Hisahiro Tamano

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NORMALITY AND PRODUCT SPACES

H. TAMANO

Kyoto

In 1951, C. H. Dowker [2] proved that the product $X \times Y$ is normal for any compact metric space Y if X is normal and countably paracompact and that the normality of $X \times I$, where I denotes the closed unit interval implies the countable paracompactness of X. His results may be stated as follows:

Theorem A. *X is normal and countably paracompact if and only if the product $X \times Y$ is normal for any compact metric space Y.*

In 1960, the author [12] proved that the normality of $X \times \beta X$, where βX denotes the Stone-Čech compactification of X, implies the paracompactness of X. On the other hand it is a well known theorem due to Dieudonné [1] that the product $X \times Y$ of a paracompact space with any compact space Y is paracompact. Thus, we have

Theorem B. *X is paracompact if and only if $X \times Y$ is normal for any compact space Y.*

Shortly after that, the author found that a number of topological properties of completely regular spaces can be characterized in terms of modified conditions of normality assumed on the product space $X \times Y$ for suitably chosen Y. The following are typical results:

Theorem 1. *X is second countable if and only if $X \times BX$ is perfectly normal for some compactification BX of X* ([13]; p. 181).

Theorem 2. *X is collectionwise normal if and only if every closed subset of the form $F \times \beta X$ is normally embedded*[1]) *in $X \times \beta X$* ([13]; p. 184).

It is worthwhile to note that the paracompactness of X is implied by the condition stated in the following theorem ([13]; p. 177).

Theorem 3. *X is paracompact if and only if $X \times C$ and Δ are functionally separated in $X \times BX$ for any compact set $C \subset BX \setminus X$, where BX is any compactification of X.*

Topologically complete spaces, realcompact spaces and Lindelöf spaces can be characterized in a similar fashion (cf. [13]). The following two theorems are concerned with countable compactness and pseudo-compactness [13]:

[1]) A subset E of X is said to be normally embedded if every bounded continuous function on E can be extended to a continuous function on X.

Theorem 4. *Let Y be any compact metric space containing infinitely many points. Then, X is countably compact if and only if two disjoint closed sets $X \times G$ and F in $X \times Y$ are separated by open sets of the form $X \times U$. In other words, the projection $Pr_Y : X \times Y \to Y$ is a closed mapping.*

Theorem 5. *Let Y be any compact metric space containing infinitely many points. Then X is pseudo-compact if and only if two disjoint closed sets $X \times G$ and $Z(F)$, where $Z(F)$ denotes the zero-set of a continuous function $F \in C(X \times Y)$, are separated by open sets of the form $X \times U$. That is, $Pr_Y[Z(F)]$ is closed for each $F \in C(X \times Y)$.*

In each of the above theorems, we consider the product $X \times Y$ of X with a compact space Y suitably specified. From this point of view, the topological properties thus characterized may be considered to form a particular class or species of properties of completely regular spaces. In Theorems A, 4 and 5, the property that Y is metric plays a certain role. The following question naturally arose under some expectation of obtaining another category of topological properties: "*Which space X has the property that $X \times Y$ is normal for any metric space Y?*" K. Morita ([7], [10]) has given a beautiful answer to this question.

Theorem C. *The product $X \times Y$ is normal for any metric space Y if and only if X is a normal P-space.*

Perfectly normal spaces and paracompact G_δ-spaces [3] are P-spaces in Morita's sense and every P-space is obviously countably paracompact. In the following, another exposition of the notion of P-space will be given with a view to making the meaning of Theorem C intuitively clear.

Let T be a partially ordered index set. We call T a *tree* if the subset $S(\alpha) = \{\lambda \in T \mid \lambda < \alpha\}$ forms a chain (well ordered set) for each $\alpha \in T$. If $S(\alpha)$ is finite for each α, then every chain in T is countable, and we call T a ω-chain in this case. We shall restrict ourselves to consider such a tree that every maximal chain is infinite, in the sequel. A family $\{G_\alpha \mid \alpha \in T\}$ will be called a tree if the index set T forms a tree and $G_\alpha \subset G_\beta$ whenever $\alpha < \beta$. If each G_α is open (closed), then the family will be called an open (closed) tree. A closed tree $\{G_\alpha \mid \alpha \in T\}$ is said to be complete if $\bigcup\{G_\alpha \mid \alpha \in S(\lambda)\}$ is closed for each $\lambda \in T$. If the subfamily $\{G_\alpha \mid \alpha \in K\}$ of a tree $\{G_\alpha \mid \alpha \in T\}$ forms an open (closed) covering of X for each maximal chain K in T, then we call $\{G_\alpha \mid \alpha \in T\}$ an open (closed) covering tree and each subfamily $\{G_\alpha \mid \alpha \in K\}$ will be called an open (closed) chain covering of X. Let $\mathscr{F} = \{F_\alpha \mid \alpha \in T\}$ and $\mathscr{G} = \{G_\alpha \mid \alpha \in T\}$ be two trees. We shall say that $\mathscr{G}$ dominates $\mathscr{F}$ and write $\mathscr{G} \succ \mathscr{F}$ if $\mathrm{Int}(G_\alpha) \succ \mathrm{Cl}(F_\alpha)$ for each α. Let $\mathscr{G}$ be an open covering tree. If there is a closed covering tree $\mathscr{F}$ such that $\mathscr{G} \succ \mathscr{F}$, then we shall say that $\mathscr{G}$ is shrinkable and $\mathscr{F}$ will be called a shrink of $\mathscr{G}$. (Roughly speaking, $\mathscr{G}$ is shrinkable if every open chain covering in $\mathscr{G}$ is simultaneously shrinkable.) If the shrink $\mathscr{F}$ of $\mathscr{G}$ consists of regular closed sets (closures of open sets), then $\mathscr{F}$ is said to be regularly shrinkable. Now, Morita's result (Theorem C) may be stated as follows:

Theorem 6. *The following conditions on a normal space X are equivalent.*

1) $X \times Y$ *is normal for any metric space* Y.
2) *Every open covering ω-tree on X is shrinkable.*

Analogously, Ishikawa's result [4] on countable paracompactness can be stated in terms of chain coverings (cf. Dowker [2]).

Theorem 7. *X is countably paracompact if and only if every open ω-chain covering of X is regularly shrinkable.*

Here is another problem concerning the normality of product spaces.

(*) "*Which space X satisfies the condition that $X \times Y$ is normal for any paracompact space Y.*"

In view of Theorems B and C and the fact that both compact spaces and metric spaces are paracompact, the space having normal product with any paracompact space should have both of the properties of paracompact spaces and P-spaces. From this point of view, it would be of some interest to obtain a characterization of paracompactness similar to Theorem 7.

Theorem 8. *X is paracompact if and only if for each open chain covering $\mathscr{C}$ and for each (closed) complete chain $\mathscr{C}'$ dominated by $\mathscr{C}$, there is a regular complete chain covering $\mathscr{C}^*$ such that*

$$\mathscr{C}' \prec \mathscr{C}^0 \prec \mathscr{C}^* \prec \mathscr{C} .$$

where $\mathscr{C}^0$ is the open chain covering consisting of interiors of members of $\mathscr{C}^*$.

The proof of the present theorem is related to that of Theorem 3, and will be given in [15].

In the following, some remarks concerning the problem (*) will be given. As an immediate consequence of Theorem B, we have

Theorem 9. *The following conditions are equivalent.*

1) $X \times Y$ *is normal for any paracompact space* Y.
2) $X \times Y$ *is paracompact for any paracompact space* Y ([13]; p. 190).

A similar theorem is valid in the case where Y is metric [8].

Theorem 10. *The following conditions on a paracompact space X are equivalent.*

1) $X \times Y$ *is normal for any metric space* Y. (That is, X is a P-space.)
2) $X \times Y$ *is paracompact for any metric space* Y.

Now, let us fix a metric space Y and suppose that $X \times Y$ is normal. "*Is it true that $X \times Y$ is paracompact?*" This problem is open. However, K. Morita [10] proved that if $X \times Y$ is normal and countably paracompact, then the product must be paracompact, and the author [14] presented a theorem asserting that the countable paracompactness of $X \times Y$ implies the paracompactness of $X \times Y$. These results have the following interesting consequence: "*If there were a product space $X \times Y$ of a paracompact space X and a metric space Y, which is normal but is not paracompact, then Dowker's problem concerning the countable paracompactness of normal space would be answered negatively.*"

In the case of the product of a normal space X with a compact space, the normality of $X \times BX$, where BX denotes any compactification of X, implies the paracompactness of the product $X \times BX$. However, T. Ishiwata [5] proved that the normality of of $X \times BZ$, where Z is a dense subspace of X, does not imply the paracompactness of $X \times BZ$ in general.

The problem (*) stated above is concerned with a problem of Michael's asking whether the product $X \times Y$ of a paracompact space and a metric space is paracompact or not. This has been answered by E. Michael [6] negatively, and Theorem 10 together with the following theorem gives a complete solution to Michael's problem.

Theorem 11. *For a metric space X to have the property that $X \times Y$ is paracompact for any paracompact space Y, it is necessary and sufficient for X to be a paracompact space which is the union of a σ-locally finite collection of compact sets.*

This surprising result (necessity) has been obtained by K. Morita [8], based on the results due to Michael [6] and A. H. Stone [11].

The author assumes that Theorem C, Theorem 8 and Theorem 11 suggest something about the solution of the problem (*) given above.

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