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ON SOME CONVERGENCE CLOSURES GENERATED BY FUNCTIONS

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0. In this note we shall consider several convergences defined on a given closure space and investigate relations between corresponding convergence closures and their modifications.

Let L be a set. Let $\mathfrak{L}$ be a set of pairs $(\{x_n\}, x)$ where $x_n \in L$, $n \in N$, and $x \in L$ satisfying the following axioms

($\mathscr{L}_0$) If $(\{x_n\}, x) \in \mathfrak{L}$, $(\{x_n\}, y) \in \mathfrak{L}$ then $x = y$.

($\mathscr{L}_1$) If $x_n = x$, $n \in N$, then $(\{x_n\}, x) \in \mathfrak{L}$.

($\mathscr{L}_2$) If $(\{x_n\}, x) \in \mathfrak{L}$ and $\{n_i\}$ is any subsequence of $\{n\}$, then $(\{x_{n_i}\}, x) \in \mathfrak{L}$.

Then $\mathfrak{L}$ is called a *convergence* on L. For each $A \subset L$ let $\lambda A = \{x \mid x \in L, \exists \{x_n\}, x_n \in A, n \in N, \ni (\{x_n\}, x) \in \mathfrak{L}\}$. Then (L, λ) is a T_1-closure space. It is denoted by $(L, \mathfrak{L}, \lambda)$ and called a *convergence space* [4]. Note that in general $\lambda^2 A \neq \lambda A$ and hence a convergence space may not be a topological space. To each convergence $\mathfrak{L}$ there corresponds convergence $\mathfrak{L}^*$ inducing the same convergence closure and satisfying the Urysohn axiom

($\mathscr{L}_3$) If each subsequence $\{x_{n_i}\}$ of a sequence $\{x_n\}$ contains a subsequence $\{x_{n_{i_j}}\}$ converging to a point x, then the sequence $\{x_n\}$ itself converges to x.

Let $(L, \mathfrak{L}, \lambda)$ be a convergence space. The finest topology coarser than λ is called a *topological modification of* λ and denoted by λ^{ω_1}. Recall that a convergence space $(L, \mathfrak{L}, \lambda)$ is called a *Fréchet space* if $\lambda^{\omega_1} = \lambda$, i.e., if (L, λ) is a topological space. A topological space (P, u) is called a *sequential space* if there exists a convergence closure μ for P such that $\mu^{\omega_1} = u$, i.e., if u is a topological modification of some convergence closure. If (P, u) is a closure space we shall denote by $C(u)$ [1]) the set of all continuous real-valued functions on (P, u).

A convergence space $(L, \mathfrak{L}, \lambda)$ is called *sequentially regular* [4] if for each point $x \in L$ and each sequence $\{x_n\}$ of points of L such that $x \in L - \lambda \bigcup_{n=1}^{\infty} (x_n)$ there is a function $f \in C(\lambda)$ such that $\{f(x_n)\}$ does not converge to $f(x)$.

[1]) We write simply $C(u)$ instead of $C((P, u))$ because we shall consider different closures for the same given set P.

Let $(L, \mathfrak{L}, \lambda)$ be a convergence space such that $C(\lambda)$ separates the points of L. The finest sequentially regular convergence closure for L coarser than λ is called a *sequentially regular modification of* λ and denoted by $\hat{\lambda}$ [3]. The finest completely regular topology for L coarser than λ is called a *completely regular modification of* λ and denoted by $\tilde{\lambda}$ [3]. The following relations hold

$$\lambda < \hat{\lambda} < \hat{\lambda}^{\omega_1} < \tilde{\lambda}\,,$$
$$\lambda < \lambda^{\omega_1} < \tilde{\lambda}\,,$$
$$\tilde{\hat{\lambda}} = \tilde{\lambda}\,.$$

1. Let (P, u) be a closure space and suppose that $C(u)$ separates the points of P. Clearly (P, u) is a separated space.

Consider the following convergences on P:

$\mathfrak{P}$: $(\{x_n\}, x) \in \mathfrak{P}$ if for each neighborhood U of x we have $x_n \in U$ for nearly all $n \in N$,

$\mathfrak{P}_{C(u)}$: $(\{x_n\}, x) \in \mathfrak{P}_{C(u)}$ if $\{f(x_n)\}$ converges to $f(x)$ for each $f \in C(u)$.

Denote π and $\pi_{C(u)}$ the corresponding convergence closures. $\mathfrak{P}$ is the usual convergence on P. The convergence space $(P, \mathfrak{P}, \pi)$ was called a *convergence space associated with* (P, u) in [3] and π is called a *sequential modification of* u in [1]. The convergence $\mathfrak{P}_{C(u)}$ was introduced by J. Novák in [5] who pointed out that there are interesting relations between the closures u, π, $\pi_{C(u)}$ and their modifications. Let us define still another convergence

$\mathfrak{P}_{C(\pi)}$: $(\{x_n\}, x) \in \mathfrak{P}_{C(\pi)}$ if $\{f(x_n)\}$ converges to $f(x)$ for each $f \in C(\pi)$.

Denote $\pi_{C(\pi)}$ the corresponding convergence closure. We shall show in Example 1 that generally $C(\pi) \neq C(u)$.

Lemma 1. $\pi < \pi_{C(\pi)} < \pi_{C(u)}$.

Proof. Let $(\{x_n\}, x) \in \mathfrak{P}$. By definition of $C(\pi)$ and $\mathfrak{P}_{C(\pi)}$ we have $(\{x_n\}, x) \in \mathfrak{P}_{C(\pi)}$. Since $\pi < u$ we have $C(u) \subset C(\pi)$ and hence $(\{x_n\}, x) \in \mathfrak{P}_{C(u)}$.

Proposition 1. *The convergence spaces* $(P, \mathfrak{P}_{C(\pi)}, \pi_{C(\pi)})$ *and* $(P, \mathfrak{P}_{C(u)}, \pi_{C(u)})$ *are sequentially regular.*

Proof. The assertion follows immediately from the definition of sequential regularity and from definitions of $\mathfrak{P}_{C(\pi)}$ and $\mathfrak{P}_{C(u)}$.

In view of Corollary 3 in [3] we have

Corollary 1. *The following are equivalent*

(a) $\pi_{C(\pi)} = \pi_{C(u)}$.
(b) $\pi^{\omega_1}_{C(\pi)} = \pi^{\omega_1}_{C(u)}$.
(c) $\tilde{\pi}_{C(\pi)} = \tilde{\pi}_{C(u)}$.

The question arises whether $\pi_{C(\pi)} = \pi_{C(u)}$ does not always hold. The following example shows that we may have $\pi_{C(\pi)} \neq \pi_{C(u)}$.

Example 1. Let $P = [0, 1]$. For $x \neq 0$ let $U_n = P \cap (x - 1/n, x + 1/n)$ be the usual local base at x and let the local base at 0 consist of sets $U_{n,S} = \{0\} \cup \cup ((0, 1/n) - S)$ where $n \in N$ and $|S| \leqq \aleph_0$. Denote u the corresponding topology. Clearly $(P, \mathfrak{P}_{C(u)}, \pi_{C(u)})$ is the interval $[0, 1]$ with the usual topology. On the other hand 0 is π-isolated and therefore also $\pi_{C(\pi)}$-isolated. Hence $\pi_{C(\pi)} \neq \pi_{C(u)}$. Note that both spaces $(P, \mathfrak{P}_{C(\pi)}, \pi_{C(\pi)})$ and $(P, \mathfrak{P}_{C(u)}, \pi_{C(u)})$ are Fréchet spaces.

Proposition 2. *If $C(\pi) = C(u)$ then $\pi_{C(\pi)} = \pi_{C(u)}$.*

Proof. $C(\pi) = C(u)$ implies that $\mathfrak{P}_{C(\pi)} = \mathfrak{P}_{C(u)}$.

Corollary 2. *If (P, u) is a convergence or a sequential space then $\pi_{C(\pi)} = \pi_{C(u)}$.*

The condition in Proposition 2 is not necessary as the following example shows.

Example 2. Let $P = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} (x_{mn}) \cup (x_0)$. The points x_{mn} are isolated and the local base at x_0 consists of sets $U_{k,r} = \bigcup_{m=k}^{\infty} \bigcup_{n=r(m)}^{\infty} (x_{mn}) \cup (x_0)$ where $k \in N$ and r is any mapping of N into itself. Denote u the corresponding topology. The space (P, u) is normal. The spaces $(P, \mathfrak{P}, \pi)$, $(P, \mathfrak{P}_{C(\pi)}, \pi_{C(\pi)})$, and $(P, \mathfrak{P}_{C(u)}, \pi_{C(u)})$ are all discrete and hence $\pi_{C(\pi)} = \pi_{C(u)}$ while $C(\pi) \neq C(u)$.

Proposition 3. *If $(\{x_n\}, x) \in \mathfrak{P}$ whenever $\{f(x_n)\}$ converges to $f(x)$ for each $f \in C(u)$ then $\pi_{C(\pi)} = \pi_{C(u)}$.*

Proof. The condition implies $\mathfrak{P}_{C(u)} \subset \mathfrak{P}$ and the assertion follows by Lemma 1.

Corollary 3. *If (P, u) is completely regular then $\pi_{C(\pi)} = \pi_{C(u)}$.*

Again the condition in Proposition 3 is not necessary as the following example shows.

Example 3. Let $P = [0, 1]$. For $x \neq 0$ let $U_n = P \cap (x - 1/n, x + 1/n)$ be the usual local base at x and let the local base at 0 consist of sets $V_n = [0, 1/n) - \bigcup_{m=1}^{\infty} (1/m)$. Denote u the corresponding topology. We have $(\{1/m\}, 0) \notin \mathfrak{P}$ while $\{f(1/m)\}$ converges to $f(0)$ for each $f \in C(u)$. However, (P, u) is a Fréchet space and hence $\pi_{C(\pi)} = \pi_{C(u)}$ by Corollary 2.

Problem 1. What is the necessary and sufficient condition for the equality $\pi_{C(\pi)} = \pi_{C(u)}$?

2. Now let us characterize the convergence closures $\pi_{C(\pi)}$ and $\pi_{C(u)}$.

Theorem 1. $\pi_{C(\pi)} = \hat{\pi}$.

Proof. By Lemma 1 and Proposition 1 $\pi_{C(\pi)}$ is a sequentially regular convergence closure coarser than π. On the other hand let λ be a sequentially regular convergence closure for P coarser than π. To complete the proof we must show that $\pi_{C(\pi)} < \lambda$. Let $A \subset P$ and $x \in \pi_{C(\pi)}A$. Then there is a sequence $\{x_n\}$ of points of A which $\mathfrak{P}_{C(\pi)}$-converges to x. Hence $\{f(x_n)\}$ converges to $f(x)$ for each $f \in C(\pi)$. Since $\pi < \lambda$ it follows that $\{g(x_n)\}$ converges to $g(x)$ for each $g \in C(\lambda)$. Because λ is sequentially regular $\{x_n\}$ $\mathfrak{L}^*$-converges to x by Lemma 2 in [3]. Therefore $x \in \lambda A$.

Let (P, u) be a closure space and let w be the weak topology for P [2]. We shall denote by $(P, \mathfrak{P}_w, \pi_w)$ the convergence space associated with (P, w), i.e., π_w is the sequential modification of w.

Theorem 2. $\pi_{C(u)} = \pi_w$.

Proof. If $(\{x_n\}, x) \in \mathfrak{P}_{C(u)}$ then $\{f(x_n)\}$ converges to $f(x)$ for each $f \in C(u)$ and hence for each w-neighborhood U of x we have $x_n \in U$ for nearly all $n \in N$. Therefore $(\{x_n\}, x) \in \mathfrak{P}_w$. On the other hand if $(\{y_n\}, y) \in \mathfrak{P}_w$ then clearly $\{f(y_n)\}$ converges to $f(y)$ for each $f \in C(u)$ so that $(\{y_n\}, y) \in \mathfrak{P}_{C(u)}$.

Since (P, w) is a completely regular space it follows that if v is any of the closures u, π, $\pi_{C(\pi)}$, $\pi_{C(u)}$ or their modifications then we have $\pi < v < w$.

3. Finally let us consider the relations between the convergence closures π, $\pi_{C(\pi)}$, $\pi_{C(u)}$ and the closure u.

If (P, u) is just a closure space then the only statement we can make is the obvious $\pi < u$.

If (P, u) is a topological space then clearly $\pi < \pi^{\omega_1} < u$. However, both $\pi_{C(u)} \lneqq u$ (Example 2) and $u \lneqq \pi_{C(u)}$ (Example 3) may occur. The same holds for $\pi_{C(\pi)}$.

Finally, if (P, u) is a completely regular space, then $\tilde{\pi}_{C(u)} < u$. It follows from Lemma 6 and Theorem 6 in [3] that $\tilde{\pi}_{C(u)} = u$ if and only if $C(\pi) = C(u)$.

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