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REMARKS ON A THEOREM OF B. POSPÍŠIL

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Let $N_{\mathfrak{m}}$ be the discrete topological space of power $\mathfrak{m} \geqq \aleph_0$, let $\beta N_{\mathfrak{m}}$ be its Čech-Stone compactification and let $X_{\mathfrak{m}} = \beta N_{\mathfrak{m}} \setminus N_{\mathfrak{m}}$. The theorem of B. Pospíšil mentioned in the title says that $X_{\mathfrak{m}}$ contains $|X_{\mathfrak{m}}| = \exp \exp \mathfrak{m}$ points with the character $\exp \mathfrak{m}$. Analysing the original proof of this theorem (see [1]) we can get the following result — of which Pospíšil's theorem is an easy consequence.

Theorem 1. *Let $f: R \to R'$ be a closed continuous mapping of the space R onto R', let $y \in R'$ and suppose that $f^{-1}(y)$ is compact. Let $\mathfrak{m}, \mathfrak{n} \geqq \aleph_0$ be cardinal numbers such that $\mathfrak{m} \geqq \mathfrak{n}$ if $\mathfrak{m}$ is regular and $\mathfrak{m} > \mathfrak{n}$ otherwise. Suppose that there is a system $\mathfrak{U}$ of power $\mathfrak{m}$ of neighbourhoods of y such that no intersection of $\mathfrak{n}$ distinct neighbourhoods from $\mathfrak{U}$ and no finite union of such intersections is a neighbourhood of y. Then there is a point $x \in f^{-1}(y)$ such that*

$$\chi(x, R) \geqq \mathfrak{m} .$$

(Here — as usual — the character of x in the space R is denoted by $\chi(x, R)$).

Theorem 2. *Let R be a Hausdorff space and R' one of its Hausdorff extensions with cellularity number $\leqq \mathfrak{m} \geqq \aleph_0$ (i.e. R' does not contain a disjoint open set system of power $> \mathfrak{m}$). If $\mathfrak{q} \geqq \mathfrak{m}$, then the set of points in R' of character $\leqq \mathfrak{q}$ has a power $\leqq \exp \mathfrak{q}$.*

The proof can be based on a set theoretical lemma from [2]. An immediate corollary of this theorem is the following partial improvement of Pospíšil's result.

Theorem 3. *Let $\mathfrak{m}$ be an infinite cardinal. Then*

$$\exp [\chi(x, \beta N_{\mathfrak{m}})] = \exp \exp \mathfrak{m}$$

for almost every point $x \in X_{\mathfrak{m}}$ (or $x \in \beta N_{\mathfrak{m}}$). That means the power of the set of points $y \in X_{\mathfrak{m}}$ with

$$\exp [\chi(y, \beta N_{\mathfrak{m}})] < \exp \exp \mathfrak{m}$$

is $< \exp \exp \mathfrak{m} = |X_{\mathfrak{m}}|$.

This result is in fact stronger than Pospíšil's theorem if we assume the generalized continuum hypothesis. However, there are also other consistent conditions on the exp function under which Theorem 3 is stronger than Pospíšil's. Assume, e.g., $\exp \aleph_0 = \exp \aleph_1 = \aleph_2$ and $\exp \aleph_2 = \aleph_3 = \exp \exp \aleph_0$. Then the cardinality of the set of points of $X_{\aleph_0}$ of character $\leqq \aleph_0$ is at most $\exp \aleph_1 = \aleph_2$, hence almost every point has character $\aleph_2 = \exp \aleph_0$.

Though by Theorem 3 almost every point of $\beta N_{\mathfrak{m}}$ has a large character, i.e., in set theoretical sense there are many points of large character, topologically there are but few, namely

Theorem 4. *For all* $\mathfrak{m} \geqq \aleph_0$ *the set of points of* $X_{\mathfrak{m}}$ *of character* $\leqq \exp \aleph_0$ *contains a dense open set, i.e., the points of a character* $> \exp \aleph_0$ *are nowhere dense.*

For an arbitrary point of $X_{\mathfrak{m}}$ we can prove the following estimation.

Theorem 5. *Let* $x \in X_{\mathfrak{m}}$ *and let*

$$\mathfrak{p}(x) = \min \{|A| : A \subset N_{\mathfrak{m}} \text{ and } x \in \bar{A}^{\beta N_{\mathfrak{m}}}\}.$$

Then

$$\mathfrak{p}(x) < \chi(x, \beta N_{\mathfrak{m}}) \leqq \exp \mathfrak{p}(x).$$

Thus, assuming the generalized continuum hypothesis all the points of $X_{\mathfrak{m}}$ have characters of the form $\exp \mathfrak{q}$, where $\aleph_0 \leqq \mathfrak{q} \leqq \mathfrak{m}$, and the exact cardinality of the set of points in $X_{\mathfrak{m}}$ with the character $\exp \mathfrak{q}$ is $\mathfrak{m}^{\mathfrak{q}} \exp \exp \mathfrak{q}$.

References

[1] *B. Pospíšil:* On bicompact spaces. Publ. Fac. Sci. Univ. Masaryk *270* (1939).

[2] *A. Hajnal and I. Juhász:* Discrete subspaces of topological spaces. Indag. Math. (1966) (in print).