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# A REMARK ON MORREY TYPE REGULARITY FOR NONLINEAR ELLIPTIC SYSTEMS OF SECOND ORDER*

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**Abstract.** In this paper we discuss the problem of the regularity of the gradient of weak solutions to nonlinear elliptic systems

$$-D_\alpha a_i^\alpha(x, Du) = 0, \quad i = 1, \dots, N,$$

where the coefficients $a_i^\alpha(x, Du)$ have some special form and they may be discontinuous in general.

**Key words.** Nonlinear equations, regularity, Morrey-Campanato spaces

**AMS subject classifications.** 49N60, 35J60

**1. Introduction.** In this paper, we consider the problem of the regularity of the gradient of weak solutions to the second order nonlinear strongly elliptic system

$$-D_\alpha a_i^\alpha(x, Du) = 0, \qquad i = 1, \dots, N, \tag{1.1}$$

where $a_i^\alpha$ are Caratheodorian mappings from $\Omega \times \mathbb{R}^{nN}$ into $\mathbb{R}$, $N > 1$, $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is a bounded open set. A function $u \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^N)$ is called a weak solution of (1.1) in $\Omega$ if

$$\int_\Omega a_i^\alpha(x, Du) D_\alpha \varphi^i(x) \, \mathrm{d}x = 0, \quad \forall\, \varphi \in C_0^\infty(\Omega, \mathbb{R}^N).$$

We use the summation convention over repeated indices.

As it is known, in case of a general system (1.1), only partial regularity can be expected for $n > 2$ (see e.g. [3, 9, 11]).

For example if

$$|a_i^\alpha(x, p)| \leq L(1 + |p|),$$

$(1 + |p|)^{-1} a_i^\alpha(x, p)$ are Hőlder continuous in $x$ uniformly with respect to $p$ and $a_i^\alpha$ are differentiable functions in $p$,

$$|a^\alpha_{i, p_j^\beta}(x, p)| \leq L$$

and

$$a^\alpha_{i, p_j^\beta}(x, p) \xi^i_\alpha \xi^j_\beta \geq \nu |\xi|^2$$

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then the first derivatives of weak solution of (1.1), are Hőlder continuous in an open set $\Omega_0 \subset \Omega$. In particular $meas(\Omega \backslash \Omega_0) = 0$.

In a special case if $a_i^\alpha = a_i^\alpha(Du)$, $a_i^\alpha(0) = 0$ and $2 \le n \le 4$ then $u$ is Hőlder continuous in $\Omega$.This result is the best possible, because for $n > 4$ $u$ is only partially Hőlder continuous.

It is well known (see [2]) that in linear case

$$-D_\alpha(A_{ij}^{\alpha\beta}(x) D_\beta u^j) = -D_\alpha f_i^\alpha, \quad i = 1, \dots, N$$

the following holds: Suppose that

$$A_{ij}^{\alpha\beta}(x) \xi_\alpha^i \xi_\beta^j \ge \nu |\xi|^2, \quad a.e. x \in \Omega, \quad \forall \xi \in \mathbb{R}^{nN}; \quad \nu > 0, \tag{1.2}$$

$A_{ij}^{\alpha\beta} \in C^0(\overline{\Omega})$, $f_i^\alpha \in L^{2,\lambda}(\Omega)$, $0 < \lambda < n$. Then $Du \in L_{loc}^{2,\lambda}(\Omega, \mathbb{R}^{nN})$. Moreover if coefficients $A_{ij}^{\alpha\beta}$ belong to some Hőlder classes then the gradient of $u$ belongs (locally) to the BMO-class.

The last mentioned result has become a motive for studying Morrey regularity of gradient of weak solutions to nonlinear systems (1.1) where

$$a_i^\alpha(x, Du) = A_{ij}^{\alpha\beta}(x) D_\beta u^j + g_i^\alpha(Du). \tag{1.3}$$

**2. Notation and definitions.** We consider a bounded open set $\Omega \subset \mathbb{R}^n$, $n \ge 2$, $u : \Omega \to \mathbb{R}^N$, $N \ge 1$, $u(x) = (u^1(x), \dots, u^N(x))$ is a vector-valued function, $Du = (D_1 u, \dots, D_n u)$, $D_\alpha = \partial / \partial x_\alpha$. The symbol $\Omega_0 \subset\subset \Omega$ stands for $\overline{\Omega}_0 \subset \Omega$. For the sake of simplicity we denote by $|\cdot|$ the norm in $\mathbb{R}^n$ as well as in $\mathbb{R}^N$ and $\mathbb{R}^{nN}$. If $x \in \mathbb{R}^n$ and $r$ is a positive real number, we set $B_r(x)$ an open ball in $\mathbb{R}^n$, centered at $x$ with radius $r$, $\Omega(x,r) = \Omega \cap B_r(x)$. By $u_{x,r} = \fint_{\Omega(x,r)} u(y)\, dy$ we denote the mean value of the function $u \in L^1(\Omega, \mathbb{R}^N)$ over the set $\Omega(x,r)$. Beside the usually used space $C_0^\infty(\Omega, \mathbb{R}^N)$, the Hölder spaces $C^{0,\alpha}(\Omega, \mathbb{R}^N)$, $C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^N)$ and the Sobolev spaces $W^{k,p}(\Omega, \mathbb{R}^N)$, $W_{\mathrm{loc}}^{k,p}(\Omega, \mathbb{R}^N)$, $W_0^{k,p}(\Omega, \mathbb{R}^N)$ (see, e.g.[10]), we use the following Morrey spaces.

Definition 2.1. Let $\lambda \in [0, n]$, $q \in [1, \infty)$. A function $u \in L^q(\Omega, \mathbb{R}^N)$ is said to belong to Morrey space $L^{q,\lambda}(\Omega, \mathbb{R}^N)$ if

$$||u||_{L^{q,\lambda}(\Omega, \mathbb{R}^N)}^q = \sup_{x \in \Omega, r > 0} \frac{1}{r^\lambda} \int_{\Omega(x,r)} |u(y)|^q \,\mathrm{d}y < \infty.$$

Proposition 2.2. *For a Lipschitz domain $\Omega \subset \mathbb{R}^n$ the following hold:*

(i) *With the norm $\|u\|_{L^{q,\lambda}}$ the space $L^{q,\lambda}(\Omega, R^N)$ is a Banach space.*

(ii) *If $u \in W_{\mathrm{loc}}^{1,2}(\Omega, \mathbb{R}^N)$ and $Du \in L_{\mathrm{loc}}^{2,\lambda}(\Omega, \mathbb{R}^{nN})$, $n - 2 < \lambda < n$, then $u \in C^{0,(\lambda+2-n)/2}(\Omega, \mathbb{R}^N)$.*

For more details see [2, 9, 10, 11].

Some generalization of Campanato spaces $\mathcal{L}^{q,\lambda}$ (see [2]) are the classes $\mathcal{L}_\psi$ introduced by Spanne [12] and [13].

Definition 2.3. A function $u \in L^2(\Omega, \mathbb{R}^N)$ is said to belong to $\mathcal{L}_\psi(\Omega, \mathbb{R}^N)$ if

$$[u]_{\psi,\Omega} = \sup_{x \in \Omega, r \in (0, \operatorname{diam} \Omega]} \frac{1}{\psi(r)} \Big( \fint_{\Omega(x,r)} |u(y) - u_{x,r}|^2 \,\mathrm{d}y \Big)^{1/2} < \infty$$

and by $l_\psi(\Omega, \mathbb{R}^N)$ we denote the subspace of all $u \in \mathcal{L}_\psi(\Omega, \mathbb{R}^N)$ such that

$$[u]_{\psi,\Omega,r_0} = \sup_{x\in\Omega, r\in(0,r_0]} \frac{1}{\psi(r)} \Big( \fint_{\Omega(x,r)} |u(y) - u_{x,r}|^2 \, \mathrm{d}y \Big)^{1/2} = o(1) \text{ as } r_0 \searrow 0,$$

where

$$\psi(r) = (1 + |\ln r|)^{-1}.$$

Some basic properties of the above-mentioned spaces are formulated in the following proposition (for the proofs see [1, 12, 13]).

Proposition 2.4. *For a Lipschitz domain $\Omega \subset \mathbb{R}^n$ we have the following:*

(i) $\mathcal{L}_\psi(\Omega, \mathbb{R}^N)$ *is a Banach space with norm* $\|u\|_{\mathcal{L}_\psi(\Omega,\mathbb{R}^N)} = \|u\|_{L^2(\Omega,\mathbb{R}^N)} + [u]_{\psi,\Omega}$.

(ii) $C^0(\overline{\Omega}, \mathbb{R}^N) \setminus \mathcal{L}_\psi(\Omega, \mathbb{R}^N)$ *and*
$(L^\infty(\Omega, \mathbb{R}^N) \cap l_\psi(\Omega, \mathbb{R}^N)) \setminus C^0(\overline{\Omega}, \mathbb{R}^N)$ *are not empty.*

### 3. Results for above mentioned type of nonlinearity.

Theorem 3.1 (continuous coefficients , sublinear growth). *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a weak solution to the system* (1.1) *and the conditions* (1.2), (1.3) *be satisfied. Suppose further that $A^{\alpha\beta}_{ij} \in C^0(\overline{\Omega})$ and $g^\alpha_i$ are smooth functions such that $|g^\alpha_i(p)| \le K|p|^\gamma$ for all $p \in \mathbb{R}^{nN}$, where $\gamma < 1$ , $i,\ j = 1, \dots, N,\ \alpha,\ \beta = 1, \dots, n$. Then $Du \in L^{2,\lambda}_{loc}(\Omega, \mathbb{R}^{nN})$ , $(0 < \lambda < n)$.*

This theorem is exactly proved in [4].

Theorem 3.2 (discontinuous coefficients , sublinear growth). *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a weak solution to the system* (1.1) *and the conditions* (1.2), (1.3) *be satisfied. Suppose further that$A^{\alpha\beta}_{ij} \in L^\infty(\Omega) \cap \mathcal{L}_\psi(\Omega)$ (in general discontinuous functions) and $g^\alpha_i$ are smooth function such that $|g^\alpha_i(p)| \le K|p|^\gamma$ , $\gamma < 1$ and $g^\alpha_i(p)p^i_\alpha \ge \nu_1|p|^{1+\gamma}$, $i,\ j = 1, \dots, N,\ \alpha,\ \beta = 1, \dots, n$. Then $Du \in L^{2,\lambda}_{loc}(\Omega, \mathbb{R}^{nN})$, $(0 < \lambda < n)$.*

For proof of Theorem 3.2 see [6].

An immediate consequence of Theorem 3.1 and Theorem 3.2 is Hőlder continuity of weak solution $u$.

To do the growth conditions on $g^\alpha_i$ weaker we have to assume some structural condition:

Theorem 3.3 (discontinuous coefficients, linear growth). *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a weak solution to the system* (1.1) *and the conditions* (1.2), (1.3) *be satisfied. Suppose further that$A^{\alpha\beta}_{ij} \in L^\infty(\Omega) \cap \mathcal{L}_\psi(\Omega)$ and $g^\alpha_i$ are smooth function such that $|g^\alpha_i(p)| \le K|p|$ , and $g^\alpha_i(p)p^i_\alpha \ge \nu_1|p|^2$,$i,\ j = 1, \dots, N,\ \alpha,\ \beta = 1, \dots, n$ and*

$$\left(\frac{K}{\nu}\right)^2 \le \frac{1}{6(1 + 2^{n+1}L)\left(c(n,q) + \frac{1}{2^{n-2}}\right)(3.2^{2n+2}L)^{(n-\delta)/\delta}}$$

*with $0 < \delta < n$ (constants $L$ and $c(n,q)$ are stated in lemmas which will follow ).Then $Du \in L^{2,\lambda}_{loc}(\Omega, \mathbb{R}^{nN})$ for $\lambda < n - \delta$.*

From Theorem 3.3 it follows that for $0 < \delta < 2$ weak solution of (1.1) $u \in C^{0,\theta}(\Omega, \mathbb{R}^N)$ with $\theta < 1 - \delta/2$.

Theorem 3.3 is proved in [7] (submitted for publication) and in the following parts we give a sketch of its proof.

The main tools which we need to prove this theorem are standard Korn's device of freezing the coefficients,higher integrability of gradient of solution and some delicate estimates.

**4. Preliminary results and sketch of proof.** In this section we present the results needed for the proof of Theorem 3.3. In $B_R(x) \subset \mathbb{R}^n$ we consider a linear elliptic system

$$-D_\alpha(A_{ij}^{\alpha\beta} D_\beta u^j) = 0 \tag{4.1}$$

with constant coefficients for which (1.2) holds.

Lemma 4.1 ([2, pp. 54–55]). *Let $u \in W^{1,2}(B_R(x), \mathbb{R}^N)$ be a weak solution to the system (4.1). Then, for each $0 < \sigma \le R$,*

$$\int_{B_\sigma} |Du(y)|^2 \, dy \le L \left(\frac{\sigma}{R}\right)^n \int_{B_R} |Du(y)|^2 \, dy$$

*hold with a constant $L$ independent of the homotethie.*

In the following considerations we will use a result about higher integrability of the gradient of a weak solution to the system (1.1).We set $A = (A_{ij}^{\alpha\beta})$, $g = (g_i^\alpha)$.

Proposition 4.2 ([9, p. 138]). *Suppose that the assumptions of* Theorem 3.3 *are fulfilled and let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a weak solutions of (1.1). Then there exists an exponent $r > 2$ such that $u \in W^{1,r}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Moreover there exist constants $c = c(\nu, \nu_1, L, \|A\|_{L^\infty})$ and $\widetilde{R} > 0$ such that, for all balls $B_R(x) \subset \Omega$, $R < \widetilde{R}$, the following inequality is satisfied*

$$\Big( ⨍_{B_{R/2}(x)} |Du|^r \, dy \Big)^{1/r} \le c \Big( ⨍_{B_R(x)} |Du|^2 \, dy \Big)^{1/2}$$

In the following we will use the function

$$\ln_+ t = \begin{cases} 0 & \text{for} \quad 0 \le t < 1, \\ \ln t & \text{for} \quad t \ge 1. \end{cases}$$

Lemma 4.3 ([5, p. 531]). *Let $u \in W^{1,2q}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, $q > 1$. Then for every ball $B_{2R}(x) \subset \Omega$ and arbitrary constants $b > 0$ we have*

$$\int_{B_R(x)} |Du|^2 \ln_+(b|Du|^2) \, dy \le C \Big( ⨍_{B_{2R}(x)} \ln_+^{q/(q-1)}(4b|Du|^2) \, dy \Big)^{1-1/q} \int_{B_{2R}(x)} |Du|^2 \, dy$$

*where $C = C(n, q)$.*

As a small modification of Lemma from [9, p. 86] we can obtain

Lemma 4.4. *Let $\phi$ be a nonnegative and nondecreasing function on $(0, R_0]$ and there is a constant $\tau$, $0 < \tau < 1$ such that for every $R < R_0$*

$$\phi(\tau R) \le \tau^\alpha \phi(R) + BR^\beta$$

*where $B \geq 0$, $0 < \beta < \alpha$. Then for every $\varrho < R \leq R_0$ we have*

$$\phi(\varrho) \leq C \left\{ \left(\frac{\varrho}{R}\right)^{\beta} \phi(R) + B\varrho^{\beta} \right\}$$

*where $C$ is a constant depending on $\tau$, $\alpha$ and $\beta$.*

Let now $\Phi$, $\Psi$ be a pair of complementary Young functions

$$\Phi(t) = t \ln_+ at \quad \text{for} \quad t \geq 0, \qquad \Psi(t) = \begin{cases} t/a & \text{for} \quad 0 \leq t < 1, \\ \mathrm{e}^{t-1}/a & \text{for} \quad t \geq 1, \end{cases} \tag{4.2}$$

where $a > 0$ is a constant. Let us recall Young inequality

$$ts \leq \Phi(t) + \Psi(s), \quad t, s \geq 0. \tag{4.3}$$

Proposition 4.5 (see [8]). *Let $v \in L^2_{loc}(\Omega, \mathbb{R}^m)$, $m \geq 1$, $B(x,\sigma) \subset \Omega$, $q \in (1,\infty)$ and $b > 0$ be arbitrary. Then*

$$\int_{B(x,\sigma)} \ln_+^q \left(b|v|^2\right) \mathrm{d}x \leq q \left(\frac{q-1}{\mathrm{e}}\right)^{q-1} b \int_{B(x,\sigma)} |v|^2 \, \mathrm{d}x.$$

As a consequence of (4.2), (4.3), Lemma 4.3 and Proposition 4.5 we have:

Proposition 4.6 (see [7]). *Let $u \in W^{1,2q}_{loc}(\Omega, \mathbb{R}^N)$, $q \in (1,\infty)$ and $|g_i^{\alpha}(p)| \leq K|p|$ holds. Then, for each $\varepsilon > 0$ and all $B_R(x) \subset\subset \Omega$,*

$$\begin{aligned} \int_{B_R(x)} &|g_i^{\alpha}(Du)|^2 \, \mathrm{d}y \\ &\leq \varepsilon K^2 c(n,q) \Big(4a\varepsilon K^2 \fint_{B_{2R}(x)} |Du(y)|^2 \, \mathrm{d}y\Big)^{(q-1)/q} \int_{B_{2R}(x)} |Du(y)|^2 \, \mathrm{d}y \\ &\quad + \kappa_n \Psi\left(\frac{1}{\varepsilon}\right) R^n. \end{aligned}$$

*Proof.* [Sketch of proof of Theorem 3.3.] We set $U(r) = U(x,r) = \fint_{B_r(x)} \|Du(y)|^2 \, \mathrm{d}y$, $\phi(r) = \phi(x,r) = \int_{B_r(x)} |Du(y)|^2 \, \mathrm{d}y$. Let $B_{R/2}(x_0) \subset B_R(x_0) \subset \Omega$ be an arbitrary ball and let $w \in W_0^{1,2}(B_{R/2}(x_0), \mathbb{R}^N)$ be a solution of the following system

$$\begin{aligned} \int\limits_{B_{R/2}(x_0)} &(A_{ij}^{\alpha\beta})_{x_0,R/2} D_\beta w^j D_\alpha \varphi^i \, \mathrm{d}x \\ &= \int\limits_{B_{R/2}(x_0)} \big((A_{ij}^{\alpha\beta})_{x_0,R/2} - A_{ij}^{\alpha\beta}(x)\big) D_\beta u^j D_\alpha \varphi^i \, \mathrm{d}x - \int\limits_{B_{R/2}(x_0)} g_i^{\alpha}(Du) D_\alpha \varphi^i \, \mathrm{d}x \end{aligned} \tag{4.4}$$

for all $\varphi \in W_0^{1,2}(B_{R/2}(x_0), \mathbb{R}^N)$. An existence and unicity of such solution is known. Now we can put $\varphi = w$ in (4.4) and, using ellipticity and Hölder inequality, we get

$$\begin{aligned} \nu^2 \int\limits_{B_{R/2}(x_0)} |Dw|^2 \, \mathrm{d}x &\leq 2 \int\limits_{B_{R/2}(x_0)} |A_{x_0,R/2} - A(x)|^2 |Du|^2 \, \mathrm{d}x + 2 \int\limits_{B_{R/2}(x_0)} |g(Du)|^2 \, \mathrm{d}x \\ &= 2I + 2II. \end{aligned}$$

From Proposition 4.2 with $r = 2q > 2$, Hölder inequality ($r' = q/(q-1)$) and using the properties of matrix $A = (A_{ij}^{\alpha\beta})$ we obtain

$$I \leq c\, \psi^{1/r'}(R) \int\limits_{B_R(x_0)} |Du|^2 \, \mathrm{d}x$$

where $c = c(n, q, [A]_{2,\Psi,\Omega}, \|A\|_{L^\infty(\Omega,\mathbb{R}^{n^2N^2})})$.

We can estimate $II$ by means of Proposition 4.6 and we have

$$\begin{aligned} \int\limits_{B_{R/2}(x_0)} |Dw|^2 \, \mathrm{d}x \leq \ & \frac{c}{\nu^2} \psi^{1/r'}(R) \int\limits_{B_R(x_0)} |Du|^2 \, \mathrm{d}x \\ & + \varepsilon \frac{K^2}{\nu^2} c(n,q) \left(4a\varepsilon K^2 U(2R)\right)^{(q-1)/q} \phi(2R) + \frac{1}{\nu^2} \kappa_n \Psi\left(\frac{1}{\varepsilon}\right) R^n. \end{aligned} \tag{4.5}$$

The function $v = u - w \in W^{1,2}(B_{R/2}(x_0), \mathbb{R}^N)$ is the solution to the system

$$\int\limits_{B_{R/2}(x_0)} (A_{ij}^{\alpha\beta})_{x_0,R/2} D_\beta v^j D_\alpha \varphi^i \, \mathrm{d}x = 0, \qquad \forall \varphi \in W_0^{1,2}(B_{R/2}(x_0), \mathbb{R}^N).$$

From Lemma 4.1 we have, for $0 < \sigma \leq R/2$,

$$\int\limits_{B_\sigma(x_0)} |Dv|^2 \, \mathrm{d}x \leq 2^n L \left(\frac{\sigma}{R}\right)^n \int\limits_{B_{R/2}(x_0)} |Dv|^2 \, \mathrm{d}x. \tag{4.6}$$

By means of (4.5) and (4.6) we obtain, for all $0 < \sigma \leq R/2$ and $\varepsilon > 0$, the following estimate

$$\begin{aligned} \phi(\sigma) \leq \ & \left[2^{n+2} L \left(\frac{\sigma}{R}\right)^n + 2(1 + 2^{n+1}L) \frac{c}{\nu^2} \psi^{1/r'}(R)\right] \phi(R) \\ + \ & \frac{2(1+2^{n+1}L)K^2 c(n,q)}{\nu^2}\, \varepsilon \left(4a\varepsilon K^2 U(2R)\right)^{(q-1)/q} \phi(2R) \\ + \ & \frac{2(1+2^{n+1}L)}{\nu^2} \kappa_n \Psi\left(\frac{1}{\varepsilon}\right) R^n \end{aligned} \tag{4.7}$$

Now if we put $a = 1/(4\varepsilon K^2 U(2R))$ in (4.7) and then put $\varepsilon = 1$, $\tau = 1/(3 \cdot 2^{2n+2} L)^{1/\delta}$ and $\sigma = 2\tau R$ we immediately get

$$\begin{aligned} \phi(2\tau R) \leq \Bigg[ & \frac{1}{3} \tau^{n-\delta} + 2(1 + 2^{n+1}L) \frac{c}{\nu^2} \psi^{1-1/q}(R) + 2(1 + 2^{n+1}L) c(n,q) \left(\frac{K}{\nu}\right)^2 \\ & + \frac{2(1 + 2^{n+1}L)}{2^{n-2}} \left(\frac{K}{\nu}\right)^2 \Bigg] \phi(2R). \end{aligned}$$

From the last estimate we see that there is $R_0 > 0$ such that for every $R < R_0$

$$\phi(2\tau R) \leq \tau^{n-\delta} \phi(2R)$$

holds. Now we can use Lemma 4.4. ∎

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