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Remarks on free objects in categories

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REMARKS ON FREE OBJECTS IN CATEGORIES

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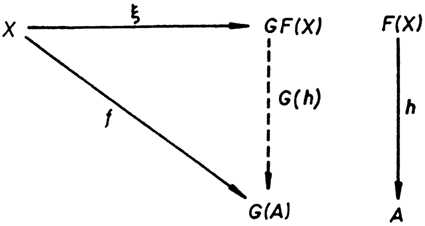
Introduction. A free Abelian group may be defined as a coproduct of copies of the infinite cyclic group Z or else as an Abelian group satisfying a certain universal property. The latter approach has found an expression in the language of adjoint functors and free-object functors as adjoints to the "underlying set" functors are now well known. On the other hand, free objects are frequently seen to be the coproducts of a certain fixed object, e.g., free topological spaces (discrete spaces) are disjoint topological sums of the single-point space. The purpose of this note is to emphasize that this is no accident. We give a "coproduct-definition" of free objects and observe that this agrees with the usual "adjoint-definition" in a fairly wide class of categories. We also establish some well known results about free and projective objects. We must point out that a "coproduct-definition" has been given by Semadeni [S1]. As far as the author knows, however, no attempt was made to connect it to the "adjoint-definition".

1. Definitions. Let $\mathbf{K}$ be a category and let $\mathbf{Ens}$ be the category of sets and functions. A covariant functor G: $\mathbf{K} \to \mathbf{Ens}$ is called a *grounding* of $\mathbf{K}$ (see Isbell [I1]). A faithfully grounded category is called a *concrete* category. A *universal element* for a grounding G is a pair (u, R) consisting of an object R of $\mathbf{K}$ and element $u \in G(R)$ with the following property: To any object K of $\mathbf{K}$ and any elements $s \in G(K)$ there is exactly one morphism $f: R \to K$ with $G(f)\, u = s$. If $\mathbf{K}(R, K)$ stands for the set of all morphisms with R as source and K as target then there is a bijection $\mathbf{K}(R, K) \simeq G(K)$; it is known that this is a natural equivalence. If $\mathbf{K}$ is concrete and the faithful grounding $G: \mathbf{K} \to \mathbf{Ens}$ has an adjoint $F: \mathbf{Ens} \to \mathbf{K}$ we say that F is a *free-object functor*. Explicit reference to G is avoided; we say free objects rather than G-free objects and define $F(X)$ to be *free* on the set X. To say that F is adjoint to G, of course, means that to any set X there is an object $F(X)$ in $\mathbf{K}$ and a function $\xi: X \to G\,F(X)$ such that given any function $f: X \to G(A)$ for an object A of $\mathbf{K}$ there is a unique morphism $h: F(X) \to A$ making the following diagram commutative



Henceforward **K** will stand for a concrete category, G: **Ens** $\to$ **K** its faithful grounding and F will denote the free-object functor. The single-point set will be denoted by a star. For a set X, $|X|$ denotes its cardinality. The covariant hom functor, for a fixed object A, will be denoted by h_A.

2. Proposition. *G : **K** $\to$ **Ens** has an adjoint F : **Ens** $\to$ **K** only if it has a universal element.*

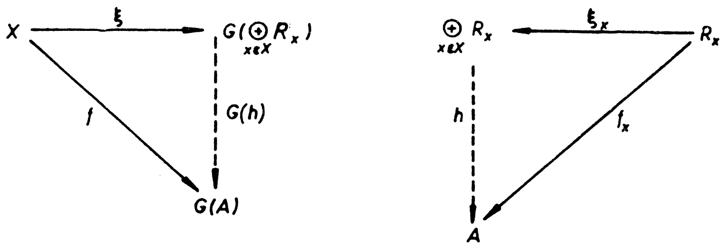
Proof. Assume that G has an adjoint F and let us consider the object $F(*)$. When we recall that an element in any set is simply a function with * as its domain, fig. 1 (on putting $\xi(*) = u$) reads: To any object A and to any point $f \in G(A)$ there is a unique morphism $h : F(*) \to A$ such that $G(h)\, u = f$. In other words, $(u, F(*))$ is a universal element for G.

3. Definition. If the universal element (u, R) for the faithful grounding G : **K** $\to$ **Ens** exists, R is called the *universal free object* and $u \in G(R)$ is called the *universal free element*. These terms will be abbreviated to ufo and ufel respectively.

A *free object* in **K** is defined to be a coproduct of copies of R.

4. Proposition. *Assume that **K** has coproducts and G has a universal element (u, R) Then the free-object functor F : **Ens** $\to$ **K** exists. Conversely, if the free object functor exists then G has a universal element (u, R) and $F(X)$ is precisely a coproduct of $|X|$ copies of R.*

Proof. Define $F(X) = \bigoplus_{x \in X} R_x$ where each R_x is a copy of R and let $\xi_x : R_x \to \bigoplus_{x \in X} R_x$ be the injections. Define $\xi : X \to G(\bigoplus_{x \in X} R_x)$ by setting $\xi(x) = G(\xi_x)(u_x)$ where $u_x \in$ $\in G(R_x)$ is the ufel.

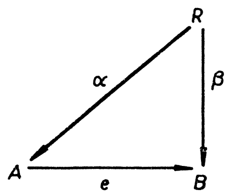


Next, let A be any object in **K** and let $f : X \to G(A)$ be any function. Then, to $f(x) \in$ $\in G(A)$ there is a unique morphism $f_x : R_x \to A$ with $G(f_x)(u_x) = f(x)$. Consequently there is a unique morphism $h : \bigoplus_{x \in X} R_x \to A$ with $h\xi_x = f_x$. Then $G(h)\,\xi(x) = G(h)\,.$ $.\, G(\xi_x)(u_x) = G(h\xi_x)(u_x) = G(f_x)(u_x) = f(x)$ so that $G(h)\,\xi = f$. Thus F is adjoint to G. The converse follows from proposition 2 and the facts that F preserves coproducts and a set X is a coproduct of $|X|$ copies of *.

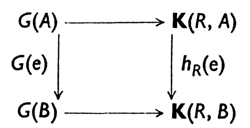
5. Definition. We say that a morphism e of $\mathbf{K}$ is a *concrete epimorphism* if $G(e)$ is a surjection. If $\alpha e = \beta e$ then $G(\alpha)\, G(e) = G(\beta)\, G(e)$ so that $G(\alpha) = G(\beta)$. Since G is faithful, $\alpha = \beta$ and e is indeed an epimorphism. An object P is called *projective* if for any concrete epi $e : A \to B$ and any morphism $\beta : P \to B$ there exists a (not necessarily unique) morphism $\alpha : P \to A$ such that $e\alpha = \beta$.

6. Proposition. *The ufo R is projective.*

Proof.



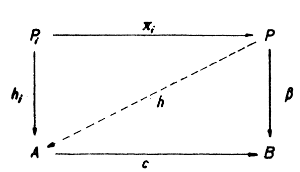
Since $G(e) : G(A) \to G(B)$ is a surjection there is some element $a \in G(A)$ with $G(e)\, a = {}$ $= G(\beta)\, u$. The unique morphism $\alpha : R \to A$ with $G(\alpha)\, u = a$ exists. To see that $e\alpha = \beta$ we recall that the bijection $\mathbf{K}(R, K) \simeq G(K)$ was natural. This means that the diagram



where the horizontal arrows are bijective, commutes. Corresponding to β in $\mathbf{K}(R, B)$ we picked the unique element $G(\beta)\, u$ in $G(B)$ and since $G(e)$ was a surjection there existed an element $a \in G(A)$ with $G(e)\, a = G(\beta)\, u$; $\alpha \in \mathbf{K}(R, A)$ was chosen via the natural bijective arrow on the top and hence $h_R(e)\,(\alpha) = \beta$ i.e. $e\alpha = \beta$.

7. Proposition. *Projective objects are closed under coproducts.*

Proof. Let P_i be a set of projective objects and let $\pi_i : P_i \to P$ be their coproduct. We want to show that P is also projective. For this let $c : A \to B$ be concrete epi and let $\beta : P \to B$ be any morphism.



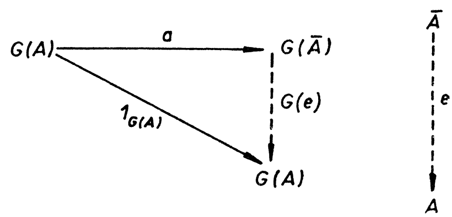
Then we have morphisms $\beta\pi_i : P_i \to B$ and since each P_i is projective, there exists morphisms $h_i : P_i \to A$ such that $ch_i = \beta\pi_i$ for every i. But then P being a coproduct of P_i there exists a unique $h : P \to A$ such that $h\pi_i = h_i$. Then $ch\pi_i = \beta\pi_i$ for each i. This implies that $ch = \beta$ since the π_i are canonical injections. Therefore P is projective.

8. Corollary. *A free object is projective.*

Proof. A free object is a coproduct of copies of the ufo.

9. Proposition. *For any object A there exists a free object $\bar{A}$ and a concrete epi $e : \bar{A} \to A$.*

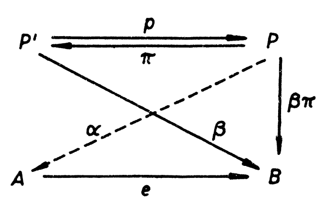
Proof. Set $\bar{A} = F\,G(A)$. The following diagram



tells us that $G(e)$ has a right inverse i.e. is surjective.

10. Proposition. *A retract of a projective object is projective.*

Proof. Let $\pi : P \to P'$ be a retraction i.e. there is $p : P' \to P$ such that $\pi p = 1_{P'}$. We shall show that if P is projective, so is P'.

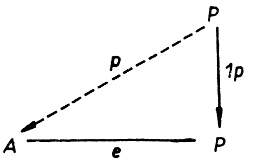


Let $e : A \to B$ be a concrete epi and let $\beta : P' \to B$ be any morphism. Then we have $\beta\pi : P \to B$ and since P is projective there is $\alpha : P \to A$ such that $e\alpha = \beta\pi$. Then $e\alpha p = \beta\pi p = \beta$ and $\alpha p : P' \to A$ is the required morphism. Thus P' is projective.

11. Proposition. *The following are equivalent*

1. *P is projective.*
2. *If $e : A \to P$ is a concrete epi then P is a retract of A.*
3. *P is a retract of a free object.*

Proof. 1 ⇒ 2. Clear from the following diagram



2 ⇒ 3). Proposition 9 tells us that there exists object $\bar{P}$ and *a* concrete epi $e : \bar{P} \to P$, this means that P must be a retract of $\bar{P}$.

3 ⇒ 1). Corollary 8 and proposition 10.

12. Examples. i) Let **Grp** (**Abg**) stand for the category of groups (Abelian groups) and homomorphisms. The infinite cyclic group Z is the ufo. A free group (a free Abelian group) is a free product (a direct sum) of copies of Z. Every group (Abelian group) is an epimorphic image of a free group (a free Abelian group).

ii) Let **Top** stand for the category of topological spaces and continuous functions. The one-point space is the ufo. A free topological space is a discrete space i.e. a disjoint topological sum of one-point spaces. Every space is the continuous image of a discrete space.

iii) In $\mathbf{Cpt}_2$, the category of compact Hausdorff spaces and continuous functions the one-point space is the ufo. A free compact space is the Stone-Čech compactification of a discrete space. Every compact space is the continuous image of a free compact space. A projective object is an extremally disconnected compact T_2 space (cf. Gleason [G2]) and is always a retract of a free compact space.

iv) In A_W the category of transition systems with input W the transition system M_W is the ufo. If A_W and B_W are two transition systems whose sets of states are A and B then their coproduct is given by the transition system whose set of states is given by the disjoint sum of A and B. A free transition system is a coproduct of copies of M_W. Other propositions also find justification. (See Giveón [G1] for details.)

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