we are ready for the following definition.

Suppose the following input data qualification : $\rho_s, c_s, n, \lambda \in L^\infty(\Omega)$, $c > 0$, $n \in (0,1)$, $q_x, q_z \in L^2(I; L^2(\Omega))$, $f_4 \in L^2(I; L^{1/2}(S_4))$, $f_5 \in L^2(I; L^2(S_5))$, $\gamma \in L^\infty(S_6)$, $T_{out} \in L^2(I; L^2(S_6))$, $T^0 \in W_2^1(\Omega)$. Then we say that the mapping $T \in L^2(I; W_2^1(\Omega))$ is the weak solution of the equation of heat transport if the following conditions hold

$$T(0) = T^0 \qquad \text{a.e. in } \Omega, \tag{15}$$

$$T(t)|_{S_4} = f_4 \qquad \text{a.e. in } I, \tag{16}$$

and the integral identity

$$\frac{d}{dt}\left(H(T),v\right)+\left(\rho c\vec{q}\cdot\nabla T,v\right)+\left(\lambda\nabla T,\nabla v\right)+\int\limits_{S_6}\gamma T v\;dS$$
$$=-\int\limits_{S_5}f_5 v\;dS+\int\limits_{S_6}\gamma T_{out}v\;dS \quad (17)$$

holds for all $v\in V_T$ in $\mathcal{D}'(I)$.

Further, let us define the space of the functions which fulfill the homogeneous stable boundary conditions in sense of traces $V_p=\{f\in W_2^1(\Omega):f|_{S_1}=0\}$. At this moment, we can formulate the following definition.

Let μ is a positive constant. Suppose the following input data qualification : $k\in L^\infty(\Omega)$, $f_2\in L^2(I;L^2(S_2))$, $\beta\in L^\infty(S_3)$, $p_{out}\in L^2(I;L^2(S_3))$ and

$$Q,\frac{\partial\rho}{\partial t}\in L^2(I;L^2(\Omega)).$$

Suppose that there exists a function $p_\Omega\in W_2^1(\Omega)$ such that $p_\Omega|_{S_1}=f_1$. Then we say that the mapping $p\in L^2(I;W_2^1(\Omega))$ is the weak solution of the equation for pressure if

$$p(t)-p_\Omega\in V_p \qquad \text{a.e. in } I,$$

and the integral identity

$$\int\limits_{\Omega}\left[\rho\frac{k}{\mu}\nabla p\cdot\nabla w-\frac{k}{\mu}\rho^2 g\frac{\partial w}{\partial z}\right]\;dxdz+\int\limits_{S_3}\beta p w\;dS$$
$$=(Q,w)-\int\limits_{S_2}f_2 w\;dS+\int\limits_{S_3}\beta p_{out}w\;dS-\left(n\frac{\partial\rho}{\partial t},w\right) \quad (18)$$

holds for all $w\in V_p$ in $\mathcal{D}'(I)$.

3 Discretization and numerical algorithm

We suggest the following combination of method of characteristics and the standard Galerkin scheme for approximation of heat transport equation (5)

$$(\mathbf{A}+\Delta t_k\mathbf{B})\mathbf{T}^{(k+1)}=\mathbf{A}\mathbf{T}^{(k)}\circ\varphi^{(k)}+\Delta t_k\mathbf{F}, \quad (19)$$

where

$$A_{ij}=\int\limits_{\Omega}\overline{\rho c}N_iN_j\;dxdz,$$
$$B_{ij}=\int\limits_{\Omega}\lambda\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z}\right)\;dxdz+\int\limits_{S_6}\gamma N_iN_j\;dS,$$
$$F_i=-\int\limits_{S_5}f_5N_i\;dS+\int\limits_{S_6}\gamma T_{out}N_i\;dS.$$

and the function

$$\varphi^{(k)}(x) := x - \Delta t_k \omega_h * \frac{\rho c \vec{q}(t_{k-1}, x)}{n\rho c + (1-n)\rho_s c_s} \tag{20}$$

is the Euler explicit approximation of characteristics. According the [3], the velocity field has to be smoothed by convolution with mollifier ω_h which is defined as

$$\omega_h(x) = \frac{1}{h^2}\omega_1(\frac{x}{h}), \qquad \omega_1(x) = \begin{cases} \kappa \exp\left(\frac{|x|^2}{|x|^2-1}\right), \text{for}|x| < 1 \\ 0 \qquad\qquad\qquad\quad \text{for}|x| \geq 1, \end{cases} \tag{21}$$

and the coefficient κ is chosen so that the integral of ω_1 over the whole support is unitary. This smoothing guarantees that the approximated characteristics do not intersect each other. This scheme is nothing but the implicit standard Galerkin approximation of the heat transport equation without convection which is included in the right hand side in terms of method of characteristics. As the Galerkin scheme solves just the diffusion problem there are no problems with the artificial oscilations in the case of dominant convection. The matrix of the system of linear algebraical equations is symmetric and if we use the mass lumping technique then it will be diagonally dominant and therefore also positively definite which is advantageous for the numerical solution of this linear algebraical system.

The equation for pressure (4) is discretized by the standard Galerkin approach. The same triangulation and the same linear basis functions as in the case of heat transport equation are used. Assuming that the heat transport equation has been solved before we can approximate the time derivative of density on the right hand side of (18) by the forward difference. The standard Galerkin discretization results to the following algebraical system for the unknown pressures at the nodes of the mesh in time t_{k+1}

$$\mathbf{A}\mathbf{p}^{(k+1)} = \mathbf{F}, \tag{22}$$

where

$$A_{ij} = \int\limits_{\Omega} \rho \frac{k}{\mu} \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z} \right) \, dxdz + \int\limits_{S_3} \beta N_i N_j \, dS, \tag{23}$$

$$F_i = \int\limits_{\Omega} \left[Q N_i - n\frac{\partial \rho}{\partial t} N_i + \frac{k}{\mu}\rho^2 g \frac{\partial N_i}{\partial z} \right] \, dxdz - \int\limits_{S_2} f_2 N_i \, dS + \int\limits_{S_3} \beta p_{out} N_i \, dS$$

are the coefficents of the matrix $\mathbf{A}$ and vector $\mathbf{F}$. As the pressure is approximed linearly on each element the pressure gradients are element-wise constant and thus, the Darcy velocity can be easily determined in terms of (3).

Assume, that we are situated in the $k-$th time level and all the quantities at time t_k are known - either from the initial condition or from the previous time step. The values of all examined quantities in time t_{k+1} can be obtained using the following steps:

- **Step 1.** The solution of the heat transport equation with the initial condition $\mathbf{T}^{(k)}$ gives us new distribution of temperature - $\mathbf{T}^{(k+1)}$.
- **Step 2.** The new layer of densities is obtained by substitution of the new temperatures to (6).
- **Step 3.** Solve the equation for pressure in order to obtain the new layer of pressures - $\mathbf{p}^{(k+1)}$. The time derivative of pressure on the right hand side of this equation can be approximated by the backward difference because we know the values of densities in time t_k and t_{k+1}.
- **Step 4.** Compute new Darcy's velocities using (3).

At this moment, we computed values of all the required quantities in time t_{k+1}. We can repeat this procedure from the step one or to stop the process if the simulation time is up.

4 Results

We simulate groundwater flow in a rectangular domain of size 100×30 m. At the begining, the temperature inside of the considered domain is 10 °C and the water does not move. The boundary conditions and the material properties will be described individualy depending on the problem solved.

Problem 1. 1DirHill : Suppose that the soil in Ω is homogeneous and isotropic. The top and bottom boundary of Ω are impermeable. On the top part of boundary and on the sides, the zero heat flux is prescribed. On the left and right side of the boundary, we prescribe the hydrostatic pressure. The water in domain Ω is heated from bottom - in this case we prescribe the temperature on the bottom side of the $\partial\Omega$. This temperature grows up linearly from 10 °C on the sides to the 90 °C in the middle of the bottom part of the boundary $\partial\Omega$. In the figure 1, we can see the situation in time of 1000 days. The colours represent the temperatures — white colour belong to the 10 °C and black represents the 90 °C. The other colours are associated with the temperatures in between. The arrows show distribution of the Darcy velocities.

Problem 2. 3DirHill : This is an example of the similar problem as described in the previous subsection. The only difference is that the temperature on the bottom part of the $\partial\Omega$ is given by a cosinus function of x-coordinate such that the temperature changes between 10 and 90 °C and has three maxima along the boundary. In the regions of these maxima, we can observe the flow to the top of the aquifer where the water is cooled and then it flows toward the bottom boundary in the regions of the minima of the temperature. The situation in Ω in time of 1000 days is shown in the figure 2. This example is useful for comparison with the problems *1Well* and *HorLayer* whose results are shown in figures 3 and 4. The first of the figures shows the same situation with added pumping while in the second figure there is a situation in which we added a horizontal layer of soil in which the permeability and the coefficient of heat conductivity are higher than in the rest of the aquifer.

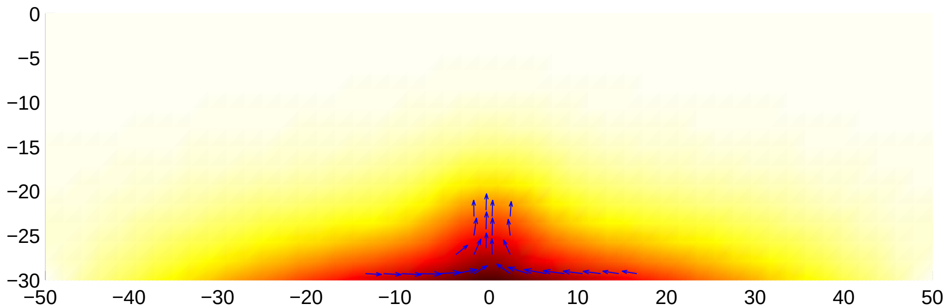

Fig. 1. Problem 1DirHill.

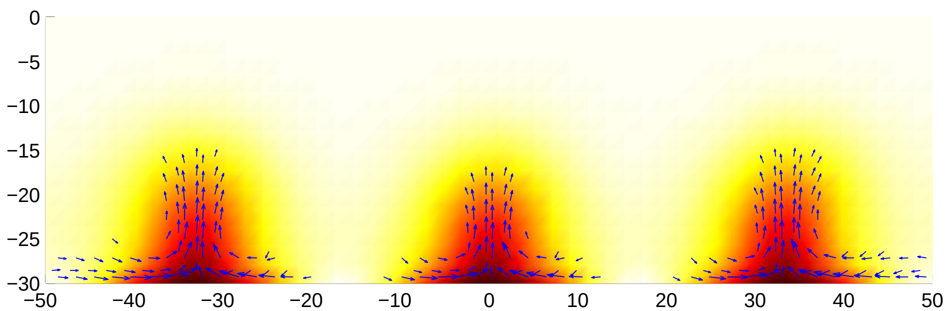

Fig. 2. Problem 3DirHill

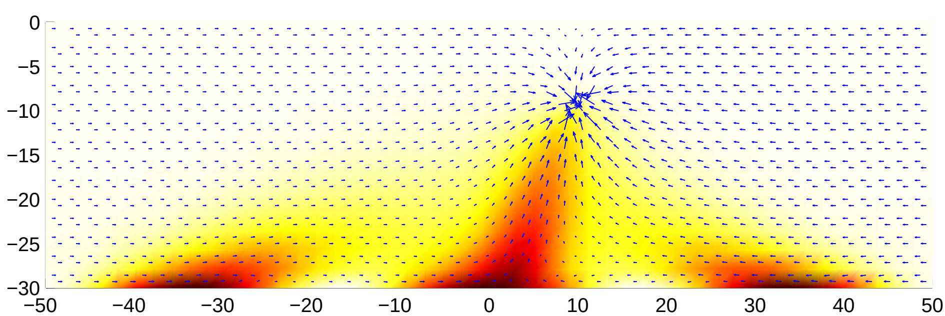

Fig. 3. Problem 1Well.

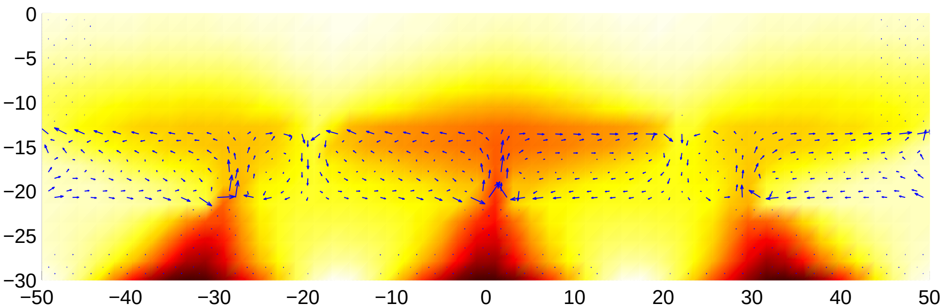

Fig. 4. Problem HorLayer.

5 Analysis of the convergence

In this section, the convergence of the proposed numerical scheme is examined. The problem 1DirHill was chosen as a suitable problem used for testing. As the exact solution is not available, we computed a numerical solution on mesh 250×75 with timestep 1 day and this solution was used for the comparison. Then, the additional solutions of the same problem were computed on coarser meshes with larger timesteps.

Our task was to measure the distances of the individual solutions from the solution on the finest mesh in the norms of several function spaces. The main problem is that we have to interpolate on two different meshes. This is solved by the point-wise projection of the solution from the coarse mesh to the finer mesh. Then, the computation of the norms of the difference is done on the finer mesh.

The results of the convergence analysis are contained in the tables 1 and 2. In the table 1, the symbol $||\cdot||_{\mathcal{X}}$ denotes the $\mathcal{X}-$norm of the difference of the temperature on the mesh 250×75 and the temperature projected from the coarser mesh to the $250 \times 75-$mesh. The same symbol used in table 2 has the analogous meaning, only the pressure difference is measured instead of the temperature.

#	**Mesh**	**Timestep**	$\|\|\cdot\|\|_{L^\infty(I;L^2(\Omega))}$	$\|\|\cdot\|\|_{L^\infty(I;W_2^1(\Omega))}$	$\|\|\cdot\|\|_{L^\infty(I;L^\infty(\Omega))}$
1	50×15	25 days	28.5641	43.1245	3.18548
2	100×30	5 days	8.33552	13.4872	0.84721
3	150×45	$2\frac{1}{2}$ days	5.41337	5.89123	0.49784
4	200×60	$1\frac{1}{3}$ days	4.09549	4.20282	0.39215

Table 1. The results of the convergence analysis for temperature

#	Mesh	Timestep	$\|\|\cdot\|\|_{L^\infty(I;L^2(\Omega))}$	$\|\|\cdot\|\|_{L^\infty(I;W_2^1(\Omega))}$	$\|\|\cdot\|\|_{L^\infty(I;L^\infty(\Omega))}$
1	50×15	25 days	438.598	658.184	66.8801
2	100×30	5 days	170.488	207.186	17.9188
3	150×45	$2\frac{1}{2}$ days	108.380	124.451	9.99112
4	200×60	$1\frac{1}{3}$ days	81.5763	93.0700	7.20534

Table 2. The results of the convergence analysis for pressure

The log-log plots of the norms contained in the previous tables as a function of the mesh size are presented in the figures 5 and 6. The slope of the curves allows to estimate the experimental orders of the convergence ($\mathcal{EOC}'s$) for temperature and pressure in the norms of the corresponding function spaces. The $\mathcal{EOC}$ between two triangulations with the mesh sizes h_1 and h_2 is defined as in [1] by

$$\mathcal{EOC} = \frac{\log E(h_1) - \log E(h_2)}{\log h_1 - \log h_2}, \tag{24}$$

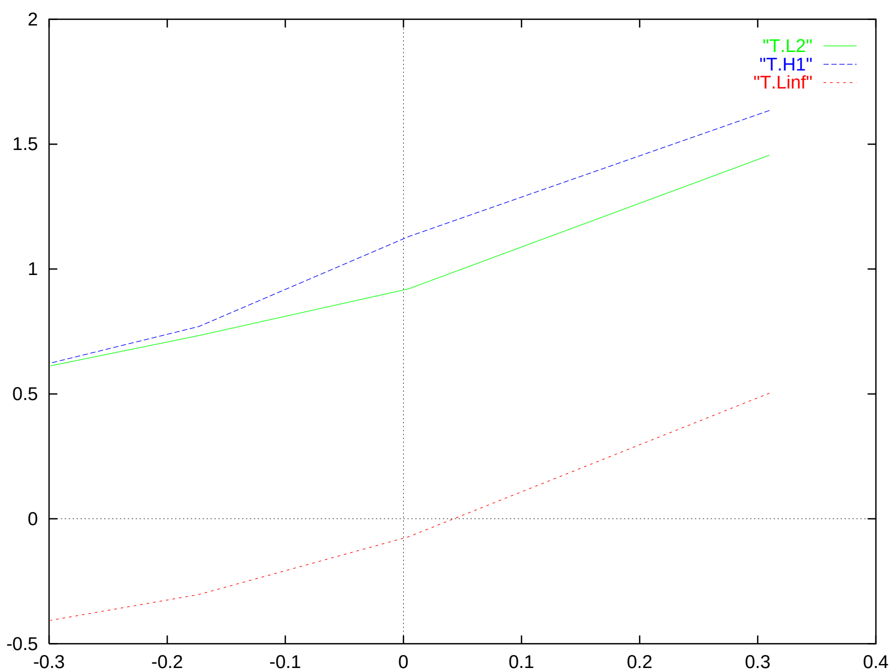

Fig. 5. Log-log plot of the convergence curves for temperature

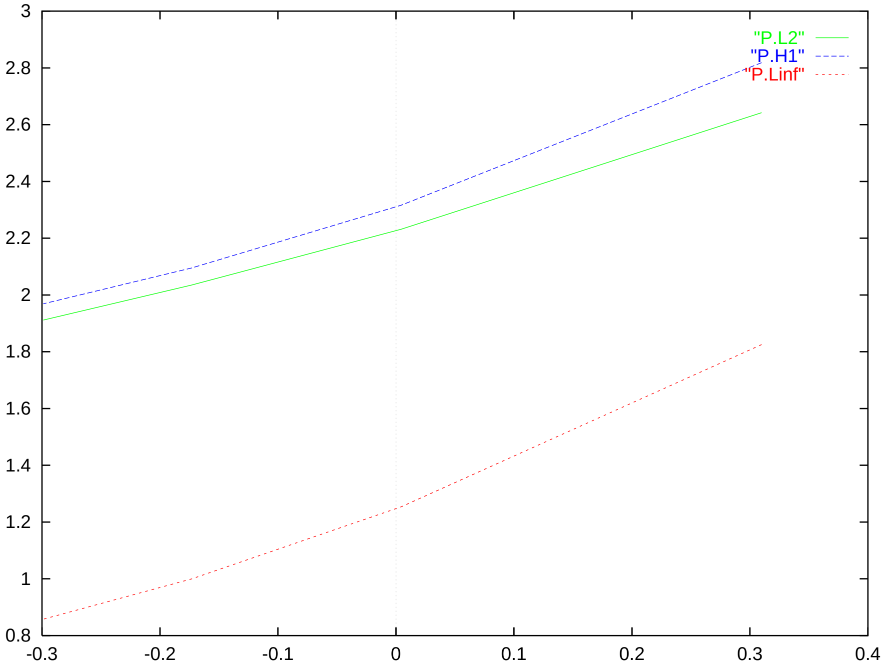

Fig. 6. Log-log plot of the convergence curves for pressure

where the symbol $E(h)$ denotes some of those norms of the difference of the solution on h-mesh and the finest mesh. The experimental orders of the convergence between the individual triangulations are summarized in the table 3.

$\mathcal{EOC}$ of $\downarrow$ between $\rightarrow$	$\#1 \mapsto \#2$	$\#2 \mapsto \#3$	$\#3 \mapsto \#4$
T in $\|\cdot\|_{L^\infty(I;L^2(\Omega))}$	1.75	1.06	0.96
p in $\|\cdot\|_{L^\infty(I;L^2(\Omega))}$	1.34	1.11	0.98
T in $\|\cdot\|_{L^\infty(I;W^1_2(\Omega))}$	1.65	2.03	1.17
p in $\|\cdot\|_{L^\infty(I;W^1_2(\Omega))}$	1.64	1.25	1.00
T in $\|\cdot\|_{L^\infty(I;L^\infty(\Omega))}$	1.88	1.30	0.82
p in $\|\cdot\|_{L^\infty(I;L^\infty(\Omega))}$	1.87	1.43	1.13

Table 3. Experimental orders of the convergence for temperature and pressure

6 Conclusion

The presented results show that the heat transport processes in porous media are relatively slow. The heat transport can be substantially faster in the fractures, so we intend to add the model of fracture flow to the current model.

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