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ORLICZ SPACE - VALUED MARTINGALES

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Let E be a metric linear space and let $(I,\mathcal{B},l)$ be the unit interval with the Borel -field and the Lebesgue measure l. It is well known that if E is not locally convex then there exists a sequence $f_n = \sum_1^{k_n} x_j^{(n)} \chi_{A_j^{(n)}}$, where $x_j^{(n)} \in E$, $A_j^{(n)} \in \mathcal{B}$, uniformly tending to zero and such that $\sum_1^{k_n} x_j^{(n)} l(A_j^{(n)})$ is not convergent to zero. So, in such spaces, the classical notion of Bochner integral cannot be used. We present an approach to the integration theory for mappings with values in Orlicz spaces (non locally convex, in particular) based on the notion of measurable stochastic process.

Let $(T,\mathcal{F},m)$ be a finite, separable measure space and let $\mathfrak{X}$ be the space of all $\mathcal{F}$-measurable real functions on T. Let ϕ be a Young function, that is, subadditive, nondecreasing continuous real function defined for $u \geqslant 0$ such that $\phi(t)=0$ iff $t=0$. Put

$$\|x\|_\phi = \int_T \phi(|x(t)|)\, m(dt) \quad , \quad x \in \mathfrak{X}.$$

Let L_ϕ be the set of all $x \in \mathfrak{X}$ such that $\|x\|_\phi < \infty$. L_ϕ is a linear space under usual addition and scalar multiplication and $\|\cdot\|_\phi$ is a (usually non-homogeneous) seminorm on L_ϕ. Under obvious identification $(L, \|\ \|_\phi)$ is a complete metric linear space called Orlicz space.

Let (Ω,Σ,μ) be a probability space. If $\xi(\omega,t)$ is a real function defined on $\Omega\times T$, measurable with respect to $\Sigma\times\mathcal{F}$ and such that $\xi(\omega,\cdot)\in L_\phi$ μ- a.e., then ξ induces, in a natural way, a mapping $\tilde{\xi}$ from Ω into L_ϕ, measurable with respect to Σ and the Borel σ-algebra $\mathcal{B}_{L_\phi}$ in L_ϕ. On the other hand, if X is a measurable mapping from (Ω,Σ) into $(L_\phi,\mathcal{B}_{L_\phi})$ then exists a $\Sigma\times\mathcal{F}$-measurable mapping ξ such that $\tilde{\xi}=X$ μ- a.e. [1, 4]. So, instead of Borel measurable mappings from Ω into L_ϕ we can consider product measurable real functions defined on $\Omega\times T$.

Now, let $\mathcal{L}_\phi$ be the set of all real $\Sigma\times\mathcal{F}$-measurable functions f defined on $\Omega\times T$ such that $[f]_\phi<\infty$, where

$$[f]_\phi = \int_T \phi(E|f|)\, dm$$

and E denote the expectation. It is easy to see that $(\mathcal{L}_\phi,[\ ]_\phi)$ is a complete metric linear space (under usual identification).

By $\mathcal{L}$ we denote the linear space of all functions f of the form

$$f(\omega,t) = \sum_{i}^{m} x_i(t)\,\chi_{A_i}(\omega)$$

where $x_i \in L_\phi$ and $A_i \in \Sigma$. It is not hard to see that $\mathcal{L}$ is a dense linear subspace of $\mathcal{L}_\phi$. If Σ_0 is a sub-σ-algebra of Σ and $f(\omega,t) = \sum x_i(t)\,\chi_{A_i}(\omega) \in \mathcal{L}$ we define

$$If = \sum_{i}^{n} x_i\,\mu(A_i \mid \Sigma_0)$$

where $\mu(\cdot \mid \Sigma_0)$ denotes the conditional probability with respect to Σ_0. Observe that $E(|I(f)|) \leqslant E(I|f|) = E|f|$, whenever $f \in \mathcal{L}$. Hence we have

$$[I(f)]_\phi \leqslant [f]_\phi .$$

Therefore I is a continuous linear mapping from $\mathcal{L}$ into $\mathcal{L}_\phi$. Since $\mathcal{L}$ is dense in $\mathcal{L}_\phi$, we can extend I to $\mathcal{L}_\phi$. This extension will be called the conditional expectation operator and will be denoted by the same symbol. I has all usual properties of conditional expectation.

Let Σ_i be an increasing sequence of sub-σ-algebras of Σ. Let f_i be a sequence of elements of $\mathcal{L}_\phi$. $\{f_i, \Sigma_i, i \in N\}$ is called an L_ϕ-valued martingale if f_i is $\Sigma_i \times \mathcal{F}$-measurable and

$$i \leqslant j \quad \Rightarrow \quad E\{f_j \mid \Sigma_i\} = f_i .$$

The following analogon of the mean convergence theorem (due to Chatterji [3], in the case of Banach-space valued martingales) holds

Theorem. Let $\{f_n, \Sigma_n, n \geqslant 1\}$ be an L_ϕ-valued martingale such that

$$f_n = E\{f \mid \Sigma_n\}$$

where $f \in \mathcal{L}_\phi$. Then

$$\lim [f_n - f_\infty]_\phi = 0$$

where $f_\infty = E\{f \mid \Sigma_\infty\}$ and Σ_∞ is the σ-algebra generated by $\bigcup_m \Sigma_n$ ($E\{\cdot \mid \mathcal{A}\}$ denotes the conditional expectation operator with respect to $\mathcal{A}$).

This theorem has applications in the probability theory on Orlicz spaces [2].

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