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THE LIMIT INFERIOR OF A FILTERED SET-FAMILY AS A SET OF LIMIT POINTS

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It is well-known (see Kuratowski [6], p. 242) that in each metric space E the limit inferior $\liminf_{n\to\infty} A_n$ of each sequence (A_n) with $A_n \subseteq E$ is the set of all $x \in E$ such that there exist an $n_0(x) \in \mathbb{N}$ and a sequence $(a_n(x))$ in E with $a_n(x) \in A_n$ for all $n \geq n_0(x)$ and $x = \lim_{n\to\infty} a_n(x)$. In this paper, we give (in Proposition 3) a modified generalization of this statement to a (general) topological space E and the limit inferior lim inf $(f, I, \mathfrak{a})$ of a filtered family $(f, I, \mathfrak{a})$ with $f(i) \subseteq E$ for all $i \in I$.

For the remainder, let (E, τ) be a topological space, $\mathfrak{V}_\tau$ the neighborhood operator, Lim_τ the limit operator, $\liminf_\tau$ the limit inferior induced by the topology τ. For abbreviation, we write just $\mathfrak{V}$, Lim, lim inf instead of $\mathfrak{V}_\tau$, Lim_τ, $\liminf_\tau$ if no confusion can arise.

1. Terminology. In every respect, we shall use the same terminology as used or introduced in [4]. For nonempty sets I and K and filters $\mathfrak{a}$ and $\mathfrak{b}$ on I and K, respectively, $\mathfrak{a} \otimes \mathfrak{b}$ denotes the ordinal product of $\mathfrak{a}$ and $\mathfrak{b}$ (which is a filter on $I \times K$)(see [3], p. 330 and p. 336, Satz 23). Given a filter $\mathfrak{a}$ on a set I and $A \in \mathfrak{a}$, $\mathfrak{a}_A$ denotes the trace of $\mathfrak{a}$ in the set A; furthermore, if, for each $i \in I$, a statement form H(i) containing i as a free variable is given, we say that H(i) holds "for $\mathfrak{a}$-almost all $i \in I$" if and only if, for some $A \in \mathfrak{a}$, H(i) holds for all $i \in A$. Given a mapping f and a set $B \subseteq \mathcal{D} f$, we denote the restriction of f to B by f_B. Recall that, for each set M, ΦM denotes the class of all filtered families in M. For each directed set $(D, \leq)$, i.e. for each set D with a reflexive and transitive relation $\leq$ such that each finite subset of D has an upper bound w.r.t. to $\leq$, we denote by $\mathfrak{F} D$ the "filter of perfinality" on D, which is defined to be the filter on D generated by the set $\{\{z \mid y \leq z \in D\} \mid y \in D\}$. Especially, for each $x \in E$, $(\mathfrak{V}x, \supseteq)$ is a directed set, and so $\mathfrak{F}(\mathfrak{V}x)$ is the filter of perfinality on $\mathfrak{V}x$.

2. Connection between lim inf and Lim. The mapping lim inf is an extension of the mapping Lim in the sense of

Proposition 1. Let κ denote the mapping $x \mapsto \{x\}$ on E into $\mathfrak{P}E$. Then

$$\mathrm{Lim}\,(f, I, \mathfrak{A}) = \liminf\,(\kappa \circ f, I, \mathfrak{A})$$

holds for all $(f, I, \mathfrak{A}) \in \Phi E$.

Proof. Use of the definitions only. □

While Proposition 1 remains true in general finitely additive quasitopological spaces (see Section 4), this is not the case for the next proposition:

Proposition 1'. In the notation of Proposition 1, one has:

$$\mathrm{Lim}\,(f, I, \mathfrak{A}) = \liminf\,(\tau \circ \kappa \circ f, I, \mathfrak{A})$$

holds for all $(f, I, \mathfrak{A}) \in \Phi E$.

Proof. Use that, for each $x \in E$, $\mathfrak{V}x$ is generated by the class of all open neighborhoods of x; furthermore that $f(i) \in \tau\{f(i)\}$ for all $i \in I$. □

One half of the statement on $\liminf_{n\to\infty} A_n$ in the introduction is still true in general topological spaces:

Proposition 2. For all $x \in E$ and all $(f, I, \mathfrak{A}) \in \Phi(\mathfrak{P}E)$, the following statement form (a) implies (b):

(a) There are an $A \in \mathfrak{A}$ and a $g \in \prod_{i \in A} f(i)$ such that $x \in \mathrm{Lim}\,(g, A, \mathfrak{A}_A)$.

(b) $x \in \liminf\,(f, I, \mathfrak{A})$.

Proof. Use of the definitions of Lim and lim inf in terms of $\mathfrak{V}$. □

The full generalization of the introductory remark on $\liminf_{n\to\infty} A_n$ is given by:

Proposition 3. For all $x \in E$ and all $(f, I, \mathfrak{A}) \in \Phi(\mathfrak{P}E)$, the following statement forms (a) through (d) are equivalent:

(a) $x \in \liminf\,(f, I, \mathfrak{A})$.

(b) There exists a mapping g on $(\mathfrak{V}x) \times I$ into E with the following property:
$g(V, i) \in f(i)$ for $((\mathfrak{F}(\mathfrak{V}x)) \otimes \mathfrak{A})$-almost all $(V, i) \in (\mathfrak{V}x) \times I$ and $x \in \mathrm{Lim}\,(g, (\mathfrak{V}x) \times I, (\mathfrak{F}(\mathfrak{V}x)) \otimes \mathfrak{A})$.

(c) is obtained from (b) by replacing the quantifier "for $((\mathfrak{F}(\mathfrak{V}x))\otimes\mathfrak{m})$-almost all " by " for $\{\mathfrak{V}x\}\otimes\mathfrak{m}$-almost all".

(d) There are a $C\in(\tilde{\mathfrak{F}}(\mathfrak{V}x))\otimes\mathfrak{m}$ and a $g\in\prod_{(V,i)\in C} f(i)$ such that $x\in \mathrm{Lim}\,(g, C, ((\mathfrak{F}(\mathfrak{V}x))\otimes\mathfrak{m})_C)$.

Proof. See [5a] . □

3. Applications of the preceding propositions. It would be desirable to regain "nice" properties of certain subspaces of $(\mathcal{P}E,\mathcal{P}\tau)$ (the power of (E,τ)) from corresponding or related properties of (E,τ), or conversely, by means of the preceding propositions. A simple step in the desired direction can be seen in the next proposition.

Proposition 4. Let $\mathfrak{M}\subseteq(\mathcal{P}E)\setminus\{\emptyset\}$ and assume $\{x\}\in\mathfrak{M}$ for all $x\in E$. Then, the subspace $(\mathfrak{M},\ (\mathcal{P}\tau)_{\mathfrak{M}})$ of $(\mathcal{P}E,\mathcal{P}\tau)$ is compact if and only if (E,τ) is compact.

Proof. One uses Propositions 1 and 2. For details, see [5a] . □

While the proof of Proposition 4 can be carried over, word by word, to finitely additive quasitopological spaces (see Section 4), this is not the case for the proof of the next proposition.

Proposition 4'. Let $\mathfrak{M}\subseteq(\mathcal{P}E)\setminus\{\emptyset\}$ and assume $\tau\{x\}\in\mathfrak{M}$ for all $x\in E$. Then, the subspace $(\mathfrak{M},\ (\mathcal{P}\tau)_{\mathfrak{M}})$ of $(\mathcal{P}E,\mathcal{P}\tau)$ is compact if and only if (E,τ) is compact.

Proof. One uses Propositions 1' and 2. For details, see [5a] . □

Remark. Proposition 4 contains the special case $\mathfrak{M}=\mathcal{P}E\setminus\{\emptyset\}$. In this case, $(\mathfrak{M},\ (\mathcal{P}\tau)_{\mathfrak{M}})$ coincides (see [5]) with the hyperspace of lower semicontinuity of (E,τ) (see Michael [7], p. 179, Definition 9.1 ("$\mathcal{P}E\setminus\{\emptyset\}$ with the lower finite topology"), and (for "closure spaces") Čech [1], p. 623, Definition 34 A.1). Proposition 4' contains the special case $\mathfrak{M}=2^E$(= set of all nonempty closed subsets of (E,τ)). In this case, $(\mathfrak{M},\ (\mathcal{P}\tau)_{\mathfrak{M}})$ coincides (see [5] or Flachsmeyer [2], p. 326, 2.1, or Poppe [8]) with Michael's space 2^E endowed with the "lower finite topology" ([7], loc. cit.). While this special case of Proposition 4' occurs in the literature (with a different proof, using Alexander's Lemma (see Flachsmeyer [2], p. 327, 2.4)), the author could not find a reference concerning the mentioned special case of Proposition 4.

In the paper [4], we have considered topological spaces (E, τ), (F, σ), (G, λ), relations $R \subseteq E \times F$, $S \subseteq F \times G$ with $\mathcal{R} R \subseteq \mathcal{D} S$, and - respectively - the canonical mapping $\hat{R}$, $\hat{S}$, $\widehat{S \circ R}$ induced by R, S, $S \circ R$, which is a mapping from (in general not on!) (E, τ), (F, σ), (E, τ) into the power $(\mathcal{P}F, \mathcal{P}\sigma)$, $(\mathcal{P}G, \mathcal{P}\lambda)$, $(\mathcal{P}G, \mathcal{P}\lambda)$ of (F, σ), (G, λ), (G, λ), as we do now for the following. We recall the validity of the logical diagram

$$\begin{array}{ccc} R \text{ and } S \text{ continuous} & \Longrightarrow & \hat{R} \text{ and } \hat{S} \text{ continuous} \\ \Downarrow & & \Downarrow \\ S \circ R \text{ continuous} & \Longrightarrow & \widehat{S \circ R} \text{ continuous} \end{array}$$

and use Proposition 3 to reprove the right-hand arrow, more precisely, to reprove the next proposition (cf. Čech [1], p. 631, Theorem 34 B. 14, and [5]; furthermore, see [4], p. 41, Proposition 7):

Proposition 5. If $\hat{R}$ is $(\tau, \mathcal{P}\sigma)$-continuous and $\hat{S}$ is $(\sigma, \mathcal{P}\lambda)$-continuous, then $\widehat{S \circ R}$ is $(\tau, \mathcal{P}\lambda)$-continuous.

Proof. See [5a]. □

4. Generalization. The Propositions 1, 2, 3, 4, 5 and their proofs remain valid if the topological spaces (E, τ), (F, σ), (G, λ) are replaced by general finitely additive quasitopological spaces (terminology: [5]) except for the following change within Proposition 3: Within (b), replace "There exists" by the words "$\mathcal{V}x$ is a filter and there exists". Within (d), replace "There are" by the words "$\mathcal{V}x$ is a filter and there are". - A quasitopological space (E, τ) is called to be compact if and only if $\operatorname{Lim}_\tau \mathfrak{a} \neq \emptyset$ for all ultrafilters $\mathfrak{a}$ in E.

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