Karel Koutský; Milan Sekanina

Modifications of topologies

In: (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the symposium held in Prague in September 1961. Academia Publishing House of the Czechoslovak Academy of Sciences, Prague, 1962. pp. [254]--256.

Persistent URL: http://dml.cz/dmlcz/700936

MODIFICATIONS OF TOPOLOGIES

K. KOUTSKÝ and M. SEKANINA

Brno

In this report we deal with the results concerning modifications of topologies. A topology on a set P is defined, according to E. Čech, as a mapping u of the set 2^P (i. e. the system of all subsets of P) into 2^P such that (1) $u\emptyset = \emptyset$, (2) $X \subset P \Rightarrow \Rightarrow X \subset uX$, (3) $X \subset Y \subset P \Rightarrow uX \subset uY$. A set together with a topology u on P is called a topological space (P, u). Throughout our report P means a fixed set.

We say that the topology v is weaker (stronger) that the topology u and we write $v \leqq u$ $(v \geqq u)$, if $X \subset P \Rightarrow vX \subset uX$ (respectively, $X \subset P \Rightarrow vX \supset uX$). It can be easily seen that the system of all topologies on P is under the relation $\leqq$ a complete lattice.

If f is a topological property, then a topology possessing f is called an f-topology.

Let u be a given topology and f a topological property. Let $\mathfrak{M}_f(u)$ $(\mathfrak{M}^f(u))$ be the system of all f-topologies which are weaker (stronger) than u. In the case that $\sup \mathfrak{M}_f(u)$ $(\inf \mathfrak{M}^f(u))$ is a f-topology we put $u_f = \sup \mathfrak{M}_f(u)$ $(u^f = \inf \mathfrak{M}^f(u))$ and $u_f(u^f)$ is called the lower (upper) f-modification of u. This concept is due to K. Koutský to whom E. Čech suggested it. For special cases of f the modifications occur often in topological studies.

We have studied lower and upper modifications for properties f given by means of the well-known topological axioms expressing the closeness of the closure (axiom U), the closeness of the point (axiom B), additivity (axiom A), separation axioms (H, $\overline{\text{H}}$, $\overline{\overline{\text{H}}}$), regularity (axiom R and $\overline{\text{R}}$), etc. For description of R and $\overline{\text{R}}$-modifications the introduction of a new topological property B* seemed to be convenient: u is called B^*-topology, if

$$x \in P \Rightarrow u(x) = s_u(x)\,,$$

where $s_u(x)$ is the intersection of all neighbourhoods of x. This property is the only topological property in our research, for which both modifications exist.

The first of our common papers [1] is in fact a continuation of one of the former papers of K. Koutský. It deals with conditions for u under which the given modification exists. The constructions of these modifications have been given either by means of closures either by means of the neighbourhoods.

Here we shall present the results concerning R-modifications. Recall that R-topology is a topology where for each point x and each neighbourhood O of x a neighbourhood O_1 of x exists, for which $uO_1 \subset O$. Following theorems hold.

(i) for lower R-modification:

Let u be a topology. Then $u_{B*} = \bigcup v$, where v runs over all R-topologies weaker than u. u_{B*} is a R-topology if and only if every non-isolated point x of the space (P, u) satisfies following conditions:

a) *for every neighbourhood O of x in (P, u) there exists a neighbourhood O_1 of x in (P, u) such that $uO_1 \subset O$.*

b) $y \in s_u(x) - u(x) \Rightarrow y$ *is isolated in* (P, u).

(ii) for upper R-modification:

Let (P, u) be a topological space. We shall define $v_\xi M$ for each $M \subset P$ and each ordinal number $\xi > 0$ in following way: (1) $v_1 M = uM$, (2) if $\xi = \eta + 1$, $\eta > 0$, then $v_\xi M = \sigma_{v_\eta} M$ ($\sigma_u M$ denotes, for a topology u on P and a set $M \subset P$, the intersection of all uO where O runs over all neighbourhoods of M), (3) if ξ is a limit ordinal number, then $v_\xi M = \bigcup_{\eta < \xi} v_\eta M$.

Then the supremum of all topologies v_ξ is exactly the upper R-modification of the topology u.

In the second paper [2] there are studied systems $\mathfrak{T}_f(u)$ ($\mathfrak{T}^f(u)$) of those topologies v for which $v_f = u$ ($v^f = u$), where u is a given f-topology. Suprema and infima of these systems have been constructed, for most cases the constructions of maximal (minimal) elements of these systems have been given, and in some cases a general construction of the elements of studied systems has been described. As a typical result we present again the theorems concerning R-topologies:

(i) for lower modification:

Let u be an R-topology. Then

(1) $v \in \mathfrak{T}_R(u)$ *if and only if the following conditions are satisfied:*

(a) *if $M \subset P$ is not a one-point set, then $vM = uM$,*

(b) $u \leqq v$,

(c) $x \in P \Rightarrow N_u(x) \cap R_u(x) \subset N_v(x) \cup R_v(x)$,

where $N_u(x)$ is the set of those points $y \in P$ for which $P - (x)$ is their minimal neighbourhood in (P, u), and $R_u(x)$ is the set of those points $y \in P$ for which $P - (y)$ is the minimal neighbourhood of x in (P, u).

(2) *Above every element $v \in \mathfrak{T}_R(u)$ there exists a maximal element of $\mathfrak{T}_R(u)$.*

(3) *If $w = \sup \mathfrak{T}_R(u)$, then $x \in P \Rightarrow w(x) = u(x) \cup N_u(x)$, and $wM = uM$ whenever M is not a one-point set.*

(ii) for upper modification.

Let u be an R-topology. Then $\inf \mathfrak{T}^R(u)$ *is a topology w defined as follows*

$$X \subset P \Rightarrow wX = X \cup u(o_u X)$$

where $o_u X$ *is the largest open set contained in* X *(when* u *is a* U*-topology, then* $o_u X = \mathrm{int}_u X$).

The third paper [3] is devoted to the following question:

If f and g are two topological properties and the modifications u^f, u^g, $(u^f)^g$, $(u^g)^f$ exist, there is a problem, under which conditions the equation $(u^f)^g = (u^g)^f$ is valid (similarly for the lower modifications). For the properties A, U, B this problem has been proposed by E. ČECH [4]. The general reply follows from the assertion that $(u^f)^g = (u^g)^f \Leftrightarrow (u^f)^g$ and $(u^g)^f$ are gf-topologies. Quite similar result holds for the lower modification. We have studied all pairs of above mentioned topological properties. Especially, we obtained interesting results in the cases, when the topological properties had been considered in regard to the points of P.

References

[1] *K. Koutský, M. Sekanina*: On the *R*-Modification and Several Other Modifications. Publ. Fac. Sci. Univ. Brno, *410* (1960), 45–64.

[2] *K. Koutský, M. Sekanina*: On the System of Topologies with a Given Modification. Publ. Fac. Sci. Univ. Brno, *418* (1960), 425–464.

[3] *K. Koutský, V. Polák, M. Sekanina*: On the Commutativity of the Modifications of Topologies. To appear in Publ. Fac. Sci. Univ. Brno.

[4] *E. Čech*: Topologické prostory. Čas. pěst. mat. a fys. *6* (1937), D 225–263.