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A TECHNIQUE FOR CONSTRUCTING EXAMPLES

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Ostaszewski introduced a basic technique for constructing examples in topology in which one well-orders the underlying set, and then defines a neighborhood base for each point with transfinite recursion, [O]. His example needs $\diamondsuit(=\clubsuit+\mathrm{CH})$. In a modification of this technique, due to Kunen, one constructs a finer topology on a hereditarily separable space, using hereditary separability in an essential way, [JKR]. This requires CH.

We pursue this line and construct finer topologies on $\mathbb{R}$, the reals. No additional axioms are needed, instead we use the idea of a classical construction of Bernstein. The basic idea is to make sure that sufficiently many countable sets which have 2^{ω} cluster points in $\mathbb{R}$, get sufficiently many cluster points in the new topology.

EXAMPLE 1. A normal, countably paracompact, ω_1-compact, separable, locally compact, locally countable space Λ which is not quasi-developable (not even weakly $\delta\theta$-refinable), and a metrizable space P such that $\Lambda \times P$ is not normal.

EXAMPLE 2. An orthocompact, locally compact, locally countable space Σ which is not countably metacompact.

We briefly sketch what these spaces look like. Recall that the topology of either space is finer ($\equiv$ more open sets) than the topology of $\mathbb{R}$.

Λ has the property that

(a) for all $F,G \subseteq \Lambda$, if $|Cl_{\mathbb{R}}F \cap Cl_{\mathbb{R}}G| = 2^{\omega}$ then $|Cl_{\Lambda}F \cap Cl_{\Lambda}G| = 2^{\omega}$.

Since $\mathbb{R}$ is hereditarily separable, it suffices to make sure that (a) holds for *countable* $F,G \subseteq \Lambda$. Condition (a) implies that Λ is normal, countably paracompact, ω_1-compact and also that Λ is not the union of countably many relatively discrete subsets. Since Λ is locally countable, the latter fact implies that Λ is not weakly $\delta\theta$-refinable. The rationals $\mathbb{Q}$ are dense in Λ, and Λ has a relatively discrete subset D of cardinality 2^{ω}. Then the subspace $\Pi = \mathbb{Q} \cup D$ of Λ is not normal. If P is $\mathbb{Q} \cup D$ as subspace of $\mathbb{R}$, then the "diagonal" of $\Lambda \times P$ is a closed homeomorph of Π, hence $\Lambda \times P$ is not normal.

One can make sure that $\Lambda \times \Lambda$ is normal.

Σ has a disjoint family $\{L_n : n \in \omega\}$ of subsets such that if $D = \cup\{L_n : n \geq 1\}$ then

(b) D is closed discrete;

(c) all points of $\Sigma - D$ are isolated;

(d) if $K \subseteq L_0$ is countable, and $|Cl_{\mathbb{R}}K| = 2^{\omega}$, then $Cl_{\Sigma}K$ intersects L_n for each $n \geq 1$.

Then Σ is orthocompact by (b) and (c), but is not countably metacompact since the open cover $\{L_n \cup (\Sigma - D) : n \geq 1\}$ has no point-finite refinement, by (d).

Details of the construction will appear in [vD]. There we also give applications, and list references of further applications of this technique.

REFERENCES

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[O] A. Ostaszewski, *On countably compact perfectly normal spaces,* J. London Math. Soc. (to appear).