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THE FIXED POINT PROPERTY OF SUPEREXTENSIONS

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Recent results have shown that superextensions are surprisingly nice spaces (e.g. VAN MILL [6], VERBEEK [8]). As another example of this nice behaviour we now give (an outline of) a proof that the superextension of a connected normal T_1 space has the fixed point property.

We assume throughout that X is *a connected normal* T_1 *space*.

1. CONVEX SETS AND PARTIAL ORDERINGS ON $\lambda(X)$.

$\lambda(X)$ consits of all *maximal linked systems* of closed subsets in X. If $A \subset X$ is closed, then

$$A^+ = \{M \in \lambda(X) | A \in M\}.$$

The collection of all A^+, with $A \subset X$ closed, is a closed subbase for the $\lambda(X)$-topology. A nonempty set $C \subset \lambda(X)$ is called *convex* if it equals an intersection of subbasic closed sets. We let $K(\lambda(X))$ denote the subspace of $H(\lambda(X))$ (the hyperspace of $\lambda(X)$), consisting of all convex sets in $\lambda(X)$.

Example. If $M,N \in \lambda(X)$, then the convex set

$$I(M,N) = \cap\{P^+ | P \in M \cap N\}$$

is called the *interval joining* M *and* N. (BROUWER and SCHRIJVER [2])

Theorem. (i) the interval map $I : \lambda(X) \times \lambda(X) \to H(\lambda(X))$ is continuous;

(ii) for each $M \in \lambda(X)$ there is a dense topological partial order on $\lambda(X)$, $\leq_M$, such that for each $N \in \lambda(X)$, $I(M,N)$ equals the set of all $\leq_M$-predecessors of N;

(iii) A closed set $C \subset \lambda(X)$ is convex iff for each $M \in \lambda(X)$ there exists a $\leq_M$-smallest element in C, denoted by $p(M,C)$. The resulting map

$p: \lambda(X) \times K(\lambda(X)) \to \lambda(X)$

is continuous.

This map p will be called *the nearest point map* of $\lambda(X)$. As a consequence of the above theorem we have the following

Corollary. The subspace $K(\lambda(X))$ of $H(\lambda(X))$ is compact, and it is densely ordered by inclusion.

2. PSEUDO-CONTRACTIONS OF $\lambda(X)$

Let $J \subset K(\lambda(X))$ be a maximal linearly ordered space w.r.t. inclusion. Then J is compact and, as $K(\lambda(X))$ is densely ordered, J also connected. The restriction

$$p^*: \lambda(X) \times J \to \lambda(X)$$

of the nearest point map p is a "pseudo-contraction" of $\lambda(X)$. In fact, J contains some singleton $\{M_o\} \subset \lambda(X)$ as well as $\lambda(X)$ itself by maximality. By the very definition of the map p, $p^*(-,\{M_o\}$ = contant map onto M_o, and $p^*(-,\lambda(X))$ = identity

Theorem. $\lambda(X)$ is acyclic.

3. RETRACTING $\lambda(X)$ ONTO BASICAL NEIGHBOURHOODS.

If $C \subset \lambda(X)$ is convex, then the restriction

$$p(-,C) : \lambda(X) \to \lambda(X)$$

of the nearest point map is easily seen to be a retraction of $\lambda(X)$ onto C. Using the normality of X, it can be proved that $\lambda(X)$ is *locally convex* in the sense that each point of $\lambda(X)$ has a neighbourhood base consisting of convex sets.

Combining this with the above acyclicity result yields the following.

Theorem. $\lambda(X)$ is an lc-space (BEGLE [1]).

In particular, $\lambda(X)$ is an acyclic Lefschetz space (terminology of BROWDER [3]) and it consequently has the fixed point property.

4. CONCLUDING REMARKS

(i) VAN MILL has proved that $\lambda(X)$ is an AR (compact metric) if X is a metric continuum (see [6]). This result can also be obtained through the above techniques: X being compact and metric, it follows that $J \subset K(\lambda(X)) \subset H(\lambda(X))$ are metrizable. Hence J is homeomorphic to the unit interval, and the nearest point map p yields and ordinary contraction $\lambda(X) \times J \to \lambda(X)$. Retracting $\lambda(X)$ onto basical neighbourhoods, and applying partial realisation techniques as in DUGUNDJI [4] then shows that $\lambda(X)$ is an AR.

(ii) If X is a T_2-continuum with a normal binary subbase (definitions to be found in VERBEEK [8]), then X is a retract of $\lambda(X)$ (VAN MILL [5]). Consequently, X has the fixed point property.

(iii) In a forthcoming paper, [7], of VAN MILL and the author, a general notion of *convexity relative to closed subbases* has been introduced. Among other things, a technique has been developed, proving that the hyperspace of convex sets is compact, not only in $\lambda(X)$ with its canonical subbase, but also in spaces which carry a normal binary subbase. A nearest point map can also be constructed on such spaces, and the above presented ideas can now be applied directly on spaces with a normal binary subbase.

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