V. A. Efremovič; A. G. Vainstein
New results in uniform topology

In: (ed.): General topology and its relations to modern analysis and algebra IV, Proceedings of the fourth Prague topological symposium, 1976, Part A: Invited papers. Springer, Berlin, 1977. Lecture Notes in Mathematics, 609. pp. [66]--80.

Persistent URL: http://dml.cz/dmlcz/700997

NEW RESULTS IN UNIFORM TOPOLOGY

V.A.Efremovič (Jaroslavl) and A.G.Vaĭnšteĭn (Moscow)

0. This paper contains a short review of some achievements attained in the last 5 years in uniform topology (or proximity geometry) of geodesic spaces by a group of mathematicians of Moscow, Voronež, Gorky and Jaroslavl. We remind that a geodesic space (g.s) (X, ρ) is a complete metric space such that every two points $x, y \in X$ may be connected by a rectilinear segment (i.e., an isometric image of a closed interval in R).

These results may be naturaly grouped into three directions (which are, however, closely connected):

1. Proximity invariants of geodesic spaces.
2. Extension of equimorphisms (i.e., uniform isomorphisms) of geodesic spaces.
3. Applications of uniform topology to differentiable dynamics.

Our main point is to describe the results and to discuss the related unsolved problems, while the proofs will be merely outlined.

Most of the results listed below being obtained in the closest co-operation with D.A.DeSpiller, L.M.Lerman, É.A.Loginov and E.S.Tihomirova, we are deeply grateful to them.

1. Proximity Invariants in Geodesic Spaces

That is one of the oldest problems in proximity geometry of g.s. We cannot even list the main results obtained here and only refer to papers [1 - 9]. But we would like to mention two circumstances concerning this topic; first, that most of the proximity invariants known are, in fact, invariants of the uniform homotopy type (u.h.t., see [7,9]). Second, that these invariants have been successfully used not only to distinguish homeomorphic and not equimorphic g.s., but also to solve some problems concerning uniform retraction [8] and behaviour of equimorphism at infinity [5,6].

2. Extension of Equimorphism of Geodesic Spaces

Motivation. The investigation of this topic has been started on the base of the results concerning the behaviour of equimorphisms of the n-dimensional Lobačevsky space Λ^n (see [10]):

Theorem 2.1. Let B^n be an n-dimensonal ball representing the Poincaré model of Λ^n. Then every equimorphism of Λ^n can be extended to a homeomorphism of the closed n-ball $\bar{B}^n$.

Here we shall discuss a recent generalisation of this theorem [11-13] .

Remarks. a) The points of the Poincaré sphere at infinity S^{n-1} ($= \bar{B}^n \setminus B^n$) are naturally identified as rectilinear rays starting at the origin 0 of the Poincaré model B^n . Below we shall also deal with the constructions using rectilinear rays.
b) Let us consider two uniformities in B^n : one of them may be called the standard uniformity and corresponds to the inclusion $B^n \subset \bar{B}^n$, while the other is defined by the Poincaré metric

$$ds^2 = (1 + r^2)^{-2} (dr^2 + r^2 \, d\theta^2)$$

and will be called the Lobačevsky uniformity. Thus Theorem 2.1 is equivalent to the following

Theorem 2.1'. Every homeomorphism of B^n which is an equimorphism relative to Lobačevsky uniformity, has the same property relative to the standard uniformity.

This approach suggests the follwoing unsolved

Problem: To describe effectively all pre-compact uniformities compatible with the topology of B^n which may replace the standard uniformity in Theorem 2.1 .

To generalize Theorem 2.1 we shall need the following notions.

Fiberings and Compactifications. Let X be a g.s. such that every bounded subset of X is totally bounded. Let us suppose that, for a certain open bounded $Q \subset X$, $X \setminus Q$ can be fibered into a set Ξ of rectilinear rays ξ in the following sense: Ξ is given a compact topology and the map $x \mapsto (\xi, h)$ is a homeomorphism between $X \setminus Q$ and $\Xi \times R^+$, where $x \in X \setminus Q$, $\xi = px$ is the ray of the fibering running through x and $h = h(x)$ is the distance between x and the starting point of ξ .

Definition 2.2. A sequence $x_n \in X$ will be called Ξ-directed to the ray $\xi \in \Xi$ iff $h(x_n) \to \infty$ and $px_n \to \xi$.

Definition 2.3. A compactification $\bar{X}_{\Xi}$ corresponding to a fibering Ξ is the space $X \cup \Xi$, the topology of this union being defined by convergence of Ξ -directed sequences.

Definition 2.4. A fibering Ξ in a g.s. X will be called exact iff for every other fibering Ξ' in X the identity map $id_X : X \to X$ may be extended to a continuous map $\tau : \bar{X}_{\Xi'} \to \bar{X}_{\Xi}$.

Obviously $\bar{X}_{\Xi_1}$ and $\bar{X}_{\Xi_2}$ are homeomorphic if Ξ_1 and Ξ_2 are both exact; therefore any compactification of X corresponding to an exact fibering will be denoted by $\bar{X}$.

Remark: It is easy to see that Definition 2.4 is equivalent to the following condition: for every fibering Ξ in X every Ξ'-directed sequence is also Ξ -directed.

Examples of Fiberings. 1) Let $X = M^k$ be a complete simply connected Riemannian manifold of non-positive 2-curvature, $Q \subset X$ an arbitrary ball. We can obtain an exact fibering Ξ in X consisting of all rectilinear rays orthogonal to ∂Q . $\bar{X}_{\Xi}$ is obviously homeomorphic to the k-ball B^k .
2) Let $X = \Lambda^2$ be the Lobačevsky plane, $Q \subset \Lambda^2$ be a bounded open subset with C^1 -smooth boundary which consists of two horocycle arcs, belonging to a pair of different horocycles orthogonal to the same straight line in Λ^2 , and connected by a pair of circle arcs. The fibering Ξ consisting of all the rays ξ orthogonal to ∂Q is not exact.

Equivalent and Separated Sequences. Sufficient Conditions for Exactness of a Fibering. Since Ξ is compact, it posesses the only uniformity $\mathcal{U}$ compatible with its topology.

Definition 2.5. Let Ξ be a fibering in a g.s. X, $x_n, y_n \in X$ two sequences going to infinity when $n \to \infty$. We will call them Ξ -equivalent ($x_n \sim y_n$ rel. Ξ) iff $\forall U \in \mathcal{U}\ \exists N \in Z^+ : \forall n > N\ (x_n, y_n) \in U$.

If, for every increasing sequence of integers n_k, $x_{n_k} \nsim y_{n_k}$ rel.

Ξ , then we will call them Ξ-separated ($x_n \nsim y_n$ rel. Ξ).
The following statement is quite obvious.

Proposition 2.6. Let Ξ be an exact fibering and Ξ' an arbitrary one. Then a) $x_n \sim y_n$ rel. Ξ' implies $x_n \sim y_n$ rel. Ξ ; b) $x_n \nsim y_n$ rel. Ξ implies $x_n \nsim y_n$ rel. Ξ' .

Thus we shall call two sequences equivalent if they are equivalent relative to any exact fibering.

Most of the results listed below are based upon the following simple but important

Lemma 2.7. Let Ξ be an arbitrary fibering in a g.s. X, $x_n, y_n \in X$, $x_n \sim y_n$ rel. Ξ . Then the entire rectilinear segment $x_n y_n$ tends to infinity with n, i.e. every compactum $K \subset X$ intersects only a finite number of these segments.

Sketch of the proof. Suppose $x_n \sim y_n$ rel. Ξ , $z_n \in \overline{x_n y_n}$, $h(z_n) \leq$ $\leq c \ \forall n$. Let $d = \operatorname{diam}\{\Xi \times \{0\}\}$, $\xi_n = px_n$, $\eta_n = py_n$. Then for some n the distance $\rho((\xi_n, c + d + 1),(\eta_n, c + d + 1)) < 1$. Obviously $\rho(x_n, y_n) \geq h(x_n) + h(y_n) - 2(c + d)$, but, on the other hand, $\rho(x_n, y_n) < h(x_n) + h(y_n) - 2(c + d) - 1$.

Lemma 2.7 implies the following

Theorem 2.8. a) If a fibering Ξ in a g.s. X satisfies the following Condition A , then Ξ is exact.

Condition A . For every pair of sequences x_n, y_n, $x_n \nsim y_n$ rel. Ξ , there exists a fixed ball $K \subset X$ such that $\overline{x_n y_n} \cap K \neq \emptyset$.

b) If some fibering Ξ in X satisfies Condition A , then any exact fibering in X also satisfies the same condition.

Thus we may say that the space X satisfies Condition A , if any fibering in X satisfies this condition.

Geometry implied by Condition A may be illustrated by the following

Theorem 2.9. If a g.s. X satisfies Condition A , then every two points $x, y \in \overline{X}$ can be connected by a rectilinear path (relative to the metric ρ).

<u>Sketch of the proof.</u> The only case to study is that of two points $x, y \in \overline{X} \setminus X$. Let $x_n, y_n \in X$, $x_n \to x$, $y_n \to y$ in $\overline{X}$. Then $x_n \sim y_n$ rel. Σ and thus Condition A implies the existence of a point $z_n \in x_n y_n$ such that $h(x_n)$ is bounded. It is easy to choose an increasing sequence of integers n_k such that $\overline{x_{n_k} y_{n_k}}$ tends to a straight line $\Gamma \subset X$ linking x with y in $\overline{X}$.

<u>Examples.</u> 3) The Lobačevsky space Λ^n satisfies Condition A ; Theorem 2.9 describes the <u>limiting line effect</u>.

4) Let X be a surface of revolution with the Riemannian metric given by the formula

$$ds^2 = dr^2 + f^2(r)\, d\theta^2,$$

$$0 \le \theta \le 2\pi, \quad r \in R^+ ;$$

a) if $f(r) = r^a$, $a \ge 1$, than X satisfies Condition A iff $a > 1$;

b) if $f(r) = r \log r$, then X satisfies Condition A .

<u>The Notion of a λ-Chain. Equimorphisms of Spaces with Exact Fiberings.</u> To describe the behaviour of the image of a rectilinear ray under an arbitrary equimorphism, we shall introduce the following

<u>Definition 2.10.</u> Let $\lambda \in R$, $\lambda > 0$, $x, y \in X$. A finite set $Z = \{z_j \in X \mid 0 \le j \le m\}$ will be called a <u>λ-chain, linking x with y</u>, iff

a) $z_0 = x$, $z_m = y$; b) $1 \le \rho(z_{j-1}, z_j) < 2$; c) $\sum_{j=k}^{1} \rho(z_{j-1}, z_j) \le$

$$\le \lambda \rho(z_{k-1}, z_1) \quad \forall k, 1 : 1 \le k \le 1 \le m .$$

The properties of uniformly continuous maps proved in [1] imply the following important

<u>Lemma 2.11.</u> Let X be a g.s., $f : X \to X$ an equimorphism. Then there exist $\lambda \ge 1$, $c > 0$ such that, for every $x, y \in X$, either $\rho(x,y) < c$ or there exists a λ-chain $Z \subset f(\overline{xy})$ linking $f(x)$ with $f(y)$.

Now we shall strengthen Condition A replacing rectilinear segments by λ-chains. Thus we obtain the following

<u>Condition B.</u> For an arbitrary $\lambda \leq 1$ and for an arbitrary sequence of λ-chains $Z_n = \{x_n = z_{n0}, z_{n1}, \ldots, z_{nm_n} = y_n\}$ such that $x_n \sim y_n$, there exists a fixed ball $K \subset X$ which intersects all these chains.

<u>Remark.</u> Obviously Condition A may be replaced by Condition B in Theorem 2.8.

Now we shall formulate the central results of this section.

<u>Theorem 2.12.</u> If a g.s. X satisfies Condition B, then every equimorphism $f : X \to X$ preserves the equivalence relation between the sequences in X.

<u>Theorem 2.13.</u> If a g.s. X satisfies Condition B, then every equimorphism $f : X \to X$ can be extended to a homeomorphism $\overline{f} : \overline{X} \to \overline{X}$,

Both these theorems are obviously implied by Lemmas 2.7 and 2.11. We shall call stable a g.s. X satisfying the conclusion of Theorem 2.13.

<u>A Metric Condition for Stability.</u> To check whether Condition B is satisfied, we shall introduce the following construction. Let $d(\cdot, \cdot)$ be any metric on Ξ compatible with its topology. We determine a function $\mu : R^+ \to R^+$ by the following formula:

$$\mu(t) = \inf \{ \rho(x,y)/d(\xi, \eta) \mid \xi = px, \ \eta = py, \ h(x) \geq t, \ \rho(x,y) \geq 2 \} . \tag{3}$$

<u>Theorem 2.14.</u> If the infinite integral

$$J = \int_1^\infty (\mu(t)^{-1})dt \tag{4}$$

converges, then Ξ satisfies Condition B.

<u>Sketch of the proof.</u> Suppose that $Z_n = \{x_n = z_{n0}, \ldots, z_{nm_n} = y_n\}$ is a sequence of λ-chains, Z_n goes to infinity with n, $x_n \sim y_n$ rel. Ξ. Let us denote $\xi_n = px_n$, $\eta_n = py_n$, $h_n = h(x_n)$, $h_n^k = (z_{nk})$, $0 \leq k \leq m_n$, $L = 3\lambda + 1$; we may suppose that $h_n \leq h_n^k$. For all n

sufficiently large

$$h_n^k \geq (h_n + k)/\ L \quad ; \qquad (5);$$

on the other hand

$$d(\xi_n, \eta_n) \leq \sum_{k=1}^{m_n} \rho(z_{n,k-1}, z_{nk}).(\mu(h_n^k - 2))^{-1} \ ; \qquad (6);)$$

since μ does not decrease (5) and (6) imply

$$d(\xi_n, \eta_n) \leq \int_{h_n/L-2}^{\infty} (\mu(t))^{-1}\, dt \ ;$$

thus $d(\xi_n, \eta_n) \rightarrow 0$, since J converges.

<u>Examples.</u> 5) Λ^n satisfies Condition B since μ grows exponentially; this proves Theorem 2.1.
6) Let X be a surface of revolution described in Example 4 a); if $a > 1$ then X satisfies Condition B .
7) Let X be a surface revolution described in Example 4 b) ; it is easy to see that the map $f : X \rightarrow X$ given by the formula

$$f(r, \theta) = \begin{cases} (r, \theta), & r \leq e \\ (r, \theta + \log\log r), & r > e \end{cases}$$

is an equimorphism of X . Since f cannot be extended to $\bar{X}$, X does not satisfy Condition B ; thus A does not imply B .

<u>Exactness of the "Central" Fibering.</u> To prove that the "central" fibering of Example 1 is exact, we shall introduce a new condition of exactness.

<u>Theorem 2.15.</u> If a fibering Ξ in a g.s. X satisfies the following Condition C , then Ξ is exact.

<u>Condition C.</u> For a fibering Ξ in a g.s. X

(C1) If $x_n, y_n \in X$, $h(x_n) \to \infty$, $\rho(x_n, y_n) <$ const, then

$x_n \sim y_n$ rel. Ξ.

(C2) Every rectilinear ray in X is Ξ-directed.

(C3) If a rectilinear ray $\Gamma \subset X$ is Ξ-directed to a $\xi \in \Xi$ then the distance between a point $x \in \Gamma$, going to infinity along Γ, and ξ does not increase.

The classical results by J.Hadamard and E.Cartan obviously imply that the "central" fibering satisfies Condition C.

The proof of Theorem 2.15 is based upon the following trivial

<u>Lemma 2.16.</u> Let Ξ be an arbitrary fibering in a g.s. X, $x_n \in X$, let x_n be Ξ-directed to $\xi \in \Xi$, $\xi_n = px_n$. Then $\forall z \in \xi, \forall \varepsilon > 0$ $\exists N = N(z, \varepsilon) : \forall n > N \quad \exists y_n \in \xi_n : \rho(y_n, z) < \varepsilon$.

<u>Sketch of the proof of Theorem 2.15.</u> Suppose that Ξ and Ξ' are two fiberings in a g.s. X, Ξ satisfies Condition C, $x_n \in X$, x_n is Ξ-directed to $\xi' \in \Xi'$, $\xi'_n = p'x_n$; then ξ'_n, ξ' are Ξ-directed to $\xi_n, \xi \in \Xi$, respectively. Using Lemma 2.16 we may find a sequence $y_n \in \xi'_n$, $h(y_n) \to \infty$, such that $\rho(y_n, \xi') \to 0$. (C3) implies that $\rho(y_n, \xi) <$ const for $\forall n$. Let $u_n, v_n \in \xi_n$ be the nearest points to x_n, y_n, respectively, and let $w_n \in \xi$ be the nearest point to y_n. Then $y_n \sim w_n$ rel. Ξ, $x_n \sim u_n$ rel. Ξ and $v_n \sim y_n$ rel. Ξ. Since obviously $u_n \sim v_n$ rel. Ξ we have $x_n \sim w_n$ rel. Ξ.

<u>Remark.</u> We do not know if Theorem 2.8 b) may be extended to Condition C; this is true in any complete, simply connected Riemannian manifold of non-positive 2-curvature.

<u>Adjoining Results.</u> These concern mainly two problems: a) equimorphisms of Λ^n and strong rigidity theorems, see [9, 14-16]; b) equimorphisms of R^n, see [17], where also some "rigidity theorems" may be obtained. We shall just mention one of them which is in fact rather trivial.

Set M_α, $\alpha \in R$, a 3-manifold with locally Euclidean Riemannian metric obtained from $\{(x,y,z)\in R^3 \mid 0\le z\le 1\}$ by identifying $(x,y,0)$ with $(x\cos\alpha - y\sin\alpha,\ x\sin\alpha + y\cos\alpha,\ 1)$

Theorem 2.17. M_α and M_β are equimorphic iff $|\alpha|\equiv|\beta| \bmod 2\pi$.

3. Applications of Uniform Topology to Differentiable Dynamics

Results listed below have been obtained in collaboration with L.M.Lerman.

Motivation. The problem has been suggested by the new approach developed by L.M.Lerman and L.P.Šilnikov [18]. To study time-dependent vector fields on a compact smooth manifold M, they considered $M\times R$ as posessing the Cartesian product uniformity. Using this uniformity and the 1-foliation into integral curves of a time dependent vector field $\vec{v}(x,t)$, $x\in M$, $t\in R$, $\vec{v}\in T_xM$, most of the qualitative properties of such a vector field may be expressed. The triple ($M\times R$, Cartesian product uniformity, 1-foliation into integral curves of a time-dependent bounded vector field $\vec{v}$) will be called an integral portrait of $\vec{v}$. Equivalence of two vector fields has been defined in such a way, and some structural stability theorems have been proved.

Here uniformity is essential, since any time dependent vector field generates an integral portrait topologically equivalent to the trivial 1-foliation $L : M\times R = \bigcup_{x\in M} (\{x\}\times R)$.

Let M be a compact C^∞-manifold, $f\in \mathrm{Diff}^1(M)$. We are going to produce a triple (topological space $M\times R$, a certain compatible uniformity in $M\times R$, 1-foliation L) such that most of the qualitative properties of f may be studied using this triple. All we have to do is to define a compatible uniformity in $M\times R$. Define $f : M\times R \to M\times R$ by the following formula:

$$\hat{f}(x,t) = (f(x),\ t-1),\ x\in M,\quad t\in R .$$

Lemma 3.1. There exists a weakest compatible uniformity $\mathcal{U}_f$ on $M\times R$ such that $\hat{f}$ generates a uniformly equicontinuous group relative to $\mathcal{U}_f$.

<u>Sketch of the proof.</u> $M \times R/\{z = f(z)\}$ is a compact C^∞-manifold M_f, and the natural projection $M \times R \to M_f$ is a (smooth) covering. The only compatible uniformity in M_f can be naturally <u>lifted up to</u> $M \times R$ to produce the desired uniformity $\mathcal{U}_f$.

<u>Remark.</u> We can easily see that $\mathcal{U}_f$ is determined by a certain C^∞ Riemannian metric on $M \times R$.

Later the pair $(M \times R, \mathcal{U}_f)$ will be abbreviated to $\widetilde{M}_f$ and the triple $(M \times R, \mathcal{U}_f, L)$ to the pair $(\widetilde{M}_f, L_f)$. $\widetilde{M}_f$ will be called a non-autonomous suspension over f .

<u>Definition 3.2.</u> $f, g \in \mathrm{Diff}^1(M)$ are called δ-<u>equivalent</u> iff there exists an equimorphism $\Phi: \widetilde{M}_f \to \widetilde{M}_g$ such that $\Phi(L_f) = L_g$.

This definition is motivated by the following trivial

<u>Proposition 3.3.</u> Two leaves $L(x), L(y) \in L_f$ such that $L(x) \ni (x,0)$, $L(y) \ni (y,0)$, are in proximity relative to $\mathcal{U}_f$ iff the same holds for the orbits of x and y under f relative to the only compatible uniformity on M .

<u>Classification of Non-Autonomous Suspensions.</u> Now it seems quite natural to suggest the following

<u>Problem:</u> When are M_f and M_g, f,g $\mathrm{Diff}^1(M)$, equimorphic? This problem may be considered as directly related to those of the first section of this paper, but we have described above its relation to dynamics.

To illustrate the problem, let us consider some

<u>Examples.</u> 1) Let $M = S^n$ be an n-dimensional sphere. Since the group Homeo (S^n) consists of only two path connection components, every $f \in \mathrm{Diff}^1(M)$ gives a non-autonomous suspension equimorphic to $S^n \times R$.

2) Let M be an arbitrary smooth compact manifold, and let f generate an equicontinuous group of homeomorphisms of M . Then $\widetilde{M}_f$

is equimorphic to $M \times R$.

Remarks: a) It is easy to prove that $f \in \mathrm{Diff}^1(M)$ is δ-equivalent to id_M iff f generates an equicontinuous group.
b) The previous remark is of special interest when compared to Theorem 2.17.
3) Let $M = T^2$ be a 2-torus, f_0, f_1, f_2, f_3 its linear diffeomorphisms corresponding to matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} .$$

$\widetilde{M}_{f_j}$ will be abbreviated here to $\widetilde{M}_j$. To distinguish $\widetilde{M}_j$ neither uniform homology [3,4] nor the volume invariant [1] are sufficient. Using the ideas of [2] one may check, however, that $\widetilde{M}_i$ is not equimorphic to $\widetilde{M}_j$, $i = 0,1$, $j = 2,3$. But this is not sufficient to distinguish $\widetilde{M}_0$ from $\widetilde{M}_1$ and $\widetilde{M}_2$ from $\widetilde{M}_3$.

Uniform Homotopy Type of Non-Autonomous Suspensions. Luckily, uniform homotopy type of non-autonomous suspensions over diffeomorphisms can be studied in rather an explicit way (see [19,20]).

Theorem 3.4. $\widetilde{M}_f, \widetilde{M}_g$ have the same u.h.t. iff there exist $n, m \in Z\setminus\{0\}$, $\varphi : M \to M$ a homotopy equivalence, such that $\varphi \circ f^m$ and $g^n \circ \varphi$ are homotopic.

The proof will be outlined below; first we shall list some corollaries.

Corollary 3.5. If $\widetilde{M}_f$ is equimorphic to $M \times R$, then there exists a $k \in Z\setminus\{0\}$ such that f^k is homotopic to the identity map.

Corollary 3.6. If $f \in \mathrm{Diff}^1(M)$ is Anosov (see [21]), then $\widetilde{M}_f$ is not equimorphic to $M \times R$.

This corollary is implied by the previous one and the following important

Theorem 3.7. No Anosov diffeomorphism of a compact manifold is homotopic to the identity map.

The proof of this theorem belongs completely to algebraic topology and is therefore omitted here.

Corollary 3.8. Let $M = T^s$ be an s-torus, $f,g \in GL(s,Z)$ its linear diffeomorphisms. Then T^s_f and T^s_g are equimorphic iff there exist $n,m \in Z\setminus\{0\}$ and $h \in GL(s,Z)$ such that $h \circ f^m = g^m \circ h$.

To prove this corollary one should apply Theorem 3.4 and consider the natural action of f,g,φ in the fundamental group of M. Notice, that this corollary implies the non-existence of an equimorphism between $\widetilde{M}_i$ and $\widetilde{M}_j$, $i \neq j$, described above in Example 1.

The purely algebraic problem, involved now with the uniform classification of non-autonomous suspensions over linear diffeomorphisms of tori, seems rather difficult, and we do not know the complete solution even for $s = 2$!

Corollary 3.8 may be easily generalized to the algebraic diffeomorphisms of infra-nilmanifolds (see [22]). Thus the problem of uniform classification of non-autonomous suspensions over all Anosov diffeomorphisms, known to us, is reduced to a purely algebraic problem (cf. [23]).

Sketch of the Proof of Theorem 3.4. The proof of Theorem 3.4 is based upon two propositions listed below.

Lemma 3.9. Let K be a compactum, N a Riemannian manifold, A an arbitrary infinite set of indices, $\{f_\alpha : K \to N \mid \alpha \in A\}$ a precompact set in $C(K,N)$ (relative to the usual topology). Then there exists $\alpha^* \in A$ such that for an infinite subset $B \subset A$ f_{α^*} and f_β are homotopic whenever $\beta \in B$.

This is quite obvious, but implies some interesting corollaries, e.g.

Corollary 3.10. Let M be a compact Riemannian manifold, and let $f \in H_0meo(M)$ generate an equicontinuous group. Then f^k is homotopic to id_M for some integer $k \neq 0$.

Lemma 3.11. Let $f,g \in Diff^1(M)$, $\Phi : M_f \to M_g$ be a uniform homotopy (u.h.) equivalence, $M_0 = M \times \{0\} \subset M_f$. Then there exists a sequence of integers $l(k)$ such that

a) $\hat{g}^{l(k)} \circ \bar{\Phi} \circ f^k|_{M_0} : M \to \widetilde{M}_g$ is a precompact set of mappings, $k \in Z$; b) there exists $L \geq 1$ such that $L^{-1} \leq |l(k)/k| \leq L$.

To prove this lemma one should use the properties of u.h. equivalences of g.s. described in [9] ; these properties imply that $l(k)$ may be defined by the formula $(M \times \{t\} \subset \widetilde{M}_g)$:

$$l(k) = - \text{entier} (\inf \{t \mid M \times \{t\} \cap \bar{\Phi} \circ \hat{f}^k(M_0) \neq \phi \}).$$

As far as the necessity of its conditions is concerned, Theorem 3.4 is obviously implied by the two lemmas above. Sufficiency is obvious.

Adjoining Results. These concern mainly necessary conditions for two diffeomorphisms to be δ-equivalent. We shall mention two theorems to illustrate this approach.

Theorem 3.12. Let $f, g \in \mathrm{Diff}^1(M)$, and let f and g be δ-equivalent; if topological enthropy of f is 0 , then the same holds for g .

Theorem 3.13. Under the assumptions of the above theorem there exist integers $m, n \neq 0$ and $\varphi \in \mathrm{Homeo}(M)$ such that $\varphi^{-1} \circ g^{-n} \circ \varphi \circ f^m$ belongs to the path connection component of id_M in Homeo (M).

We hope to achieve further results in this direction involving structural stability of wider classes of time dependent diffeomorphisms and vector fields.

References

[1] V.A.Efremovič: Proximity Invariants,(Russian) Ivanov.Gos. Ped. Inst.Učen. Zap. 31 (1963), vyp. mat.,74-81. MR 47# 9505.

[2] A.S.Švarc: A volume invariant of coverings. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 32-34. (Russian) MR 1 p. 18 .

[3] E.S.Tihomirova: Uniform homology groups. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 26 (1962), 865-876. MR 27# 6263.

[4] E.S.Tihomirova: The spectrum of uniform homologies. Dokl. Akad. Nauk SSSR 191 (1970), 1235-1237 (Russian); translated as Soviet Math. Dokl. II (1970), 538-541. MR 44 # 1021.

[5] V.A. Efremovič, E.S.Tihomirova: Volumes and capacities in proximity geometry. (Russian) Dokl. Akad. Nauk SSSR 214 (1974), 29-32. MR 50 # 3167.

[6] E.S.Tihomirova: Invariants of equimorphisms in Riemannian manifolds. All -Union Sci. Conf. in Non-Euclidean Geometry" 150 Years of Lobačevsky Geometry", Abstracts, 195, Kazan (1976).

[7] A.G.Vaĭnšteĭn: Uniform homotopy and proximity invariants.(Russian. English summary) Vestnik Moskov. Univ. Ser. I Mat. Meh. 25 (1970), no. 1, 17-20.MR 43 # 5531.

[8] A.G.Vaĭnšteĭn: Uniform homology group and uniform retracts.(Russian.English summary) Vestnik Moskov. Univ. Ser. I. Mat. Meh.26 (1971), no. 4, 59-62. MR 45 #4407.

[9] A.G.Vaĭnšteĭn: Equimorphisms of geodesic spaces and uniform homotopy. (Russian) Vestnik Moskov. Univ. Ser. I Mat. Meh. 28 (1973), no. 3, 66-69.

[10] V.A.Efremovič, E.S.Tihomirova: Equimorphisms of hyperbolic spaces. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 28 (1964), 1139-1144. MR 29 # 6374.

[11] V.A.Efremovič, È.A.Loginov, E.S.Tihomirova: Equimorphisms of Riemannian manifolds. (Russian) Dokl. Akad. Nauk SSSR 197 (1971), 25-28. MR 44 # 2171.

[12] V.A.Efremovič, È.A.Loginov: Extension of equimorphisms in Riemannian geometry. (Russian) Uspehi Mat. Nauk 27 (1971), no.6 , 237-238.

[13] A.G.Vaĭnšteĭn, V.A.Efremovič, È.A.Loginov: Infinity in geodesic spaces. (Russian) Dokl. Akad. Nauk SSSR 220 (1975), no.3, 505-508.

[14] D.A.De-Spiller: Equimorphisms, and quasiconformal mappings of the absolute. (Russian) Dokl. Akad. Nauk SSSR 194 (1970), 1006-1009. MR 42 # 6698.

[15] G.A.Margulis: The isometry of closed manifolds of constant negative curvature with the same fundamental group. (Russian) Dokl. Akad. Nauk SSSR 192 (1970), 736-737. MR 42 # 1012.

[16] G.D.Mostow: Strong rigidity of locally symmetric spaces. Princeton Univ. Press, 1973. MR 52 # 5874.

[17] A.G.Vaĭnšteĭn: Uniform classification of isometries in Euclidean and Lobačevsky spaces. (Russian) Uspehi Mat. Nauk 29 (1974),no.5, 217-218,Erratum, ibidem 30 (1975), no. 4, 301.

[18] L.M.Lerman, L.P.Šilnikov: The classification of structurally stable nonautonomous second order systems with a finite number of cells. (Russian) Dokl. Akad. Nauk SSSR 209 (1973), 544-547. MR 48 # 3077.

[19] A.G.Vaĭnšteĭn, L.M.Lerman: Proximity geometry and non-autonomous suspensions over diffeomorphisms. (Russian) Uspehi Mat. Nauk 31 (1976), no. 5, 231-232.

[20] A.G.Vaĭnšteĭn, L.M.Lerman: Uniform topology and time dependent flows. All-Union Sci.Conf. in Non-Euclidean Geometry " 150 Years of Lobačevsky Geometry " , Abstracts, 38, Kazan (1976).

[21] D.V.Anosov: Geodesic flows on closed Riemannian manifolds of negative curvature. (Russian) Trudy Mat. Inst. Steklov 90 (1967), 209 pp. MR 36# 7157.

[22] Z.Nitecky: Differential dynamics. MIT Press, 1971.

[23] A.Manning: There are no new Anosov diffeomorphisms on tori. Amer. J. Math. 96 (1974), 422-429. MR 50# 11324.