TOPOLOGICAL EMBEDDINGS IN EUCLIDEAN SPACE

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Here we want to give an account of the work on geometric topology which has been carried out in our seminar at the Steklov Institute (Moskva). It can be divided into three parts:

– structure of the homeomorphism group of R^n and locally flat embeddings of manifolds in R^n.

– topological embeddings of manifolds, polyhedra and compacta in R^n.

– monotone mappings of manifolds.

1. Homeomorphisms of R^n and locally flat embeddings

Let us recall that a homeomorphism h of R^n is said to be *stable* if it is a finite product of homeomorphisms, each of which leaves fixed all points of an open set. At present a most important problem in geometric topology is to decide whether every orientation-preserving homeomorphism of R^n is stable. There are many reasons for this. The only known orientation-preserving homeomorphisms are stable. The stable homeomorphisms enjoy many properties one might desire: they can be approximated with piecewise linear ones (for $n \neq 4$), they can be connected with the identity with a continuous one-parameter family of homeomorphisms g_t ($g_0 = 1$, g_1 is a given homeomorphism), i.e., an isotopy, etc. Also, the positive answer to this problem will imply many important consequences. This problem being unsettled as yet for $n > 3$, let us weaken the definition.

We call a homeomorphism *k-stable* if it can be decomposed into a product $h = h_s \dots h_1$ of homeomorphisms each of which has at least a linear k-simplex of fixed points. This is the same as saying that h is a product : $h = h_k h'$ of a stable homeomorphism h' and of one homeomorphism h_k with a k-simplex of fixed points. The strongest result obtained up to the present is the following:

Theorem 1.1 (Černavskiĭ [20]). *Each homeomorphism of R^n is $(n-3)$-stable.*

This result has applications to the embedding problem. We shall recall that a cell Q embedded in R^n is said to be *trivial* if there exists a homeomorphism h of R^n onto itself such that hQ is a standard simplex. One says that a manifold M in R^n is *locally flat* if each point in M has a neighbourhood in M which is a cell trivially embedded in R^n. It has been shown by B. Mazur [14] and M. Brown [2] (for the case $k = n - 1$) and by J. Stallings [16] (for the case $k \leqq n - 3$) that locally flat k-spheres in R^n are unknotted if $k \neq n - 2$, i.e., such a sphere can be transformed into a standard one by an ambient homeomorphism. Now we have the following

Corollary. *If $k \leqq n - 3$ and S is a locally flat k-sphere in R^n, then there exists an ambient isotopy of R^n which transforms S into a standard sphere* [20].

The proof of theorem 1.1 is based on two recent results. The first one, a result obtained by T. Homma [3], asserts that *topological embeddings of k-cells in R^n can be approximated by piecewise linear (simplicial) ones if $k \leqq n - 3$.*

The second result is the theorem on the union of locally flat cells:

Theorem 1.2 (Černavskiĭ [19]). *If two locally flat cells in R^n, B_1^k and B_2^k, have only their boundaries in common and if $k \neq n - 2$, $n \geqq 5$, then their union $B_1^k \cup B_2^k$ is a locally flat sphere.*

As a matter of fact both results have been proved in a much more general form, but we pass it here.

Up to the present we have been discussing embeddings in codimensions more than two ($n - k > 2$). The codimension two can be called a "knotting" codimension: because by the results of Stallings, Mazur and Brown it is the unique codimension where knotting is possible. (Of course, in the above we mean locally flat spheres.) Here we can state a result of Sosinskiĭ. He has defined the connected sum of locally flat knots and has proved the

Theorem 1.3 (Sosinskiĭ [15]). *Each knot can be decomposed into a finite sum of indecomposable summands.*

The definition is based on the fact that for each locally flat knot there exists an equivalent knot which contains a linear $(n - 2)$-simplex. Then the definition may be given as usual by putting together the knots along a common linear simplex.

Defining, further, the notion of infinite connected sum which can be clarified by an illustration (page 219); he has proved that this sum (always equal to a sphere locally flat at all points except possibly the limit point) is also locally flat at the limit point if and only if it contains only a finite number of knotted summands.

This also gives first examples of isolated wild points of embeddings in codimension 2, $n > 3$, that is of points where the embedding is not only non-flat, but also is not equivalent to a polyhedral embedding. With this construction he has obtained (see [15]) some examples of *a union of two flat cells in codimension 2, $n > 3$, with only their boundaries in common, which is a sphere with a unique (or with many if you wish) wild points.*

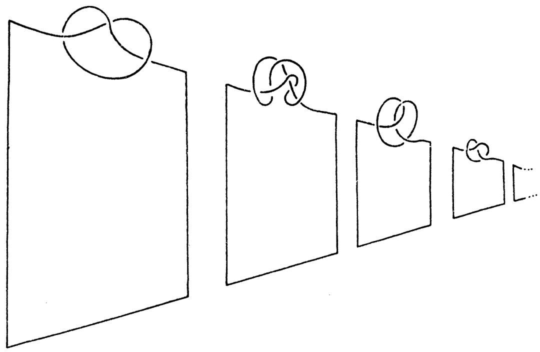
We should like to note here the following

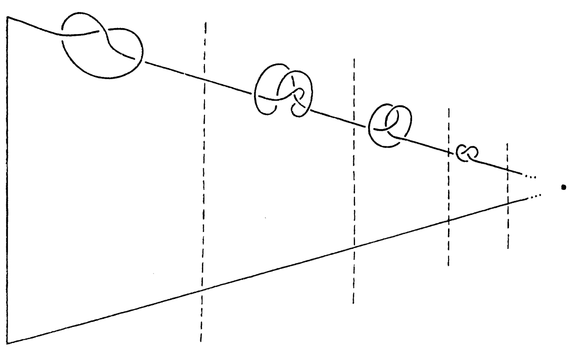
Corollary to Theorem 1.2 (see [19]). *No isolated wild point can occur in codimension distinct from* 2.

By similar methods Černavskiĭ has proved some propositions about embeddings of polyhedra. Here one encounters additional difficulties and the results have been obtained only for $k < \frac{2}{3}n - 1$.

Theorem 1.4 (Černavskiĭ [18], [23]). *If P is a k-polyhedron topologically embedded in R^n, where $k < \frac{2}{3}n - 1$, and if for a triangulation of P all open simplexes are locally flat in R^n, then there exists a homeomorphism of R^n, h, such that hP is a rectilinear polyhedron in R^n.*

One can choose this homeomorphism to be an ε-homeomorphism for any given $\varepsilon > 0$.





2. Embeddings in R^3

If the embedding of a surface M^2 is "wild" (not locally flat), then there exists no homeomorphism of R^3 onto itself which sends M^2 onto a polyhedral surface. Let us recall now that a *pseudoisotopy* of a space X is a one-parameter continuous collection of maps of X onto itself, g_t, where g_t is a homeomorphism for all $t < 1$. As usual, g_0 is the identity. For metric space X g_t is called an ε-pseudoisotopy if $d(x, g_t(x)) < \varepsilon$ for every $x \in X$ and for all $t \in [0, 1]$. Now we can state the following

Theorem 2.1 (L. Keldyš [6], [7]). *For each surface M^2 (with or without boundary) embedded in R^3 (in an arbitrary way) and for any $\varepsilon > 0$ there exists a polyhedral surface $M_1^2 \subset R^3$ and an ε-pseudoisotopy g_t of R^3 such that*

1) $g_1 M_1^2 = M^2$;

2) *the set of all points of R^3 which have nondegenerate inverse images of points, $g_1^{-1}x$, is a zero-dimensional subset of M^2 contained in the set of wild points of M^2.*

If M^2 is a topological 2-sphere, then this result implies (with the aid of Alexander's theorem) the existence of an ambient pseudoisotopy which sends the standard sphere onto M^2, but, of course, it is not an ε-pseudoisotopy.

For spheres there are related results obtained by the American school (Bing [1], Linninger [13] and others), but the results obtained deal with domains bounded by a sphere (crumpled cubes) and not with pseudoisotopies of the ambient space. L. Keldyš has obtained the same result (but without the condition concerning the dimension of the set of nondegenerate inverses) for topological k-polyhedra in R^n for all n in the trivial range ($k \leqq n/2 - 1$) [6]. Previously, L. Keldyš [5] had proved a similar result for the embedding of 0-dimensional compacta in R^n.

As regards the embedding problem in three-dimensional space it may be said that the situation is fairly clear now after the results of Bing, Moise and others, but the methods of proof remain as yet very complicated. It is our hope that there are ways of simplifying constructions and clarifying the basic ideas.

Now we come to results which deal with embedding of compacta in R^n. Let us say that a compactum in R^n is *cellularly divided* if it is an intersection of a decreasing sequence of disjoint finite unions of cells:

$$K = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{n_i} Q_{i,j}\,, \quad \text{each } Q_{i+1j} \text{ belongs to a } Q_{i,j'}\,;$$

$$Q_{ij} \cap Q_{ij'} = \emptyset\,, \quad i \neq j\,.$$

For example a zero-dimensional compactum satisfies this condition if it is tame and a continuum does if it is cellular in the sense of M. Brown [2].

Theorem 2.2 (Štan'ko [21]). *Among all one-dimensional compacta, those which can be embedded in R^3 in a cellularly divided manner are exactly tree-like compacta* (i.e., compacta which are projective limits of finite sums of trees).

Theorem 2.3 (Štan'ko [21]). *For each tree-like compactum K in R^3 there exists a homeomorphic ε-push h of K into R^3 such that hK is cellularly divided in R^3.*

Štan'ko has found a criterion for a compactum in R^3 to be cellularly divided and we state here the following

Corollary (Štan'ko). *In R^3 each subcontinuum of a cellular one-dimensional continuum is cellular itself.*

3. Monotone mappings

We recall that a mapping of a manifold is *point-like* if the inverse images of points are cellular. A mapping of a manifold onto a manifold may be called *cellular* if the inverse images of cellular sets are cellular.

There are some problems concerning point-like mappings. It is well known that (if $n \geqq 3$) the image of a manifold under a point-like mapping need not be a manifold. Suppose we have a point-like mapping f of a manifold M onto a manifold N.

1. Is N homeomorphic to M? (Bing)
2. If N is homeomorphic to M, is f pseudoisotopic to a homeomorphism? Or, in another form, can f be approximated by a homeomorphism? (The inverse statement is correct, and if the answer is positive then we have a nice characterization of mappings which can be approximated by homeomorphisms.)
3. Is a point-like mapping of a manifold onto a manifold cellular?

In this direction some results have been obtained by V. Kompaniec. In a joint paper by Černavskiĭ and Kompaniec point-like mappings of a sphere onto itself have been considered, and it was proved:

Theorem 3.1 (Kompaniec and Černavskiĭ [9]). *A point-like map of a sphere onto itself is cellular* ($n \neq 4$).

Later, Kompaniec has obtained a general criterion for cellular mappings of piecewise linear manifolds. It is of a homotopic character:

Theorem 3.2 (Kompaniec [11]). *If $f : M \to N$ is a point-like mapping, then for any open set in N f induces an isomorphism between its homotopic groups and those of its preimage. If $f : M \to N$ is a mapping and if for any open set U in N its preimage $f^{-1}U$ has the same homotopic groups as U, then the mapping f is cellular.*

The following corollaries are worth mentioning:

Cellular mappings induce homotopic equivalence.

If $n \neq 4$, then point-like mappings are cellular.

If $n \geqq 5$ and one of the two manifolds is R^n or S^n, then so is the other.

Previously Kompaniec has proved that a mapping of a sphere onto itself is point-like if there are only countably many nondegenerate inverse images of points. This generalizes the results of M. Brown [2], Kwun [12].

Now we recall L. Keldyš's example of a monotone mapping f of a 3-cube I^3 onto any cube of a higher dimension k [4]. Later L. Keldyš observed that the mapping she had constructed was the limit of a sequence of embeddings of I^3 into I^k of a special kind.

An embedding h of I^p in I^q is called ε-dense if the ε-neighbourhood of $h(I^p)$ coincides with I^q. It turns out that for any $\varepsilon > 0$ I^3 may be embedded in I^k, $k > 3$, ε-densely and in such a way that each pair of points in $h(I^3)$ whose distance is less than ε can be connected by an arc in $h(I^3)$ of diameter less than 2ε. The mapping f is the limit of such embeddings.

For a square such embeddings cannot exist for small ε, and this is the reason why I^2 cannot be mapped onto I^k, $k > 2$, in a monotone way.

This construction has been generalized by Černavskiĭ who proved that I^p can be mapped onto any cube of higher dimension with preimages of points acyclic up to the dimension r, where $2r \leqq p - 3$. This result is precise because of a theorem of Frum-Ketkov [17]. These mappings can also be obtained as limits of special embeddings. They may be changed into open mappings by general method of L. Keldyš [8].

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