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EXOTIC TOPOLOGIES FOR LINEAR SPACES

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0. Introduction

This is an elementary study of the structure of topological linear spaces (t. l. s.), with special emphasis on those which are *not* locally convex (l. c.). It was motivated in part by the following question of ALEX and WENDY ROBERTSON: If a t. l. s. admits a separating family of continuous linear functionals, is the same true of its completion? We hoped also to discover some sort of "structure theorems" about general t. l. s. Those t. l. s. which are not l. c. are generally regarded as mathematical curiosities rather than as objects of serious interest, probably because they have so little connection with interesting problems in analysis.[2]) But even though they may be quite "pathological" as t. l. s., surely they are unusually "smooth" as topological groups, and thus it is irritating (at least to the author) that so little is known about them. The present approach is not conspicuously successful and the irritation is only slightly ameliorated. Several unsolved problems are posed. We are able to answer (negatively) the Robertsons' question, and in doing so are led to the amusing notion of the orthogonality of two topologies. (Two topologies τ_1 and τ_2 for the same set are said to be *orthogonal* provided every nonempty τ_1-open set meets every nonempty τ_2-open set.)

The prefix "*h*-" before a topology or uniformity will indicate that it fulfills the Hausdorff separation axiom. A topology τ for a (real) linear space L will be called *admissible* provided (L, τ) is a t. l. s. For technical reasons, it is convenient here to consider all admissible topologies rather than merely the admissible *h*-topologies. Seven types of admissible topology seem especially worthy of study, and these will now be defined.

An admissible topology will be called *weak* provided every neighborhood of the origin 0 contains a linear subspace of finite deficiency, *convex* provided every neighborhood of 0 contains a convex neighborhood of 0, and *nearly convex* provided each point of $L \sim \text{cl}\,\{0\}$ can be separated from 0 by a continuous linear functional. It will

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[2]) A conspicuous exception is the space S of all measurable functions on [0, 1], in the metric topology corresponding to convergence in measure — $d(f, g) = \int_0^1 |f - g|/(1 + |f - g|)$. There are a few other exceptions.

be called *nearly exotic* provided L admits no nontrivial continuous linear functional, and *exotic* provided no proper closed neighborhood of 0 contains an absorbing convex set. We follow IVES [5] in calling a set $X \subset L$ *β-convex* provided X is starshaped from $0([0, 1]\, X = X)$ and $\beta(X + X) \subset 2X$; a set is *semiconvex* provided it is β-convex for some $\beta > 0$. (Alternatively, we could use a similar notion due to M. LANDSBERG [11, 12]). An admissible topology will be called *semiconvex* provided every neighborhood of 0 contains a semiconvex neighborhood of 0, and *strongly exotic* provided no proper closed neighborhood of 0 contains an absorbing semiconvex set. These are the seven types of topology to be studied here, and the terms are applied to the space (L, τ) as well as to the topology τ. Since an admissible topology is nearly convex if and only if each point of $L \sim \mathrm{cl}\,\{0\}$ can be separated from 0 by a convex neighborhood of 0, we might define analogously the notion of a *nearly semiconvex* topology, but this seems to be of only marginal interest while the other types of topology do appear in connection with well-known t. l. s.

Note that for every linear space L, the concrete topology $\{\emptyset, L\}$ has all the properties mentioned above except that of being an h-topology. Every infinite-dimensional normed linear space is convex but not weak. For $0 < p < 1$, the space l^p is nearly convex and semiconvex but not convex and the space L^p is semiconvex and exotic but not strongly exotic. (These spaces are discussed in [3, 5, 10, 11, 12, 14].) The space S is strongly exotic, as can be seen by adapting the proof of S. MAZUR and W. ORLICZ [14] that S is nearly exotic. (Ives [5] has a more detailed discussion of the space S and a related example.) If J is an $\aleph_0$-dimensional dense subspace of $L^p (0 < p < 1)$ or of S, then J (in the relative topology) is nearly exotic but it is not exotic and in fact admits no nonconcrete exotic topology, for the finest admissible topology in an $\aleph_0$-dimensional space is known to be convex [7]. The space J is semiconvex in the first case (for L^p) but not in the second (for S).

In the sequel, L will always denote a (real) linear space, E a t. l. s., and E^* the space of all continuous linear functionals on E. The real number space will be denoted by R. The symbol I' will denote the set of indices $\{a, w, c, sc, nc, ne, e, se\}$, with $I = I' \sim \{a\}$. For $\iota \in I$, Γ_ι will denote the class of all t. l. s. which are (in order) weak, convex, semiconvex, nearly convex, nearly exotic, or strongly exotic. For a linear space L and $\iota \in I$, $\sigma_\iota L$ will denote the supremum of all topologies τ for which $(L, \tau) \in \Gamma_\iota$; $\sigma_a L$ is the supremum of all admissible topologies for L.

Proofs will sometimes be abbreviated or omitted. For basic results on t. l. s., sometimes employed here without specific reference, N. BOURBAKI [2] and G. KÖTHE [10] are recommended; for topology, N. Bourbaki [1].

1. Some characterizations

For a t. l. s. E, E_h will denote the corresponding h. l. s. The following obvious fact can often be used to reduce considerations to h. l. s.:

1.1. Remark. *For each* $\iota \in I$, $E \in \Gamma_\iota$ *if and only if* $E_h \in \Gamma_\iota$.

The following characterization is hardly surprising:

1.2. Proposition. *A t. l. s. E is weak if and only if every neighborhood of* 0 *contains a set of the form* $\bigcap_1^j f_i^{-1}]-1, 1[$ *for* $f_i \in E^*$.

Proof. For the "only if" part, consider an arbitrary neighborhood U of 0. Let V be a neighborhood of 0 such that $V + \mathrm{cl}\, V \subset U$, F a subspace of finite deficiency in V, and φ the natural homomorphism of E onto the quotient space $Q = E/\mathrm{cl}\, F$. Then Q is finite-dimensional and φV is a neighborhood of the origin in Q, so there exist $g_i \in Q^*$ such that $\bigcap_1^j g_i^{-1}]-1, 1[\subset \varphi V$. With $f_i = g_i \varphi$ we have $f_i \in E^*$ and

$$\bigcap_1^j f_i^{-1}]-1, 1[\subset \varphi^{-1}\varphi V \subset V + \mathrm{cl}\, F \subset U .$$

The next result was stated by Mackey [13], and can be deduced almost at once from the fact that $\dim E^* > \aleph_0$ when E is an infinite-dimensional normed linear space.

1.3. Proposition. *If E is convex and* $\dim E^* \leqq \aleph_0$, *then E is weak.*

The following remark is useful:

1.4. Lemma. *In a t. l. s. E of the second category, a closed absorbing set X has nonempty interior; if X is semiconvex,* $0 \in \mathrm{int}\, X$.

Proof. Since $E = \bigcup_{n=1}^{\infty} nX$ and each set nX is closed, some nX has an interior point p and then $n^{-1}p \in \mathrm{int}\, X$. Now suppose further that X is semiconvex, whence $\beta(X + X) \subset X$ for some $\beta > 0$, and let $Y = X \cap -X$. Then Y has an interior point q and $0 = \beta q + \beta(-q) \in \beta(\mathrm{int}\, Y + \mathrm{int}\, Y) \subset \mathrm{int}\, \beta(Y + Y) \subset \beta(X + X) \subset X$.

Using 1.2, 1.4, and the Hahn-Banach theorem, we find

1.5. Proposition. *Every weak topology is convex; every convex topology is semiconvex and nearly convex. Every strongly exotic topology is exotic and every exotic topology is nearly exotic. Every nearly exotic topology of the second category is exotic.*

In connection with 1.5, recall the examples in the Introduction. For 1.6 below, recall the definition of orthogonality of two topologies.

1.6. Theorem. *An admissible topology for L is* $\left\{\begin{matrix}\textit{nearly exotic}\\ \textit{exotic}\\ \textit{strongly exotic}\end{matrix}\right\}$ *if and only if it is orthogonal to every* $\left\{\begin{matrix}\textit{weak}\\ \textit{convex}\\ \textit{semiconvex}\end{matrix}\right\}$ *topology for L.*

Proof. For the "only if" part for Γ_{ne}, suppose L admits a weak topology τ_2 which fails to be orthogonal to the admissible topology τ_1. Then there exist nonempty disjoint τ_i-open sets U_i in L with $0 \in U_2$. Let F be a subspace of finite deficiency contained in U_2, F_1 the τ_1-closure of F, and G a subspace supplementary to F_1 in L. Then G is a h. l. s., $0 < \dim G < \aleph_0$, and (L, τ_1) is the direct sum of F_1 and G. This implies that τ_1 is not nearly exotic.

It is easily seen that if τ_1 is not exotic, then it is not orthogonal to the convex topology $\sigma_c L$. Now assume, on the other hand, the existence of a convex topology τ_2 for L not orthogonal to τ_1. Let U_i be nonempty disjoint τ_i-open sets with $0 \in U_2$, let C be a convex τ_2-neighborhood of 0 such that $C \subset U_2$, let V_1 be a nonempty τ_1-open set whose τ_1-closure lies in U_1, and let W denote the τ_1-closure of the set $L \sim V_1$. Then W is a τ_1-closed neighborhood of 0 and $0 \in C \subset W \neq L$, so τ_1 is not exotic. This takes care of Γ_e, and the argument for Γ_{se} is essentially the same.

By 1.6, a weak topology for L must be orthogonal to every nearly exotic topology for L. We have been unable to decide in general whether this property characterizes the weak topologies. To prepare for a partial result in that direction, we recall two notions of D. T. FINKBEINER and O. M. NIKODYM [4]. *A* set $C \subset L$ is a *Hamel body* provided there is a basis $B \subset L$ such that $C = \operatorname{conv}(B \cup -B)$. A set is *linearly bounded* provided its intersection with each line lies in some segment.

1.7. Proposition. *A linear space L is of countable dimension if and only if every symmetric linearly bounded convex body in L is contained in a Hamel body.*

Proof. Suppose first that $\dim L > \aleph_0$. Let X be a basis for L and let U be the set of all points of the form $\sum_{x \in X} (fx)\, x$ where f is a finitely supported real-valued function on X and $\sum_{x \in X} (fx)^2 \leqq 1$. Then U is a symmetric linearly bounded convex body in L, and we claim that no Hamel body contains U.

Suppose U lies in a Hamel body C determined by a basis B for L. Let L_1 denote the space L as normed by the gauge functional of C, L_2 the same space as normed by the gauge functional of U, and note that the identity mapping T in L is a continuous linear transformation of L_2 onto L_1. Since the set B is uncountable, there exists a finite number n for which the set $B \cap nU$ is infinite. Let M denote the linear extension of $B \cap nU$ and let M_i denote the normed linear space obtained by restricting to M the norm of L_i. Since $C = \operatorname{conv}(B \cup -B)$, we have $C \cap M \subset nU$ and thus the restriction T' of T to M is a linear homeomorphism of M_2 onto M_1. Then of course T' can be extended to a linear homeomorphism which carries the completion of M_2 onto that of M_1, and this is impossible for the first completion is reflexive (being an l^2 space) and the second is not (being an l^1 space). The "only if" part of 1.7 has been established.

To complete the proof of 1.7, we must show that if E is an $\aleph_0$-dimensional normed linear space with unit cell $U = \{x \in E : \|x\| \leqq 1\}$, then U is contained in some Hamel body in E. Let the sequence x_α form a Hamel basis for E. It is not difficult to produce a sequence f_α in E^* such that $f_i x_j = 0$ for $i \neq j$, and always $f_i x_i > 0$ and $\sup f_i C = 2^{-i}$. Let G denote the linear space of all eventually-zero sequences of real numbers and for each $x \in E$ let

$$\varphi x = (f_1 x, f_2 x, \ldots) \in G .$$

Then φ is an algebraic isomorphism of E onto G, and since always $\sup f_i U = 2^{-i}$ it is easy to see that the set φU lies in the Hamel body C determined by the natural basis $\{\delta_i\}_1^\infty$ of G. Thus $\varphi^{-1} C$ is a Hamel body containing U and the proof of 1.7 is complete.

1.8. Theorem. *A convex topology for a linear space L is weak if and only if it is orthogonal to every nearly exotic topology for L.*

Proof. Only the "if" part requires discussion. Suppose the convex topology τ is not weak, whence there is a symmetric closed convex neighborhood U of 0 which contains no subspace of finite deficiency. The union L' of all lines through 0 which lie in U is a subspace of infinite deficiency, and thus there are supplementary linear subspaces L_1 and L_2 of L for which $L' \subset L_1$ and $\dim L_2 = \aleph_0$. It is easily verified that $U \subset L_1 + (U \cap L_2)$ and that the set $U \cap L_2$ is a symmetric linearly bounded convex body in L_2, whence by 1.7 it lies in some Hamel body C in L_2.

Let η be a nearly exotic h-topology for an $\aleph_0$-dimensional space J and V a non-empty η-open subset of J whose closure misses 0. Since the finest admissible topology $\sigma_a J$ for J is convex [7], there exists an absorbing convex set W in J such that $W \cap \cap V = \emptyset$. It is evident that W must contain a Hamel body C' in J [4] and that there is an algebraic isomorphism T of J onto L_2 which carries C' onto C. Now let ζ be the family of all subsets of L of the form $L_1 + TY$ for η-open $Y \subset J$. Then ζ is nearly exotic because η is nearly exotic. However, the set $L_1 + TV$ is ζ-open and misses the τ-open set U, so τ is not orthogonal to every nearly exotic topology for L and the proof of 1.8 is complete.

Are there characterizations of convex or semiconvex topologies which have a similar relationship to 1.7? Of course there exist nonconvex admissible topologies which are orthogonal to every exotic topology, for an $\aleph_0$-dimensional space admits no nonconcrete exotic topologies. However, it may be that an admissible topology τ is coarser than a convex topology if and only if τ is orthogonal to every exotic topology. This would imply that an admissible topology of the second category is convex if and only if it is orthogonal to every exotic topology, and weak if and only if it is orthogonal to every nearly exotic topology.

2. Preservation of type

The following two assertions are easily verified:

2.1. Proposition. *For each $\iota \in I$, if the t. l. s. E is the direct sum or product of the t. l. s. E_α, then $E \in \Gamma_\iota$ if and only if $E_\alpha \in \Gamma_\iota$ for all α.*

2.2. Proposition. *Let σ denote the supremum of admissible topologies τ_α for a linear space L. If $\iota \in I \sim \{ne\}$ and $(L, \tau_\alpha) \in \Gamma_\iota$ for all α, then $(L, \sigma) \in \Gamma_\iota$.*

In particular, $(L, \sigma_\iota L) \in \Gamma_\iota$ for each $\iota \in I \sim \{ne\}$. This fails for $\iota = ne$, and it seems conceivable even that $\sigma_{ne} L = \sigma_a L$ when L is infinite-dimensional. We have been unable to settle this, but shall prove the following:

2.3. Theorem. *If L is infinite-dimensional, the topology $\sigma_{ne} L$ is finer than the topology $\sigma_w L$.*

Proof. Let f be a nontrivial linear functional on L. We wish to show that f is continuous for the topology $\sigma_{ne}L$, and shall in fact describe two nearly exotic h-topologies ζ_1 and ζ_2 for L such that f is continuous for the topology $\sup(\zeta_1, \zeta_2)$.

Note that an $\aleph_0$-dimensional linear space admits a nearly exotic h-topology, whence the same is true of every infinite-dimensional linear space (for it may be regarded as the direct sum of a family of its $\aleph_0$-dimensional subspaces).

Let τ be a nearly exotic h-topology for the space L under consideration. Let $y \in L$ with $fy = 2$ and consider the following two topologies τ_1 and τ_2 for the space $L \times R$: a set is τ_1-open (resp. τ_2-open) provided it has the form $U \times \{0\} + R(0,1) = U \times R$ (resp. $U \times \{0\} + R(y, 1)$) for some τ-open set $U \subset L$. Then τ_1 and τ_2 are both nearly exotic topologies for $L \times R$. For each $x \in L$, let $\xi x = (x, fx) \in L \times R$ and let ζ_i be the topology for L which is determined by specifying that ξ shall be a homeomorphism of L onto ξL in its relative topology ζ_i' induced by τ_i. Since ξL is dense in $L \times R$ under the topology τ_i, each topology ζ_i is a nearly exotic topology for L. Although the τ_i are not h-topologies, it is easily verified that the topologies ζ_i' (and hence ζ_i) do satisfy the separation axiom. To show that f is continuous for the topology $\sup(\zeta_1, \zeta_2)$, it suffices to produce τ_i-neighborhoods V_i of 0 in $L \times R$ such that $(x, r) \in V_1 \cap V_2$ implies $r \neq 1$, for then the functional $f\xi^{-1}$ fails to assume the value 1 on the set $V_1 \cap V_2 \cap \xi L$, the same must be true of f on the set $\xi^{-1}(V_1 \cap L) \cap \xi^{-1}(V_2 \cap L)$, and since the latter set is a $\sup(\zeta_1, \zeta_2)$-neighborhood of 0 in L it follows readily that f is continuous. Let U be a τ-neighborhood of 0 in L such that $y \notin U - U$, and suppose

$$(z, 1) \in (U \times \{0\} + R(0, 1)) \cap (U \times \{0\} + R(y, 1)).$$

Then we have $z \in U$ and $z \in U + y$, whence $y \in U - U$ and the contradiction completes the proof of 2.3.

2.4. Theorem. *For* $1 \leqq j \leqq 5$, *let* A_j *denote the set of all* $\iota \in I$ *such that whenever* F *is a subspace of a t. l. s.* E

1) *then* $E \in \Gamma_\iota$ *implies* $F \in \Gamma_\iota$;
2) *and* F *is dense in* E, *then* $E \in \Gamma_\iota$ *implies* $F \in \Gamma_\iota$;
3) *and* $\dim E/F < \aleph_0$, *then* $E \in \Gamma_\iota$ *implies* $F \in \Gamma_\iota$;
4) *and* F *is dense in* E, *then* $F \in \Gamma_\iota$ *implies* $E \in \Gamma_\iota$;
5) *and* $\dim E/F < \aleph_0$, *then* $F \in \Gamma_\iota$ *implies* $E \in \Gamma_\iota$.

Then $A_1 = \{w, c, sc, nc\}$, $A_2 = A_1 \cup \{ne\} = I \sim \{e, se\}$, $A_3 = I$, $A_4 = I \sim \{nc\}$, *and* $A_5 = \{w, c, sc\}$.

Proof. The assertion about A_1 is obvious, as is the fact that $A_1 \cup \{ne\} \subset A_2$. To see that $\{e, se\} \subset I \sim A_2$, recall that the space S is strongly exotic and admits a dense subspace J of countable dimension, but J cannot be exotic for the topology $\sigma_a J$ is convex [7].

Note that if F is a dense subspace of E and U is a neighborhood of 0 in F, then $\mathrm{cl}\, U$ is a neighborhood of 0 in E. It follows at once that $\{c, sc\} \subset A_4$. To see that $w \in A_4$, recall the characterization 1.2 and the fact that every continuous linear functional on F can be extended to one on E. That $\{ne, e, se\} \subset A_4$ follows at once from the relevant definitions. Thus $A_4 \supset I \sim \{nc\}$. In § 3 we give an example of a dense subspace F_0 of a h. l. s. E_0 such that F_0 is nearly convex but E_0 is not, thereby showing that $nc \notin A_4$ and completing the discussion of A_4. Let $x \in E_0 \sim F_0$ with $fx = 0$ for all $f \in E^*$; then the pair F, $F + Rx$ shows that $nc \notin A_5$.

For A_3 and A_5 it suffices to consider the case in which dim $E/F = 1$. Then F must be dense or closed in E, and in the latter event E is the direct sum of F and a line. Thus the remaining assertions about A_5 are evident and for A_3 it remains only to show that $\{e, se\} \subset A_3$.

Suppose E is exotic and F is a (necessarily dense) hyperplane in E but F is not exotic. By 1.6, the topology of F cannot be orthogonal to every convex topology for F, and hence there exist a neighborhood U of 0 in E and a nonempty convex set $C \subset \subset F \sim U$ such that C is equal to its own core relative to F. Let V be a symmetric starshaped neighborhood of 0 in E such that $V + V \subset U$ and let $v \in V \sim F$. Then

$$C +]-v, v[\subset C + V \subset (F \sim U) + V \subset E \sim V.$$

Since the set $C +]-v, v[$ is convex and equal to its own core relative to E, it follows from 1.6 that E is not exotic and the contradiction implies that $e \in A_3$. A similar argument shows that $se \in A_3$ and completes the proof of 2.4.

The relationship of a t. l. s. to its $\aleph_0$-dimensional subspaces seems worthy of study. A. Pełczyński has asked whether every infinite-dimensional metric linear space has an $\aleph_0$-dimensional subspace which admits a nontrivial continuous linear functional, and in the other direction we inquire whether a nearly exotic space must have a nearly exotic $\aleph_0$-dimensional subspace. We may ask also whether membership of E in Γ_ι is implied by that of every $\aleph_0$-dimensional subspace of E. This is trivially the case for $\iota = ne$, and also for $\iota = e$ and $\iota = se$ since the only exotic topologies of countable dimension are concrete. It is not the case for $\iota = c$ or $\iota = sc$, for if dim $L > \aleph_0$ the topology $\sigma_a L$ is not semiconvex even though its restriction to each $\aleph_0$-dimensional subspace is convex. The question is of interest for $\iota \in \{w, nc\}$, and also for $\iota \in \{c, sc\}$ under additional restrictions on the space. We can report only the following partial results:

2.5. Proposition. *A semiconvex space is weak if and only if all its $\aleph_0$-dimensional subspaces are weak.*

2.6. Proposition. *Suppose $\aleph$ is an infinite cardinal number, $\aleph'$ is the first cardinal $> \aleph$ which is the limit of a sequence of its predecessors, and E is a metric linear space with $\aleph \leq$ dim $E < \aleph'$. Then E is convex, semiconvex, or nearly convex if and only if all its $\aleph$-dimensional subspaces have the corresponding property.*

Proofs. For 2.5, we consider a symmetric semiconvex neighborhood U of 0 in E and denote by F the union of all lines through 0 which lie in U. It can be verified (using semiconvexity) that F is a linear subspace of E and hence admits a supplementary subspace M. Since $U \cap M$ contains no lines through 0, either M is finite-dimensional or M is not weak, and this completes the proof of 2.5.

For 2.6, let Z denote the interval $[\aleph, \aleph'[$ of cardinal numbers, and for $\iota \in \{c, sc, nc\}$ let Z_ι denote the set of all $\zeta \in Z$ such that whenever E is a metric linear space of dimension ζ and all $\aleph$-dimensional subspaces of E are members of Γ_ι, then $E \in \Gamma_\iota$. We wish in each case to show that $Z_\iota = Z$. Suppose $Z \sim Z_\iota$ is nonempty and let δ be the first member of $Z \sim Z_\iota$. Let E be a metric linear space of dimension δ, all of whose $\aleph$-dimensional subspaces are members of Γ_ι, and let B be a Hamel basis for E. Let B be well-ordered in such a way that for each $b \in B$, the set P_b of all predecessors of b in B is of cardinality $< \delta$, and for each $b \in B$ let E_b denote the linear extension of P_b in E. From the definition of δ it follows that $E_b \in \Gamma_\iota$ for each $b \in B$, and also that no countable set is cofinal in B.

Now for the case $\iota = nc$, we wish to show that each point $p \in E \sim \{0\}$ can be separated from 0 by a continuous linear functional, whence $E \in \Gamma_{nc}$, $\delta \in Z_\iota$, and the contradiction shows that $Z = Z_\iota$. For each $b \in B$, let φb denote the supremum of the set of all numbers $r \geqq 0$ such that $\operatorname{conv}(N_r \cap E_b) \subset E \sim \{p\}$, where $N_r = \{x \in E : d(x, 0) \leqq r\}$. Then φ is an antitone mapping of B into $[0, \infty]$, and always $\varphi b > 0$ since $E_b \in \Gamma_{nc}$. Since B admits no cofinal sequence it is clear that $\inf \varphi B = 2\varepsilon > 0$ and then $\operatorname{conv} N_\varepsilon \subset E \sim \{p\}$, whence the Hahn-Banach theorem guarantees the existence of $f \varepsilon E^*$ with $fp \neq 0$. This completes the discussion for Γ_{nc}. For Γ_c the reasoning is similar, with the set $E \sim \{p\}$ replaced by an arbitrary neighborhood of 0.

For Γ_{sc}, we consider an arbitrary neighborhood U of 0 in E, and for each $b \in B$ let ψb denote the set of all $(\beta, r) \in]0, 1] \times]0, \infty[$ such that $\operatorname{conv}_\beta (N_r \cap E_b) \subset U$, where $\operatorname{conv}_\beta$ denotes the β-convex hull [5]. Each set ψb is nonempty, for $E_b \in \Gamma_{sc}$, and it is clear that if $(\beta, r) \in \psi b$, $\beta' \in]0, \beta]$, and $r' \in]0, r]$, then $(\beta', r') \in \psi b$. Choose sequences β_α and r_α in $]0, 1]$ such that $\beta_\alpha \searrow 0$ and $r_\alpha \searrow 0$. We claim that for sufficiently large i, $(\beta_i, r_i) \in \bigcap_{b \in B} \psi b$. (Then of course $\operatorname{conv}_{\beta_i} N_{r_i} \subset U$ and the semiconvexity of E is established.) Suppose the contrary, whence for each i there exists b_i having $(\beta_i, r_i) \notin \psi b$. Then with $b' \in B$ and $b_i < b'$ for all i, it follows that $\psi b'$ is empty, a contradiction completing the proof of 2.6.

2.7. Theorem. *For $j = 1, 2$, let B_j denote the set of all $\iota \in I$ such that whenever η is a continuous linear transformation of the t. l. s. E onto the t. l. s. F*

1. *then $E \in \Gamma_\iota$ implies $F \in \Gamma_\iota$;*

2. *and η is open or F of the second category, then $E \in \Gamma_\iota$ implies $F \in \Gamma_\iota$.*

Then $B_1 = \{w, ne, e, se\}$ and $B_2 = I \sim \{nc\}$.

The relevant proofs and examples are rather straightforward and will be left for the reader. Use 1.4 when F is of the second category. In [8] there are described a separable metric linear space E and supplementary closed linear subspaces M_1 and M_2 of E such that E is nearly convex but both the quotient spaces E/M_i are nearly exotic. Of course this cannot happen if E is complete. However, we do not know (in the setting of 2.7) whether near convexity of E implies that of F when η is continuous and open *and* F is of the second category. Note that if E is convex and η continuous, then

F cannot be exotic unless its topology is concrete, but F can be a nearly exotic t. l. s., as we see by letting J be an $\aleph_0$-dimensional dense subspace of S, η the identity mapping on J, $E = (J, \sigma_a J)$, and F the space J in the relative topology inherited from S.

3. Orthogonality and completeness

Two uniform structures for the same set will be called *orthogonal* provided the topologies which they generate are orthogonal.

3.1. Proposition. *Suppose $\mathcal{U}_1$ and $\mathcal{U}_2$ are orthogonal uniformities for a set X such that $(X, \sup(\mathcal{U}_1, \mathcal{U}_2))$ is a complete uniform space. Then for any two points x_1 and x_2 of X, the $\mathcal{U}_1$-closure of $\{x_1\}$ meets the $\mathcal{U}_2$-closure of $\{x_2\}$.*

Proof. Let $\mathcal{B}$ be the family of all sets of the form $U_1|x_1| \cap U_2|x_2|$ for $U_i \in \mathcal{U}_i$. Since $\mathcal{U}_1$ and $\mathcal{U}_2$ are orthogonal, each member of $\mathcal{B}$ is nonempty and it follows that $\mathcal{B}$ is the base of a filter $\mathcal{F}$. We claim that $\mathcal{F}$ is a Cauchy filter for the uniformity $\sup(\mathcal{U}_1, \mathcal{U}_2)$. Indeed, consider an arbitrary member V of $\sup(\mathcal{U}_1, \mathcal{U}_2)$, containing the set $V_1 \cap V_2$ for certain $V_i \in \mathcal{U}_i$. Let W_i be a member of $\mathcal{U}_i$ such that $W_i = W_i^{-1}$ and $W_iW_iW_iW_i \subset V_i$, and let $p \in Z = W_1|x_1| \cap W_2|x_2|$. For each $q \in Z$ we have $(p, x_i) \in W_i$ and $(x_i, q) \in W_i$, whence $(p, q) \in W_iW_i$. It follows that

$$Z \times Z \subset W_iW_iW_iW_i \subset V_i,$$

whence $Z \times Z \subset V_1 \cap V_2$. Thus $\mathcal{F}$ is a Cauchy filter for $\sup(\mathcal{U}_1, \mathcal{U}_2)$ and by hypothesis must converge to a point $z \in X$. Since each neighborhood $U_i'|z|$ (for $U_i' \in \mathcal{U}_i$) must contain $U_i|x_i|$ for some $U_i \in \mathcal{U}_i$, it is clear that z lies in the $\mathcal{U}_i$-closure of $\{x_i\}$ and this completes the proof.

3.2. Corollary. *If a linear space L is complete under the supremum of two orthogonal admissible topologies τ_1 and τ_2, then L is the linear sum of the τ_1-closure of $\{0\}$ and the τ_2-closure of $\{0\}$.*

3.3. Proposition. *Two convex topologies for a linear space are orthogonal if and only if there is no nontrivial linear functional which is continuous in both topologies.*

The result 3.3 follows at once from the separation theorem for convex bodies. The result 3.4 below is well-known and can be proved in simpler ways (cf. Theorem 15 of Mackey [13]), but the reader may find it instructive to base a proof on 3.1 and 3.3.

3.4. Proposition. *A weak h. l. s. E is complete if and only if no proper subspace of E^* separates the points of E.*

For a uniform space $(X, \mathcal{U})$, the corresponding h-uniform space will be denoted by $(X, \mathcal{U})_h$ and the completion of $(X, \mathcal{U})_h$ by $(X, \mathcal{U})_c$.

3.5. Theorem. *If $\mathcal{U}_1$ and $\mathcal{U}_2$ are orthogonal uniformities for a set X, the space $(X, \sup(\mathcal{U}_1, \mathcal{U}_2))_c$ is uniformly isomorphic with the product space $(X, \mathcal{U}_1)_c \times \times (X, \mathcal{U}_2)_c$.*

Proof. Let $\mathcal{U}_0 = \sup(\mathcal{U}_1, \mathcal{U}_2)$ and for $i = 0, 1,$ or 2 let X^i, X^i_h, and X^i_c denote respectively the spaces $(X, \mathcal{U}_i)$, $(X, \mathcal{U}_i)_h$, and $(X, \mathcal{U}_i)_c$. Let ξ_i denote the natural mapping of X^i onto X^i_h and η_i the natural mapping of X^i_h into X^i_c. For each point $y = (y_1, y_2) \in X^1_c \times X^2_c$ and for $j = i$ or 2 let $\mathscr{H}^j_y$ denote the trace on $\eta_0 X^j_h$ of the filter-base consisting of all open neighborhoods of y_j in X^j_c, and let $\mathscr{G}^j_y$ denote the image of $\mathscr{H}^j_y$ under the transformation $\xi_j^{-1}\eta_j^{-1}$. Then $\mathscr{G}^j_y$ is a filter-base consisting of $\mathscr{U}_j$-open subsets of X, and since the uniformities $\mathscr{U}_1$ and $\mathscr{U}_2$ are orthogonal, the set $\mathscr{F}_y$ of all intersections $G_1 \cap G_2$ with $G_j \in \mathscr{G}^j_y$ must also be a filter-base, and in fact clearly a $\mathscr{U}_0$-Cauchy filter-base for $\mathscr{G}^j_y$ is $\mathscr{U}_j$-Cauchy. Thus the image of $\mathscr{F}_y$ under the mapping $\eta_0\xi_0$ is a filter-base in $\eta_0 X^0_h$ converging to a unique point $\varphi y \in X^0_c$. Since each $\mathscr{U}_0$-Cauchy net is both $\mathscr{U}_1$-Cauchy and $\mathscr{U}_2$-Cauchy, it is evident that φ is a biunique transformation of $X^1_c \times X^2_c$ onto X^0_c. It is a routine matter to check that φ is a uniform isomorphism and this completes the proof of 3.5. Further, if X is a linear space and the orthogonal uniformities $\mathscr{U}_1$ and $\mathscr{U}_2$ are generated by admissible topologies for X, then φ turns out to be a linear transformation, whence –

3.6. Corollary. *If τ_1 and τ_2 are orthogonal admissible topologies for a linear space L, the h. l. s. $(L, \sup(\tau_1, \tau_2))_c$ is linearly homeomorphic with the h. l. s. $(L, \tau_1)_c \times (L, \tau_2)_c$.*

Now for the example needed in connection with 2.4, observe that if $\dim L \geqq 2^{\aleph_0}$ then L admits both a convex h-topology τ_1 and an exotic h-topology τ_2. By 3.6, the space $(L, \sup(\tau_1, \tau_2))_c$ is linearly homeomorphic with the product space $(L, \tau_1)_c \times (L, \tau_2)_c$. Of course the space $(L, \sup(\tau_1, \tau_2))$ is nearly convex, but from 2.4 (A_4) it follows that $(L, \tau_2)_c$ is exotic and hence every continuous linear functional on the product space must vanish everywhere on $\{0\} \times (L, \tau_2)_c$. Note that the topologies τ_i can be chosen to be separable and metrizable and then the same will be true of their supremum. These examples can be described more briefly by following the same ideas but suppressing some of the machinery employed above.

Now for $\iota \in I' = I \cup \{a\}$, let C_ι denote the class of all cardinal numbers $\aleph$ such that the space $(L, \sigma_\iota L)$ is complete for $\aleph$-dimensional L. It has been proved by I. Namioka (unpublished) that C_a includes all cardinals. It follows from 3.4 that C_w consists only of finite cardinals. A result of S. Kaplan [6] is that C_c includes all cardinals, which implies that C_a, C_{sc}, and C_{nc} include all countable cardinals. Of course the same is true of C_e and C_{se}, though in a trivial fashion. We conjecture that $\aleph_0 \in C_{ne}$, but this is not known and may be connected with the question as to whether $\sigma_{ne}L$ is finer than $\sigma_c L$ when $\dim L \geqq \aleph_0$. For $\aleph > \aleph_0$ nothing is known except in the cases of C_a, C_w and C_c. To what extent can arbitrary admissible topologies be represented in terms of the seven types studied here? Note that by Namioka's result in conjuction with 3.2, $\sigma_2 L$ is not the supremum of an admissible h-topology and an admissible non-concrete topology which is orthogonal to τ.

Another limitation on the representation of admissible topologies is indicated by 3.8 below. In preparation for its proof, we establish the following

3.7. Lemma. *Suppose X_1 is a symmetric starshaped subset of R^n, $X_{i+1} = X_i + X_i$ for all i, and M_i is the union of all lines through 0 which lie in $\mathrm{cl}\, X_i$. Then for some $r \leqq n$, the following three statements are true:*

(a) *for each* $i \leqq r$, M_i *contains a linear subspace of dimension* i;

(b) M_r *is an* r*-dimensional linear subspace*;

(c) *there is a compact set* K *such that* $X_1 \subset M_r + K$.

Proof. Let r be the largest integer for which (a) holds. Let L_r be an r-dimensional linear subspace contained in M_r and let F be a subspace supplementary to L_r. Let $\| \ \|$ be a norm for R^n, $U = \{x \in R^n : \|x\| \leqq 1\}$, and $Z = \{x \in F : \|x\| = 1\}$. For each $z \in Z$, let δ_z be the least upper bound of the set of all numbers $t > 0$ such that $X \cap (L_r + Rz) \subset L_r + tU$. Let $m = \sup \delta_z \in [0, \infty]$. Since L_r is a subspace and X is symmetric and starshaped, it is easy to see that $L_r +]-\delta_z, \delta_z[\, z \subset L_r + X$ for each $z \in Z$. Then if $m = \infty$ it follows from compactness of Z that $L_r + Rz_0 \subset \mathrm{cl}\,(L_r + X)$ for some $z_0 \in Z$, whence of course $L_r + Rz_0$ is an $(r + 1)$-dimensional subspace of M_{r+1}. This contradicts the definition of r, so we conclude that $m < \infty$ and $X \subset L_r + mU$, thus establishing (c). The truth of (b) is now immediate and this completes the proof. (Note, in addition, that $M_j = M_r$ for all $j \geqq r$.)

3.8. Proposition. *The usual topology* τ *of the space* l^p $(0 < p < 1)$ *is not the supremum of a convex topology and a nearly exotic topology.*

Proof. Suppose τ is the supremum of a convex topology τ_1 and a nearly exotic topology τ_2, and let γ denote the admissible topology generated by the family of all convex τ-neighborhoods of 0. Then of course γ is finer than τ_1, and since the set $U = \{x = (x_1, x_2, \ldots) \in l^p : \sum \mid |x_i|^p \leqq 1\}$ is a τ-neighborhood of 0 there exist a γ-neighborhood V' of 0 and a τ_2-neighborhood W' of 0 such that $V' \cap W' \subset U$ and hence $W' \subset U \cap (l^p \sim V')$. By a theorem of DAY [3], each continuous linear functional on l^p is a linear combination of the coordinate functionals $x_i \mid x \in l^p$, whence the topology γ is weak by 1.3 and consequently there exist $\varepsilon > 0$ and an integer $n \geqq 2$ such that

$$V' \supset V = \{x \in l^p : |x_i| < \varepsilon \text{ for } 1 \leqq i \leqq n - 1\}\,.$$

Let W_1 be a symmetric starshaped τ_2-neighborhood of 0 such that $W_n \subset W'$ (where $W_{i+1} = W_i + W_1$). Then of course $W_n \subset U \cup (l^p \sim V)$.

For each $x \in l^p$, let $\pi x = (x_1, \ldots, x_n) \in R^n$, whence $\pi U = \{x \in R^n : \sum_1^n |x_i|^p \leqq 1\}$ and $\pi V = \{x \in R^n : |x_i| < \varepsilon$ for $1 \leqq i \leqq n - 1\}$. Let $X_1 = \pi W_1$, whence X_1 is symmetric and starshaped with $X_n \subset \pi U \cup (R^n \sim \pi V)$. Let the subspace M_r of R^n be as in 3.7. Since clearly $\mathrm{cl}\, X_r \subset \mathrm{cl}\, X_n \neq R^n$, we see that $M_r \neq R^n$ and then it follows from (c) of 3.7 that $\mathrm{conv}\, X_1 \neq R^n$. This implies that $\mathrm{conv}\, W_1 \neq l^p$ and contradicts the assumption of near exoticity for the topology τ_2. The proof of 3.8 is now complete.

In connection with 3.8, note also that an $\aleph_0$-dimensional linear space admits a topology which cannot be represented as the supremum of a semiconvex, a nearly convex, and an exotic topology. However, relative to any examples known to us at the moment, each of the following hypotheses *may* be true:

each admissible topology is the supremum of a nearly convex topology and a nearly exotic topology;

each admissible topology is coarser than one which is the supremum of a convex topology and a nearly exotic topology;

each t. l. s. is linearly homeomorphic with a subspace of a t. l. s. whose topology is the supremum of a convex topology and an exotic topology.

In view of 2.1 and a result in [9], it would suffice in the last instance to consider metric linear spaces.

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