UNIVERSAL SPACES

R. M. SCHORI

Baton Rouge

Let $\mathscr{P}$ be a class of topological spaces. We say that the space U is a *universal* member of $\mathscr{P}$ if U is in $\mathscr{P}$ and each member of $\mathscr{P}$ is homeomorphic to a subspace of U.

There are some well-known examples in topology of classes $\mathscr{P}$ and corresponding universal members. For example, Menger [3] has given a space which is universal for the class of 1-dimensional compact metric spaces; and the Hilbert cube is a universal member of the class of separable metric spaces.

We shall give a summary of some results dealing with the existence of classes which contain a universal member.

Definition 1. If $\boldsymbol{v}$ is a class of compact metric spaces, then the metric space X is

i) *$\boldsymbol{v}$-expandable* iff X is the limit of an inverse sequence of members of $\boldsymbol{v}$ with bonding maps onto.

ii) *$\boldsymbol{v}$-like* iff for each $\varepsilon > 0$ there exists an ε-map from X onto some member of $\boldsymbol{v}$.

iii) *weakly $\boldsymbol{v}$-like* iff for each $\varepsilon > 0$ there exists an open cover of X with mesh less than ε whose nerve is homeomorphic to some member of $\boldsymbol{v}$.

These notions are closely related and are equivalent in the case when the members of $\boldsymbol{v}$ are connected polyhedra. See McCord [2], Mardešić and Segal [1], and Pasynkov [4] for details and further implications. Furthermore, let $\boldsymbol{v}^*$ be the set of all spaces X such that X is $\boldsymbol{v}$-expandable. Then the following is a list of some well-known examples of classes $\boldsymbol{v}$ and the corresponding classes $\boldsymbol{v}^*$.

1) If $\boldsymbol{v} = \{\text{arc}\}$, then $\boldsymbol{v}^*$ is the class of all chainable (snake-like or arc-like) continua.

2) If $\boldsymbol{v} = \{n\text{-cell}\}$, then $\boldsymbol{v}^*$ is the class of all n-cell-like continua.

3) If $\boldsymbol{v} = \{\text{all trees}\}$, then $\boldsymbol{v}^*$ is the class of all tree-like continua.

4) If $\boldsymbol{v} = \{\text{all [connected] polyhedra (of dimension} < n)\}$, then $\boldsymbol{v}^*$ is the class of all [connected] compacta (of dimension $< n$).

5) If $\boldsymbol{v} = \{\text{circle}\}$, then $\boldsymbol{v}^*$ is the class of all circle-like continua.

In [5] Schori gave the first verification that *the class of all chainable continua has a universal member*. The construction leaned heavily on the inverse limit characterization of chainable continua. In [2] McCord, and in [4] Pasynkov independently introduced appropriate definitions and generalized [5] to obtain some much stronger results.

McCord introduced the notion of an amalgamable class of polyhedra.

Definition 2. The class $\mathbf{v}$ of polyhedra is *amalgamable* if for each finite sequence $(P_1, \ldots, P_n, Q)$ of members of $\mathbf{v}$ and maps $\varphi_i : P_i \to Q$ $(1 \leqq i \leqq n)$, there exists a member P of $\mathbf{v}$ with imbeddings $f_i : P_i \to P$ and a map φ of P onto Q such that for each i, $\varphi \, . f_i = \varphi_i$.

Theorem (McCord). *If $\mathbf{v}$ is an amalgamable class of polyhedra, then $\mathbf{v}^*$ contains a universal member.*

Much of the definition of amalgamable class is of a technical nature, but in practice it is fairly routine to check this condition for many classes of polyhedra. McCord verifies that the classes $\mathbf{v}$ in (1), (2), (3), and (4) above (plus others) are all amalgamable.

Pasynkov [4] introduces the apparently cleaner notion of a finitely additive class of metric compacta.

Definition 3. A system $\mathbf{v}$ of compact metric spaces is *finitely additive* if $A, B \in \mathbf{v}$ implies there exists $C \in \mathbf{v}$ such that C contains disjoint homeomorphic copies of A and B.

Theorem (Pasynkov). *If $\mathbf{v}$ is a countable, finitely additive system of compact metric spaces, then $\mathbf{v}^*$ has a universal member.*

It is immediate to check that the classes $\mathbf{v}$ of (1), (2), (3), and (4) satisfy finite additivity. However, the exact relationship between amalgamable classes and finitely additive classes is not yet clear.

Thus far we have been concerned only with sufficient conditions for a class of spaces to have a universal member. Necessary conditions seem less accessable. In [2] McCord includes some negative results. One of them is the following.

Theorem (McCord). *If $\mathbf{v}$ is a class of closed, connected, triangulable n-manifolds, then $\mathbf{v}^*$ has no universal member.*

Thus, for example, the class $\mathbf{v}^*$ of circle-like continua of (5) has no universal member.

Problem 1. Find necessary conditions on a class $\mathbf{v}$ of polyhedra such that $\mathbf{v}^*$ contains a universal member.

Problem 2. Generalize the existing techniques to larger classes of spaces (e.g., non-metric spaces) to obtain universal members. Presumably with non-metric spaces one would have to consider inverse systems as opposed to inverse sequences.

Problem 3. Investigate and compare the notions of amalgamable class and finitely additive class.

Conjecture. If T is any non-degenerate tree other than an arc and $\boldsymbol{v} = \{T\}$, then $\boldsymbol{v}^*$ has no universal member. Generalize.

References

[1] *S. Mardešić and J. Segal:* ε-mappings onto polyhedra. Trans. Amer. Math. Soc. *109* (1963), 146—164.

[2] *M. C. McCord:* Universal $\mathscr{P}$-like compacta. Mich. Math. Jour. *13* (1966), 71—85.

[3] *K. Menger:* Über umfassendste n-dimensionale Mengen. Proc. Akad. Wetensch. Amsterdam *29* (1926), 1125—1128.

[4] *B. A. Pasynkov:* On universal compacta. (Russian.) Uspehi Math. Nauk. *21* (1966), 91—100.

[5] *R. M. Schori:* A universal snake-like continuum. Proc. Amer. Math. Soc. *16* (1965), 1313—1316.