Ivo Marek

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ITERATIONS OF LINEAR BOUNDED OPERATORS AND KELLOGG'S ITERATIONS

I. MAREK

Praha

The purpose of this paper is to show that the existence of a fixed point of a special type is a sufficient condition for the convergence of the Kellogg iteration process for determining eigenvectors and eigenvalues of linear bounded operators in Banach spaces. It will follow from arguments given below that Kellogg's and probably similar methods can be applied to such a class of problems for which the existence of fixed points of certain type is guaranteed for the corresponding operators.

It can also be shown on the example of the Kellogg iteration process, how such a general approach to the convergence problem makes it possible to drop unimportant assumptions such as the symmetry or compactness of the investigated operator.

The basis of the proofs is the application of operator calculus of linear operators in the Banach space ([2], [3], [5]).

Let X be a complex Banach space, X^* its adjoint space of continuous linear forms. We denote the null-vector of the space X by the symbol O. Let X_1 be the Banach space of linear bounded operators mapping the space X into itself. We denote the identity operator by the symbol J. Let $\sigma(T)$ be the spectrum of the operator T.

The point μ_0 is called the dominant point of the spectrum of the operator T, if

$$|\lambda| < |\mu_0|$$

holds for every point $\lambda \in \sigma(T)$, $\lambda \neq \mu_0$.

Let $\{x_m^*\}$, $\{y_m^*\}$, $\{z_m^*\}$ be such sequences of linear forms of X^* that elements $x^* \in X^*$, $y^* \in X^*$ exist for which

(1) $$x^*(x) = \lim_{m\to\infty} x_m^*(x)\,,$$
$$y^*(x) = \lim_{m\to\infty} y_m^*(x) = \lim_{m\to\infty} z_m^*(x)$$

hold for every vector $x \in X$.

Let

$$B_1 = \frac{1}{2\pi i}\int_{C_0} R(\lambda, T)\,\mathrm{d}\lambda\,, \quad B_{k+1} = (T - \mu_0 J)\,B_k\,, \quad k \geqq 1$$

where $R(\lambda, T) = (\lambda J - T)^{-1}$ and C_0 is a circle with centre μ_0 and $\overline{\operatorname{int} C_0} \cap \sigma(T) = \{\mu_0\}$.

Let the following assumptions hold in the next theorems:

(A) The operator T is a linear bounded operator mapping X into itself.

(B) The value μ_0 is a pole of the multiplicity q of the resolvent $R(\lambda, T)$.

(C) The value μ_0 is the dominant point of the spectrum of the operator T.

Theorem 1. *In the norm of the space X_1 we have*

$$\lim_{m \to \infty} m^{-q+1} \mu_0^{-m} T^m = \frac{\mu_0^{-q+1}}{(q-1)!} B_q .$$

Let $x^{(0)} \in X$ be such a vector that $B_1 x^{(0)} \neq O$, so that such an index s, $1 \leqq s \leqq q$ exists, that

$$B_s x^{(0)} \neq O , \quad B_{s+1} x^{(0)} = O , \tag{2}$$

and let

$$x^*(B_s x^{(0)}) \neq O , \quad y^*(B_s x^{(0)}) \neq O , \quad x_0 = \frac{B_s x^{(0)}}{x^*(B_s x^{(0)})} . \tag{3}$$

We can construct Kellogg's iteration process:

$$x^{(m)} = T x^{(m-1)} , \quad x_{(m)} = \frac{x^{(m)}}{x_m^*(x^{(m)})} , \tag{4}$$

$$\mu_{(m)} = \frac{z_m^*(x^{(m+1)})}{y_m^*(x^{(m)})} . \tag{5}$$

Theorem 2. *Let* (1) *hold for the forms x_m^*, y_m^*, z_m^* and let* (2), (3) *hold for the vector $x^{(0)} \in X$.*

Then

$$\lim_{m \to \infty} x_{(m)} = x_0$$

holds for the sequence (4) *in the norm of the space X and*

$$\lim_{m \to \infty} \mu_{(m)} = \mu_0 .$$

Using the operational calculus method it is possible to prove the convergence of the Schwarz – Collatz [1] and Birger – Kolomý [4] type iterations.

The results which are valid for linear bounded operators can be extended in the usual way to the case of characteristic values of equation

$$Lx = \lambda Bx ,$$

where L and B are generally unbounded linear operators mapping its domains $D(L)$, $D(B)$ into X and the inclusion $D(L) \subset D(B)$ is correct.

References

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