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In: (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the second Prague topological symposium, 1966. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1967. pp. 56--62.

Persistent URL: http://dml.cz/dmlcz/700882

A HEREDITARILY INFINITE DIMENSIONAL SPACE

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1. Introduction

In recent papers [1, 2, 3] David Henderson describes examples of infinite dimensional compact metric spaces which contain no 1-dimensional closed subsets. In this paper we modify Henderson's approach slightly to give alternative descriptions of such examples.

The plan for getting an example is to start with a Hilbert cube H. We regard H as the Cartesian product of $I_1 \times I_2 \times \ldots$ where $I_i = [-1/2^i, 1/2^i]$. The metric for H is Euclidean.

Note that H has a countable basis $U_1, U_2, \ldots$ where U_i is of the form $B^n \times \times I_{n+1} \times I_{n+2} \times \ldots$ where $i \geqq n$ and B^n is the intersection of an open round n-ball in E^n with $I_1 \times I_2 \times \ldots \times I_n$ where for the moment we regard $I_1 \times I_2 \times \ldots \times I_n$ as lying in Euclidean n-space E^n.

An example of an infinite dimensional compact metric space with no 1-dimensional closed subset is obtained as the intersection of a countable number of closed subsets $K_1, K_2, \ldots$ of H where the K_i's have two important properties.

The first of these important properties is the following:

Property 1. *Any continuum in K_i from U_i to $H - U_i$ contains a subset of* Bd U_i *of dimension greater than or equal to* 2.

This property insures that $K = K_1 \cap K_2 \cap \ldots$ contains no closed 1-dimensional subset.

The second important property of the K_i's is chosen to insure that K is not 0-dimensional or null. In fact, we choose the K_i's so that K contains a continuum that joins the first pair of opposite faces of H. This pair is $\{-1/2\} \times I_2 \times I_3 \times \ldots$ and $\{1/2\} \times I_2 \times I_3 \times \ldots$. In choosing the property we are guided by a generalization of the following interesting property of a canonical cube C^3. If X is a closed set that separates the front face from the back face of C^3 and Y is a closed set that separates the left face from the right, then $X \cap Y$ contains a continuum joining the top and bottom of C^3. See Proposition A on page 40 of [4].

If X, Y are subsets of H, we say that Y separates X wrt I_i if $X - Y$ is the union of two mutually separated sets one of which contains $X \cap (I_1 \times I_2 \times \ldots \times I_{i-1} \times$

$\times \{-1/2^i\} \times I_{i+1} \times \ldots)$ and the other of which contains $X \cap (I_1 \times I_2 \times \ldots \times \times I_{i-1} \times \{1/2^i\} \times I_{i+1} \times \ldots)$. We say that Y weakly separates X wrt I_i if $X - Y$ is the union of two mutually separated sets (either or both of which may be null) one of which misses $I_1 \times I_2 \times \ldots I_{i-1} \times \{-1/2^i\} \times I_{i+1} \times \ldots$ and the other of which misses $I_1 \times I_2 \times \ldots \times I_{i-1} \times \{1/2^i\} \times I_{i+1} \times \ldots$.

Our remarks about C^3 may be extended to H as follows. If $X_2, X_3, \ldots$ are closed subsets of H such that X_2 weakly separates H wrt I_2, X_3 weakly separates X_2 wrt I_3, X_4 weakly separates X_3 wrt $I_4, \ldots$, then it can be shown that $X_2 \cap X_3 \cap \ldots$ contains a continuum which joins the first pair of opposite faces of H. This result and related ones are given by Theorems 3, 4, 5 in Section 4.

The second important property of the K_i's is the following:

Property 2. *K_i is a closed subset of a subcontinuum R_i of H such that:*

a) *R_i weakly separates H* wrt I_{2i} *and*
b) *K_i weakly separates R_i* wrt I_{2i+1}.

It follows from Theorem 5 that $K = K_1 \cap K_2 \cap \ldots$ is not 0-dimensional since it contains a continuum joining the first pair of opposite faces of H.

Theorem 1. *The set $K = K_1 \cap K_2 \cap \ldots$ is an infinite dimensional compact metric space with no 1-dimensional closed subset.*

2. Description of the R_i's and K_i's

First we consider the case where B^n is a round ball in $I_1 \times I_2 \times \ldots \times I_n$. Let R_i be the union of $H - U_i$ and the set of all points p of U_i whose $2i$th coordinate is $1/2^{2i} \sin(1/\varrho(p, \mathrm{Bd}\, U_i))$. We use ϱ to denote the distance function. Note that R_i weakly separates H wrt I_{2i}.

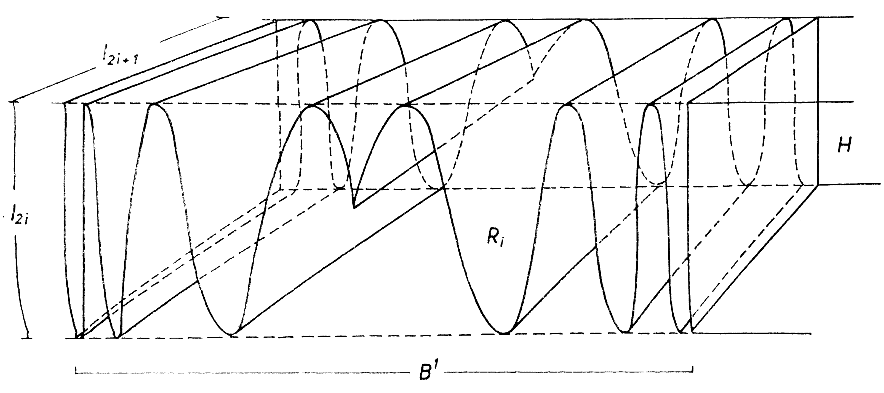

Figure 1.

Figure 1 gives a diagramatic view of R_i in case B^n is 1-dimensional and Figure 2 shows it if B^n is of dimension 2. In Figure 2 we are reminded of a vibrating drum or ripples on a pond where the period becomes short near the boundary but the amplitude remains constant.

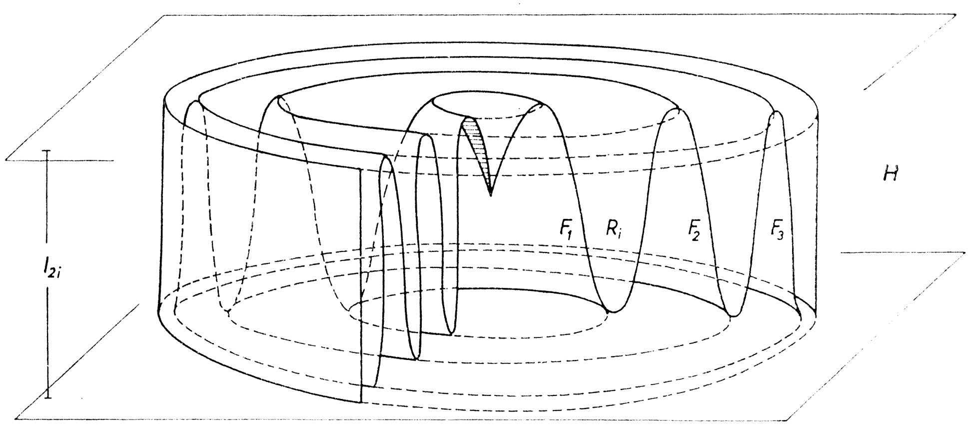

Figure 2.

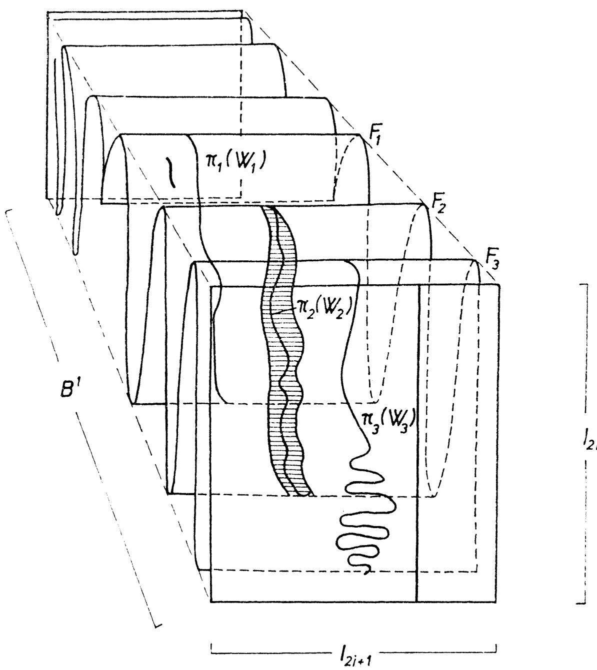

Figure 3.

By considering the variable half periods of $y = \sin 1/x$ we find that $R_i \cap U_i$ contains a countable number of mutually exclusive sets $F_1, F_2, \ldots$ such that there is a homeomorphism π_j of $\text{Bd}\, U_i$ onto F_j such that if $\pi_j(x_1, x_2, \ldots) = (y_1, y_2, \ldots)$, then $y_k = x_k$ if $k > n$ and $(y_1, y_2, \ldots, y_n)$ is between $(x_1, x_2, \ldots, x_n)$ and the center of B^n. Any infinite subsequence of the F_j's will converge to $\text{Bd}\, U_i$ and if C is a continuum in R_i from U_i to $H - U_i$, then for j sufficiently large, $F_j \cap C$ will contain a continuum that joins the $2i$th pair of opposite faces of H.

Consider the set of all closed subsets of $\text{Bd}\, U_i$ that separate $\text{Bd}\, U_i$ wrt I_{2i+1}. If these closed sets are metrized with the Hausdorff metric, they become the points of a separable metric space. Let $W_1, W_2, \ldots$ be a dense set of these separators. Let K_i be the set of all points p such that either $p \in \bigcup \pi_j W_j$ or p is a point of the closure of $R_i - \bigcup F_j$. Note that K_i weakly separates R_i in H wrt I_{2i+1}. It is shown at the end of Section 3 that K_i has Property 1.

Figure 3 shows half of some of the $\pi_j(W_j)$'s in case B^n is 1-dimensional. In case $n = 2$, F_j resembles a pipe with length the $2i$ direction and thickness the $2i + 1$ direction, where $\pi_j(W_j)$ is a set separating the inside lateral surface of the pipe from the outside lateral surface.

In case B^n is not a subset of $I_1 \times I_2 \times \ldots \times I_n$, we let t be a positive number so small that $tB^n \subset I_1 \times I_2 \times \ldots \times I_n$. If R_i' (K_i') is the set like R_i (K_i) we get by using tB^n, then R_i (K_i) is the set of all points $(x_1, x_2, \ldots)$ of H such that $(tx_1, tx_2, \ldots, tx_n, x_{n+1}, \ldots)$ is a point of R_i' (K_i').

3. Preventing 1-dimensionality

How does one prove that a set is of dimension greater than 1? In Lemma 2 of [2], Henderson used the criteria that a set is of dimension greater than 1 if there is an essential map of it onto a square. In this section we use a modification of Lemma 2 proposed by Harry Row.

Theorem 2. *Suppose in a metric space X that A_1, A_2 are mutually exclusive closed sets; B_1, B_2 are mutually exclusive sets; and Y is a compact set such that each subset of X that separates A_1 from A_2 in X intersects Y in a set containing a continuum from B_1 to B_2. Then dimension $Y \geqq 2$. In fact, if W separates A_1 from A_2 in X then there is a point of $W \cap Y$ at which Y is of dimension greater than* 1.

Proof. That dimension $Y \geqq 2$ follows immediately from Proposition B on page 34 of [4] but we include another proof. Let W' be a closed subset of W such that $X - W'$ is the union of two mutually separated sets V_1, V_2 containing A_1, A_2 respectively. Assume Y is of dimension less than or equal to 1 at each point of $Y \cap W'$.

At each point p of $Y \cap W'$, let O_p be an open set such that $p \in O_p$, dimension $(Y \cap \text{Bd}\, O_p) \leqq 0$, and $\bar{O}_p \cap (A_1 \cup A_2) = \emptyset$. Let $O_1, O_2, \ldots, O_n$ be a finite number of O_p's covering $Y \cap W'$.

Let $W'' = (W' - \bigcup_{i=1}^{n} O_i) \cup \bigcup_{i=1}^{n} \text{Bd } O_i$. Note that W'' separates A_1 from A_2 since $X - W''$ is the union of the mutually separated sets $V_1 - \bigcup_{i=1}^{n} \bar{O}_i$ and $(V_2 \cup \bigcup_{i=1}^{n} \bar{O}_i) - \bigcup_{i=1}^{n} \text{Bd } O_i)$. Since dimension $(W'' \cap Y) \leqq 0$, $(W'' \cap Y)$ does not contain a continuum from B_1 to B_2. This contradiction resulted from the assumption that Y is of dimension less than or equal to 1 at each point of $Y \cap W'$.

Theorem 2 and Property 1. It follows from the above Theorem 2 that if K_i is as described in Section 2 and C is a continuum in K_i from U_i to $H - U_i$, then dimension $(C \cap \text{Bd } U_i) \geqq 2$. To see this, let Bd U_i of Section 2 be the X of Theorem 1, the intersection of Bd U_i with the $2i$th pair of opposite faces of H be B_1, B_2, the intersection of Bd U_i with the $(2i + 1)$st pair of opposite faces of H be A_1, A_2, and $C \cap \text{Bd } U_i$ be Y. To see that if W is a subset of Bd U_i that separates Bd U_i wrt I_{2i+1}, then $W \cap C$ contains a continuum joining the $2i$th pair of opposite faces of H we proceed as follows. Let $W(n_1)$, $W(n_2)$, ... be a subsequence of W_1, W_2, ... converging to W. For k sufficiently large, there is a continuum $C(n_k)$ in $C \cap \pi_{n_k} W(n_k)$ joining the $2i$th pair of opposite faces of H. Some subsequence of $C(n_1)$, $C(n_2)$, ... converges to a continuum in W. This continuum lies in $W \cap C$ and joins the $2i$th pair of opposite faces in H.

4. Essential maps

A map f of a set X onto a cell B is said to be inessential if there is a map $g : X \to \text{Bd } B$ such that $f = g$ on f^{-1} Bd B. If there is no such map g, we say that f is essential. In the following theorem we use I^n to denote an n-cell and I^1 to denote $[0, 1]$.

Theorem 3. *Suppose f is an essential map of a metric space X onto $I^{n+1} = I^n \times I^1$ and π is the projection map of $I^n \times I^1$ onto I^n. If Y is a subset of X that separates $f^{-1}(I^n \times \{0\})$ from $f^{-1}(I^n \times \{1\})$, then $\pi f/Y$ is an essential map of Y onto I^n.*

Proof. Suppose $X - Y$ is the union of mutually separated sets U, V where $f^{-1}(I^n \times \{0\}) \subset U$ and $f^{-1}(I^n \times \{1\}) \subset V$. We suppose Y is closed since by using the fact that X is hereditarily normal, we find that if an arbitrary set in X separates two sets in X, a closed set in the arbitrary set separates the same two sets in X.

Assume $\pi f/Y$ is inessential. Then there is a map $g' : Y \to \text{Bd } I^n$ such that $g' = \pi f$ on $Y \cap (\pi f)^{-1} \text{Bd } I^n$. Let $g : Y \cup f^{-1} \text{Bd } (I^n \times I^1) \to \text{Bd } (I^n \times I^1)$ be such that $g = f$ on f^{-1} Bd $(I^n \times I^1)$ and for each point $x \in Y$, $g(x) = (g'(x), t)$ where t is the second coordinate of $f(x)$. Let $\bar{g}$ be an extension of g to all of X such that $\bar{g}(U) \subset (I^n \times \{0\}) \cup (\text{Bd } I^n) \times I^1$ and $\bar{g}(V) \subset (I^n \times \{1\}) \cup (\text{Bd } I^n) \times I^1$. Note that $\bar{g}$ is an extension of the map f/f^{-1} Bd $(I^n \times I^1)$. The assumption that $\pi f/Y$ is inessential led to the contradiction that f is inessential.

Theorem 3 can be extended as follows.

Theorem 4. *If in Theorem 3 we supposed merely that $X - Y$ was the sum of two mutually separated sets one of which missed $f^{-1}(I^n \times \{0\})$ and the other of which missed $f^{-1}(I^n \times \{1\})$, the conclusion still holds that $\pi f/Y$ is an essential map of Y onto I^n.*

Proof. The proof is the same as the proof of Theorem 3 except that instead of being able to require that $g = f$ on $f^{-1}\,\mathrm{Bd}\,(I^n \times I^1)$, we would merely suppose that $g = f$ on $f^{-1}((\mathrm{Bd}\,I^n) \times I^1)$, $g(f^{-1}(I^n \times \{0\})) \subset I^n \times \{0\}$, and $g(f^{-1}(I^n \times \times \{1\})) \subset I^n \times \{1\}$. Since $g/f^{-1}\,\mathrm{Bd}\,(I^n \times I^1) \to \mathrm{Bd}\,(I^n \times I^1)$ is homotopic to f and g can be extended to take X onto $\mathrm{Bd}\,(I^n \times I^1)$, the Homotopy Extension Theorem (see Theorem VI 5 of [4]) says that $f/f^{-1}\,\mathrm{Bd}\,(I^n \times I^1)$ can be so extended.

Theorem 5. *If $X_2, X_3, \ldots$ is a sequence of sets in H such that X_2 weakly separates H* wrt *I_2, X_3 weakly separates X_2* wrt *I_3, X_4 weakly separates X_3* wrt *$I_4, \ldots$ then $X_2 \cap X_3 \cap \ldots$ contains a continuum joining the first pair of opposite faces of H.*

Proof. Let $\pi_{n,j}$ be the projection of H onto $I_1 \times I_{n+1} \times \ldots \times I_{n-1+j}$. It follows from Theorem 4 and induction on n that $\pi_{n,j}/X_2 \cap X_3 \cap \ldots \cap X_n$ is an essential map of $X_2 \cap X_3 \cap \ldots \cap X_n$ onto $I_1 \times I_{n+1} \times \ldots \times I_{n-1+j}$.

Since $\pi_{n,1}/X_2 \cap X_3 \cap \ldots \cap X_n$ takes $X_2 \cap X_3 \cap \ldots \cap X_n$ essentially onto I_1, there is a continuum C_n in $X_2 \cap X_3 \cap \ldots \cap X_n$ joining the first pair of opposite faces of H. Some subsequence of $C_2, C_3, \ldots$ converges to a continuum in $X_2 \cap X_3 \cap \cap \ldots$ and this continuum joins the first pair of opposite faces of H.

5. Variations in the definition of K

The proof of Theorem 3 is slightly easier than that of Theorem 4 so it would have been easier to prove that dimension $(K_1 \cap K_2 \cap \ldots) \geqq 2$ if we had replaced "weakly separates" by "separates" in Property 2. We could do this by taking a new R_i whose points have coordinates the same as the points of the old R_i except that the $2i$th coordinate is divided by 2. To get a new K_i we would divide both the $2i$th and the $(2i + 1)$st coordinates of points of the old K_i by 2. While this variation simplifies the proof, it complicates the construction.

Another variation of description of K_i permits us to avoid using the projection π_j. Instead of letting $W_1, W_2, \ldots$ be a dense set of separators of $\mathrm{Bd}\,U_i$ wrt I_{2i+1} and projecting these separators onto the F_i's to obtain K_i we could have let $W_1', W_2', \ldots$ be a dense set of separators of R_i wrt I_{2i+1} and used $W_i' \cap F_i$ instead of $\pi_i W_i$ in defining K_i. Instead of proving that dimension $(C \cap \mathrm{Bd}\,U_i) \geqq 2$ we would have shown that dimension $C \geqq 2$ but this would have been just as good.

6. Infinite dimensional continuous curves

Of course, $K = K_1 \cap K_2 \cap \ldots$ is not locally connected or it would contain an arc. However it is possible to change any compact metric space to a continuous curve (Peano continuum) by adding to it the union of a null sequence of arcs so that no two of them intersect each other except possibly at an end point of each. Hence such an addition would convert K into an infinite dimensional continuous curve with no 2-dimensional subcontinuum.

By picking a nondegenerate component K', adjoining a null sequence of mutually exclusive arcs to it to convert K' to a continuous curve C and then shrinking the arcs to points, there results an infinite dimensional continuous curve C' with no 2-dimensional subcontinuum. Actually C' results from a decomposition of K' whose nondegenerate elements are point pairs. Also C' has the property that each open subset of it is infinite dimensional.

7. Questions

What conditions imposed on an infinite dimensional compact metric space implies that the space contains a 1-dimensional subcontinuum? Would the triviality of the 1st homology (Čech) imply this? Would the triviality of all the homology groups imply it?

The triviality of the 0-th homotopy implies that a space is arcwise connected. Does some restriction on the global and local homotopy groups of an infinite dimensional compact metric space imply that it contains a 2-dimensional subcontinuum? Need an infinite dimensional compact metric space contain a 2-dimensional closed subset even if it is an absolute retract?

References

[1] *David Henderson:* Finite dimensional subsets of infinite dimensional spaces. Wisconsin Topology Conference, Summer 1965, Annals of Mathematics Studies, Princeton University Press (1966), 141—146.

[2] *David Henderson:* An infinite-dimensional compactum with no positive-dimensional compact subsets — a simpler construction. American Journal of Mathematics *89* (1967), 105—121.

[3] *David Henderson:* Each strongly infinite-dimensional compactum contains a hereditarily infinite-dimensional compact subset American Journal of Mathematics *89* (1967), 122—123.

[4] *W. Hurewicz and H. Wallman:* Dimension Theory. Princeton University Press, 1941.

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