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A HEREDITARILY NORMAL STRONGLY ZERO - DIMENSIONAL SPACE CONTAINING SUBSPACES OF ARBITRARY LARGE DIMENSION.

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1. It was an old problem raised by E. Čech [2] whether the covering dimension dim is monotone in the class of hereditarily normal spaces; the analogous problem for the large inductive dimension Ind was investigated by C. H. Dowker [3] (cf. also [1] and [12]).

Under the assumption of an existence of Souslin's continuum V. V. Filippov [8] solved these problems in the negative exhibiting a hereditarily normal space X with $\dim X = 0$ containing for $n = 1,2,\ldots$ a subspace X_n with $\dim X_n = \operatorname{Ind} X_n = n$.

A year ago we constructed [15] using only the usual set theory a hereditarily normal space Z with $\dim Z = 0$ containing a subspace Y with $\dim Y = \operatorname{Ind} Y = 1$, and quite recently we improved this construction [16] to get a hereditarily normal space X with $\dim X = 0$ containing subspace X_n with $\dim X_n = \operatorname{Ind} X_n = n$ for $n = 1,2,\ldots$

It is worth while to notice that compact hereditarily normal spaces missing the monotonicity of dimensions dim and Ind were constructed recently by V. V. Fedorčuk [5],[6] and A. Ostaszewski [13], under some hypothesis stronger than the continuum hypothesis, and more recently, by V. V. Fedorčuk [7] and E. Pol [14], under the continuum hypothesis.

In this note we shall briefly discuss the main idea of our constructions. Our topological terminology will follow [4].

2. B. Knaster and K. Kuratowski gave in their classical work on connectedness [10] the following beautiful construction of a hereditarily disconnected, but not totally disconnected space K: let P be irrationals and Q rationals from the unit real interval I, and let E be a subset of P which is not an F_σ - set in P; then define $K = E \times P \cup (P \setminus E) \times Q \subset P \times I$ (compare with the Knaster - Kuratowski Broom [1],[4],[9; p. 22]).

The standard proof that $\dim K = 1$ relies on the verification that the sets $(P \setminus E) \times \{0\}$ and $(P \setminus E) \times \{1\}$ can not be separated by the empty set.

However, one can also observe that each G_σ - set in $P \times I$ containing the space K contains a set $\{p\} \times I$ for a point $p \in P$ and this property suggests the following construction. Let us split

P into $n+1$ disjoint Bernstein sets $B_0, B_1, \ldots, B_n$ (i.e., each B_m intersects each subspace of P homeomorphic to P [11; § 40]), let, for $0 \le m \le n$, R_n^m be the set of the points in the n-dimensional cube I^n exactly m of whose coordinates are rational (i.e., we consider the standard decomposition of I^n into $n+1$ zero - dimensional sets [9; p. 19]; note that $R_1^0 = P$ and $R_1^1 = Q$) and let us define $K_n = \bigcup_{m=0}^{n} B_m \times R_n^m$. Then each G_σ - set in $P \times I^n$ containing K_n contains also a set $\{p\} \times I^n$ for a point $p \in P$ and hence $\dim K_n = n$.

We have seen that the dimension of the spaces K or K_n was designated by the Borel properties of the sets E or B_m, respectively.

Following this observation, in order to obtain a perfectly normal space with local dimension less than dimensions dim or Ind (which in fact is just what we need) we would like to get through constructions analogous to that of K or K_n with an appropriate perfectly normal space B, in which local Borel properties of sets differ from global, instead of P. We shall describe a correspondent space B in the next section.

3. Let ω_1 be the set of all countable ordinals with the discrete topology, let N be the set of natural numbers and let $B(\aleph_1) = \omega_1^N$ be the Baire space of weight $\aleph_1$. To define the space B we give the set ω_1^N a topology finer than the topology of $B(\aleph_1)$, by taking as a base the sets $U \cap \{x \in \omega_1^N : x(n) < \alpha \text{ for } n \in N\}$, where U is an open set in the Baire space $B(\aleph_1)$ and $\alpha < \omega_1$.

The space B is perfectly normal, but not paracompact. Although B is locally second - countable and thus its local structure is quite different from those of $B(\aleph_1)$, the topology of B is closely related to the topology of $B(\aleph_1)$ and, in particular, the Borel sets of B and of $B(\aleph_1)$ coincide. The detail informations about B the reader can find in [17].

4. A. H. Stone [19] defined a non - Borel subset E of the Baire space $B(\aleph_1)$ such that each separable subspace of E is countable. The set E considered as the subspace of B is also non - Borel, however E is locally countable in B.

We let (compare with the definition of K in sec. 2) $Y = E \times P \cup (B \setminus E) \times Q \subset B \times I$. Since E is locally countable, we infer from the sum theorem that the local dimension $\operatorname{loc} \dim Y = 0$, whereas $\dim Y = \operatorname{Ind} Y = 1$, by the same arguments as in case of K.

The space Z we are looking for can be now defined as follows: let

$Z = Y \cup \{p\}$ where $p \notin Y$, Y is an open subspace of Z and the basic neighbourhoods of the point p are taken in the form $\{p\} \cup U$, where the space $Y \setminus U$ is open - and - closed in Y, second countable and zero - dimensional. One can easily verify that the space Z is hereditarily normal, Lindelöf and $\dim Z = 0$.

5. The construction of the spaces X_n will imitate the definition of the spaces K_n given in the section 2, where the space B (sec. 3) will be used in place of irrationals P.

For the purpose we shall exploit the decomposition $B_0, \ldots, B_n$ of the Baire space $B(\chi_1)$ into $n+1$ disjoint non-separable analogues to the Bernstein sets in P, constructed in [18].

The sets B_m considered in the space B are locally F_σ-sets, but their global properties are similar to those of the Bernstein sets.

We let $X_n = \bigcup_{m=0}^{n} (B_m \times R_n^m) \subset B \times I^n$.

By the sum theorem we have $\operatorname{loc} \dim X_n = 0$. However, as in the case of the space K_n, each G_σ-set in $B \times I^n$ containing the space X_n, contains a set $\{p\} \times I^n$ for a point $p \in B$ and this easily implies that the projection of X_n onto the n-dimensional cube I^n is an essential mapping (cf. [1]). Thus $n \leq \dim X_n \leq \operatorname{Ind} X_n$, while $\operatorname{Ind} X_n \leq \operatorname{Ind} B + \operatorname{Ind} I^n = n$.

The hereditarily normal space X with $\dim X = 0$ containing all of the spaces X_n can be obtained in the same way as the space Z from the preceding section, where we let Y to be the free union of the spaces X_n.

REFERENCES.

[1] P.S.Aleksandrov, B.A.Pasynkov, Introduction to dimension theory (in Russian), Moskva 1973.

[2] E.Čech, Problem 53, Coll. Math. 1 (1948), 332.

[3] C.H.Dowker, Local dimension of normal spaces, Quart. Journ. Math. Oxford 6 (1955), 101-120.

[4] R.Engelking, General topology, Warszawa 1976.

[5] V.V.Fedorčuk, Compatibility of some theorems of general topology with the axioms of set theory (in Russian), DAN SSSR 220 (1975), 786-788.

[6] V.V.Fedorčuk, Fully closed mappings and compatibility of some theorems of general topology with the axioms of set theory (in Russian), Mat. Sb. 99 (1976), 3-33.

[7] V.V.Fedorčuk, On the dimension of hereditarily normal spaces (preprint).

[8] V.V.Filippov, On the dimension of normal spaces (in Russian), DAN SSSR 209 (1973), 805-807.

[9] W.Hurewicz, H.Wallman, Dimension theory, Princeton 1941.

[10] B.Knaster, K.Kuratowski, Sur les ensembles connexes, Fund. Math. 2 (1921), 206-255.

[11] K.Kuratowski, Topology, vol. I, New York 1966.

[12] K.Nagami, Dimension theory, New York 1970.

[13] A.Ostaszewski, A perfectly normal countably compact scattered space which is not strongly zero - dimensional (preprint).

[14] E.Pol, Remark about Juhász - Kunen - Rudin construction of a hereditarily separable non - Lindelöf space, Bull. Acad. Pol. Sci. (to appear).

[15] E.Pol, R.Pol, A hereditarily normal strongly zero - dimensional space with a subspace of positive dimension and an N - compact space of positive dimension, Fund. Math. (to appear).

[16] E.Pol, R.Pol, A hereditarily normal strongly zero - dimensional space containing subspaces of arbitrary large dimensions, Fund. Math. (to appear).

[17] R.Pol, A perfectly normal, locally metrizable, non - paracompact space, Fund. Math. (to appear).

[18] R.Pol, Note on decompositions of metrizable spaces II, Fund. Math. (to appear).

[19] A.H.Stone, On σ-discretness and Borel isomorphism, Amer. Jour. Math. 85 (1963), 655-666.

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