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SCATTERED SPACES OF POINT-COUNTABLE TYPE

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1. Introduction. We announce here a number of results concerning the structure of σ-scattered q-spaces. Part of the interest in such results arises from the large number of important counter-examples which are spaces of this kind. We indicate in our exposition how various strengthenings of these conditions affect the structure of spaces. Full proofs will be given in an expanded version of this paper. There are two aspects of structure which arise: completeness and some sort of uniform first-countability such as is exemplified by having a base of countable order. One of the earliest results of the type considered is that of Kuratowski [7]: Every separable metric scattered space is an absolute G_δ. Telgársky [10] proved that every T_2 paracompact first countable scattered space is an absolute G_δ. The authors showed [13] that every T_1 first countable scattered space has λ-bases hereditarily. Both the results of Kuratowski and Telgársky follow from this result with the aid of known theorems. In this paper we obtain results for spaces which are σ-scattered and satisfy the condition of being of point-countable type or the weaker condition of q-space. We also consider the case of σ-(closed-and-scattered) spaces. The conditions of scattered, σ-(closed-and-scattered), and σ-scattered result in respectively weaker structures on the spaces.

2. Definitions and terminology. We use ω to denote the non-negative integers. Recall [7] that a topological space X is scattered if and only if X has no nonempty dense-in-itself subspace. A space X is σ-scattered (σ-(closed-and-scattered)) if and only if $X = \bigcup\{S_n : n \in \omega\}$, where each S_n is scattered (closed-and-scattered). A space is said to have a base of countable order [1] if and only if it has a base $\mathcal{B}$ such that the range of every decreasing sequence $\langle B_n : n \in \omega \rangle$ of distinct sets in $\mathcal{B}$ is a base at all points of $\bigcap\{B_n : n \in \omega\}$. A space X has a λ-base [12] if and only if it has a base $\mathcal{B}$ of countable order such that for every decreasing sequence $\langle B_n : n \in \omega \rangle$ of distinct sets in $\mathcal{B}$ there exists $p \in X$ such that every open set containing p includes some B_n. A q-space [8] is a space X such that for every $p \in X$ there is a sequence $\langle V_n : n \in \omega \rangle$ of neighborhoods of p such that if $y_n \in V_n$ for all $n \in \omega$, then $\langle y_n : n \in \omega \rangle$ has a cluster point. A space X is of point-countable type [2] if and only if it is the union of compact sets of countable character. We note that first countable implies point-countable type implies q-space. Let X be a space and let $\mathfrak{G} = \langle \mathfrak{G}_n : n \in \omega \rangle$ be a sequence of bases for X. A decreasing representative of $\mathfrak{G}$ is a sequence $\langle G_n : n \in \omega \rangle$ such that for all $n \in \omega$, $G_{n+1} \subset G_n \in \mathfrak{G}_n$. Such a representative is fixed if $\bigcap\{G_n : n \in \omega\} \neq \emptyset$. If every fixed decreasing representative $G = \langle G_n : n \in \omega \rangle$ has the property that if $y_n \in G_n$ for all $n \in \omega$ implies that

$\langle \mathcal{G}_n : n \in \omega\rangle$ clusters, then X is called monotonically quasi-complete. If X is monotonically quasi-complete and every decreasing representative G of $\mathcal{G}$ has the property that $\bigcap\{\overline{G_n} : n \in \omega\} \neq \emptyset$, then X is called countably monotonically Čech complete. When X is regular, these two concepts are equivalent respectively to the concepts of β_c-space and λ_c-space defined in [11]. If for every filter base $\mathcal{A}$ having the property that if there is a (fixed) decreasing representative G of $\mathcal{G}$ whose range is a subset of the filter generated by $\mathcal{A}$ then $\bigcap\{\overline{A} : A \in \mathcal{A}\} \neq \emptyset$, then X is called a (monotonic p-space) monotonic Čech complete space. These names are used in the regular case in [5] and results there show that what is stated here is an equivalent definition. In the case where X is regular these correspond respectively to the β_b- and λ_b-spaces of [11].

Certain base conditions related to the preceding ones involve well-ordered collections of open sets. If Z is a well ordered collection and $p \in Z$, then $F(p, Z)$ denotes the first element of Z that contains p. Suppose X is a space and $\langle \mathcal{W}_n : n \in \omega\rangle$ is a sequence of well ordered open covers of X. If for all $p \in X$, $\bigcap\{F(p, \mathcal{W}_n) : n \in \omega\} = \{p\}$, X is said to have a primitive diagonal. If for all $p \in X$, $\{F(p, \mathcal{W}_n) : n \in \omega\}$ is a local base at p, then X has a primitive base. If for all $p \in X$, every sequence $\langle y_n : n \in \omega\rangle$ such that $y_n \in F(p, \mathcal{W}_n)$ for all $n \in \omega$ has a cluster point, then X is called primitively quasi-complete[15]. If every filter base $\mathcal{A}$ for which some $\{F(p, \mathcal{W}_n : n \in \omega\}$ is a subset of the filter generated by $\mathcal{A}$ has the property that $\bigcap\{\overline{A} : A \in \mathcal{A}\} \neq \emptyset$, then X is said to have a primitive p-structure. For relations among these concepts and those of the preceding paragraph and such spaces as p-spaces and quasi-complete spaces the reader is referred to [14, 15, and 16].

3. Theorems and examples. A basic result enhancing the utility of the results is the following.

Theorem 3.1. Every locally σ-scattered space is σ-scattered.

An analogue of this result is apparently known for scattered spaces. Because of 3.1 we can use the word "locally" in the hypotheses of the theorems which follow.

Theorem 3.2. Suppose X is a locally scattered T_1 space.

(a) If X is first countable, then X has a λ-base [13].

(b) If X is of point-countable type, then X is monotonically Čech complete. If X is also regular, then X is a λ_b-space.

(c) If X is a q-space, then X is countably monotonically Čech complete.

Theorem 3.3. Suppose X is a locally σ-(closed-and-scattered) T_1 space.

(a) If points are G_δ's in X, then X has diagonal a set of interior condensation [14].

(b) If X is first countable, then X has a base of countable order.

(c) If X is regular and of point-countable type, then X is a β_b-space.

(d) If X is a q-space, then X is monotonically quasi-complete. If X is also regular, then X is a β_c-space.

Theorem 3.4. Suppose X is a locally σ-scattered space.

(a) If points are G_δ's in X, then X has a primitive diagonal.

(b) If X is first countable, then X has a primitive base.

(c) If X is of point-countable type, then X has a primitive p-structure.

(d) If X is a q-space, then X is primitively quasi-complete.

That the classes of scattered, σ-scattered, σ-(closed-and-scattered) first countable T_1 spaces are distinct is shown by the following examples.

Example 3.5. The space Q of rationals with the usual topology is a σ-(closed-and-scattered) space which is not scattered. It does not have a λ-base since it does not have the Baire property.

Example 3.6. The so-called Michael line [9] in which the underlying set is the reals with the topology generated by the usual topology and all subsets of the irrationals is an example of a σ-scattered firstcountable T_2 paracompact space which is not σ-(closed-and-scattered).This follows from the fact that it cannot have a base of countable order since it is T_2 paracompact but not metrizable.

Remark 3.7. That no stronger property such as being developable or even quasi-developable is implied by being scattered, normal, and first countable is shown by the example of ω_1 with the order topology.

4. Applications. We apply the theorems of Section 3 and some known theorems to obtain the following applications.

Theorem 4.1. Let X be T_1 locally σ-scattered q-space.

(a) If points are G_δ's and X is hereditarily weakly θ-refinable [4], then X is quasi-developable.

(b) If closed sets are G_δ's, and X is weakly θ-refinable, then X is developable.

(c) If X is collectionwise normal, closed sets are G_δ's, and X is weakly θ-refinable, then X is metrizable.

Theorem 4.2. Suppose X is a locally σ-scattered space such that points are G_δ's. Then X has a base of countable order if and only if X is monotonically a β-space in the sense of [5].

Theorem 4.3. Let X be a locally σ-(closed-and-scattered) T_1 q-space. Then if X is θ-refinable it is a p-space.

Theorem 4.4. Let X be a scattered Tychonoff space of point-countable type. If X is θ-refinably embedded [16] in X, then X is Čech complete.

Theorem 4.5. Suppose X is a locally countable T_1 q-space. Then X has a base of countable order.

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