TYPES OF ULTRAFILTERS ON COUNTABLE SETS[1])

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Two ultrafilters x and y on countable sets X and Y are said to be of the same type if there exists a bijective mapping f of X onto Y such that $f[x] = y$ (or equivalently, $f^*x = y$ where x and y are considered as points of the Čech-Stone compactifications of X and Y, and f^* is the Stone-Čech extension of f). Let T be a set and τ be a mapping of the class of all ultrafilters on countable sets onto T such that $\tau x = \tau y$ iff x and y are of the same type.

Let N be the discrete space of the counting numbers. If $x \in \operatorname{cl} X - X$ where X is a discrete countable subset of $\beta\mathsf{N}$, then the intersections of the neighbourhoods of x with X form an ultrafilter on X which will be denoted by x_X. The type τx_X is called the type of x with respect to X; τx_{N} is called the type of x, and the types with respect to subsets of $\beta\mathsf{N} - \mathsf{N}$ are called the relative types of x.

By [2] the producing relation Φ is defined to be the set of all $\langle t, t' \rangle \in \mathsf{T} \times \mathsf{T}$ such that $\tau x_{\mathsf{N}} = t'$, $\tau x_X = t$ for some $x \in \beta\mathsf{N} - \mathsf{N}$ and $X \subset \beta\mathsf{N} - \mathsf{N}$. If $\langle t, t' \rangle \in \Phi$ then we say that "t produces t'" or "t' is produced by t".

Theorem. A. *If $\langle t_1, t_2 \rangle \in \Phi$, $\langle t_2, t_3 \rangle \in \Phi$ then $\langle t_1, t_3 \rangle \in \Phi$.*

B. *No type is produced by itself, i.e. $\langle t, t \rangle \in \Phi$ for no t.*

C. *Any type is produced by at most* $\exp \aleph_0$ *types.*

A is simple, B and C are rather profound (B was proved in [3], C in [2]). It should be remarked that B is equivalent to the following theorem on fixed points: No homeomorphism of $\beta\mathsf{N}$ into $\beta\mathsf{N} - \mathsf{N}$ has a fixed point.

Let us state two applications of C given in [2]: *a proof of nonhomogeneity of* $\beta\mathsf{N} - \mathsf{N}$ *without any use of the continuum hypothesis, and an example of a space X such that X^n is countably compact but X^{n+1} is not* (here n is an arbitrary counting number). A very simple proof of nonhomogeneity of $\beta\mathsf{N} - \mathsf{N}$ without the continuum hypothesis is based on B: if h is a homemomorphism of $\beta\mathsf{N} - \mathsf{N}$ into itself and $hx = y$, then the sets of the relative types of x and y coincide. Thence, according to B, if the type of x is a relative type of y then $hx = y$ for no h.

The properties of $\langle \mathsf{T}, \Phi \rangle$ and some related objects are developed in [4].

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[1]) This is a part of an invited lecture which was cancelled because of a presumed overlap with Gillman's lecture.

References

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[4] *Z. Frolík:* On βN. To appear.

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