Marián J. Fabián

On singlevaluedness and continuity of monotone mappings

In: Josef Novák (ed.): General topology and its relations to modern analysis and algebra IV, Proceedings of the fourth Prague topological symposium, 1976, Part B: Contributed Papers. Society of Czechoslovak Mathematicians and Physicist, Praha, 1977. pp. 116--118.

Persistent URL: http://dml.cz/dmlcz/700617

ON SINGLEVALUEDNESS AND CONTINUITY OF MONOTONE MAPPINGS

M. FABIAN

Praha

Let $T: X \longrightarrow 2^{X^*}$ be a monotone multivalued mapping from a real Banach space X to its dual X^* (endowed with the norm dual to the norm on X) such that its domain has nonempty interior, i.e., int $D(T) \neq \emptyset$. For sake of simplicity, we denote the following assertion by (A).

The set of all those $x \in$ int $D(T)$ for which Tx is a singleton and T is upper semicontinuous at x, i.e., to each $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $u \in D(T)$, fulfilling $\|u-x\| < \delta$, the set Tu is included in the ε-neighbourhood of Tx, is dense residual in int $D(T)$.

Up to now the following theorems on singlevaluedness and continuity of T are known.

THEOREM 1 (Robert [5]). X^* is separable $\Longrightarrow$(A).

THEOREM 2 (Author [1]). X is reflexive $\Longrightarrow$(A).

THEOREM 3 (Author [2]). X^* is strictly convex and has the property (H_ω) (see below) $\Longrightarrow$(A).

THEOREM 4 (Kenderov, Robert [4]). X^* has the property (H) (see below) $\Longrightarrow$(A).

Thanks to the renorming statement of John and Zizler [3], Theorems 1 and 2 are included in Theorem 3 or 4.

In this communication, we outline the way how to obtain Theorem 3.

1. Let P be a metric space, X a real normed linear space, X^* its topological dual endowed with the norm dual to the norm on X. We say that X^* has the property (H) (resp. (H_ω)) if for every $w \in X^*$ and every net (resp. sequence) $\{w_\alpha\} \subset X^*$ the following implication holds

$$(w_\alpha \longrightarrow w \text{ (weakly}^*\text{)}, \ \|w_\alpha\| \longrightarrow \|w\|) \Longrightarrow w_\alpha \longrightarrow w.$$

Throughout the section, $T: P \longrightarrow 2^{X^*}$ will be a demiclosed multivalued mapping, i.e., (we do not distinguish a mapping from its graph)

$$\forall u \in P \ \ \forall w \in X^* \ \ \forall \text{net } \{(u_\alpha, w_\alpha)\} \subset T$$

$$(u_\alpha \longrightarrow u,\ w_\alpha \longrightarrow w \text{ (weakly}^*),\ \sup\|w_\alpha\| < +\infty) \Longrightarrow (u,w) \in T.$$

A singlevalued mapping $T_1: P \longrightarrow X^*$, having the same domain as T, i.e., $D(T_1)=D(T)$, and such that $T_1 \subset T$, is called a selection of T. If, moreover, $(u,w) \in T$ implies $\|w\| \geqslant \|T_1 u\|$, then T_1 is called a lower selection of T. Let $f_T: P \longrightarrow (-\infty, +\infty]$ be the function defined by

$$f_T(u) = \inf\{\|w\| \mid w \in Tu\}, \quad u \in P.$$

Finally set (T_1 being a selection of T)

$$C(f_T) = \{u \in D(T) \mid f_T \text{ is continuous at } u\},$$

$$C(T_1) = \{u \in D(T) \mid T_1 \text{ is continuous at } u\},$$

$$C^d(T_1) = \{u \in D(T) \mid T_1 \text{ is demicontinuous at } u\},$$

where demicontinuity means continuity from the metric topology to the weak* topology.

Under the above notations and assumptions the following lemmas are valid:

LEMMA 1.1. f_T is a lower semicontinuous function.

LEMMA 1.2. The set $C(f_T)$ is residual in $D(T)$.

LEMMA 1.3. If there exists a unique lower selection T_o of T, then $C(f_T) \subset C^d(T_o)$ and hence, $C^d(T_o)$ is residual in $D(T)$.

LEMMA 1.4. If X^* has the property (H_ω), and there exists a unique lower selection T_o of T, then $C(T_o)=C(f_T)$, and hence $C(T_o)$ is residual in $D(T)$.

2. In this section we apply the above lemmas for the study of monotone mappings. Recall that a mapping $T: X \longrightarrow 2^{X^*}$ is called monotone, if

$$\langle x^*-y^*, x-y\rangle \geqslant 0 \quad \text{for all } (x,x^*),\ (y,y^*) \in T,$$

where $\langle \cdot,\cdot \rangle$ means the duality pairing between X^* and X, and maximal monotone, if T is not properly contained in any other monotone mapping. In what follows, we shall assume that $T: X \longrightarrow 2^{X^*}$ is a maximal monotone multivalued mapping from a real Banach space X to its dual X^* such that $\operatorname{int} D(T) \neq \emptyset$. Set

$$SV(T) = \{x \in \operatorname{int} D(T) | \ Tx \text{ is a singleton}\}.$$

LEMMA 2.1. T is demiclosed, and if X^* is strictly convex, then there exists a unique lower selection T_0 of T.

LEMMA 2.2. For any selection T_1 of T the following inclusion holds

$$C^d(T_1) \cap \operatorname{int} D(T) \subset SV(T).$$

PROPOSITION 2.1. If X^* is strictly convex, then the set $SV(T)$ is dense residual in $\operatorname{int} D(T)$.

PROPOSITION 2.2. If X^* is strictly convex and has the property (H_ω), and T_0 denotes the (unique) lower selection of T, then $C(T_0)$ is residual in $D(T)$.

LEMMA 2.3. If T_1, T_2 are two arbitrary selections of T, then

$$C(T_1) \cap \operatorname{int} D(T) = C(T_2) \cap \operatorname{int} D(T).$$

Now all is prepared for the proof of Theorem 3.

REFERENCES

[1] M. FABIAN, On strong continuity of monotone mappings in reflexive Banach spaces, to appear in Acta Polytechnica.

[2] M. FABIAN, On singlevaluedness and (strong) continuity of maximal monotone mappings, to appear in Comment. Math. Univ. Car.

[3] K. JOHN, V. ZIZLER, A renorming of dual spaces, Israel J. Math. 12(1972), 331-336.

[4] P. KENDEROV, R. ROBERT, Nouveaux résultats génériques sur les opérateurs monotones dans les espaces de Banach, to appear in C. R. Acad. Sci. Paris.

[5] R. ROBERT, Une généralisation aux opérateurs monotones des théorèmes de differentiabilité d'Asplund, Analyse convexe et ses Applications, Lecture Notes in Economics and Mathematical System 102, 168-179.