Jiří Vinárek

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ON SUBDIRECT IRREDUCIBILITY IN SEMIREGULAR CATEGORIES

by

JIŘÍ VINÁREK

Sometimes an important subcategory of a given concrete category can be characterized by non-existence of a subobject or all the subobjects of a given type. E.g. the category of reflexive directed graphs contains all the directed graphs which do not contain any one-point graph without a loop as a subgraph. The similar situation is for antireflexive graphs, symmetric graphs, graphs without triangles. T_0-topological spaces are all the topological spaces which do not contain the two-point indiscrete space as a subspace, torsion groups are all the groups which do not contain the additive group of integers as a subgroup ,etc.

For an object A of a concrete category $(\underline{C},U)$ denote by $A\neg(\underline{C},U)$, shortly $A\neg\underline{C}$, the full concrete subcategory of $(\underline{C},U)$ generated by all the objects B such that there is no subobject $A \rightarrowtail B$, and by the terminal object of $\underline{C}$. Every such a category is closed to subobjects. Now, the very natural question arises : for a concrete category characterize all its objects A such that the subcategory $A\neg\underline{C}$ is closed to products. This problem was discussed by A.Pultr and the author of this note in [7]. It was proved that for the most of "every-day life" categories the productivity of $A \urcorner \underline{C}$ is for a finite A equivalent to the natural extension of Birkhoff's concept of subdirect irreducibility to categories. We are going to give some characterizations also for infinite objects.

Definition 1 (cf. [7]). A subobject in a concrete category $(\underline{C}, U)$ is a monomorphism $\mu : A \longrightarrow B$ such that for every $f : UC \longrightarrow UA$ for which there is a $\psi : C \longrightarrow B$ with $U\psi = U\mu \circ f$, there exists a $\varphi : C \longrightarrow A$ with $U\varphi = f$.

Definition 2. Let $(\underline{C}, U)$ be a concrete category. For a set X define a preordered class $\underline{C}UX := (\{A \mid UA = X\}, \prec)$ putting $A \prec B$ iff there is an $\alpha : A \longrightarrow B$ with $U\alpha = 1_X$.

An object A is said to be weakly maximal if for every $B \succ A$ there exists a subobject $A \longrightarrow B$.

An object A is said to be meet-irreducible (weakly meet-irreducible resp.) whenever $A = \bigwedge_{i \in I} A_i$ in $\underline{C}UX$ implies existence of an $i_0 \in I$ such that $A = A_{i_0}$ (such that there exists a subobject $A \longrightarrow A_{i_0}$ resp.).

Definition 3. A concrete category $(\underline{C}, U)$ is said to be semiregular if it has the following properties :

(S 1) U preserves limits.

(S 2) If X is a set and $f : X \longrightarrow UA$ an invertible mapping then there is an isomorphism φ with $U\varphi = f$.

(S 3) If α is an isomorphism and $U\alpha = 1_{UA}$, then $\alpha = 1_A$.

(S 4) Every $\underline{C}UX$ is a set.

(S 5) For every morphism φ there is a subobject decomposition $\varphi = \mu\varepsilon$ with μ a subobject and $U\varepsilon$ onto.

Definition 4. a) An object A of a concrete category ($\underline{C}$,U) is said to be subdirectly irreducible (SI; cf.[1],[7],[8]) if for every subobject

$$\mu : A \longrightarrow \prod_J A_i$$

such that all $p_i\mu$ are onto (p_i are projections), at least one $p_i\mu$ is an isomorphism.

b) An object A is said to be weakly subdirectly irreducible (WSI) if $A \urcorner \underline{C}$ is closed to products.

Theorem 1(cf.[7]) . Every subdirectly irreducible object of a semiregular category is weakly subdirectly irreducible.

Theorem 2. An object A of a semiregular productive category ($\underline{C}$,U) is subdirectly irreducible iff

either A is maximal (in $\underline{C}$U(UA)) and for any monomorphic system (i.e. a system such that $\mu_i\alpha = \mu_i\beta$ for all $i \in I \Rightarrow \alpha = \beta$) there is an $i_0 \in I$ such that μ_{i_0} is a monomorphism,

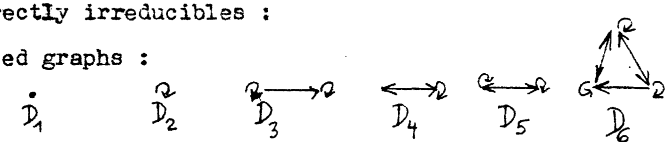
or A is not maximal, it is meet-irreducible and for any φ : A $\longrightarrow$ B with $U\varphi$ not one-to-one there is a ι: A $\longrightarrow$ C, $\iota \neq 1_A$, with $U\iota = 1_{UA}$ and $\bar{\varphi}$: C $\longrightarrow$ B such that $\bar{\varphi}\iota = \varphi$.

Theorem 3. An object A of a semiregular productive category ($\underline{C}$,U) is weakly subdirectly irreducible iff

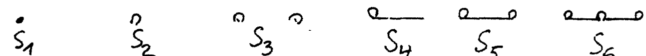
either A is weakly maximal and for any monomorphic system $(\mu_i : A \longrightarrow B_i)_{i\in I}$ there is an $i_0 \in I$ and a subobject $\nu : A \longrightarrow B_{i_0}$, or **A** is not weakly maximal, it is weakly meet-irreducible and for any $\varphi : A \longrightarrow B$ such that $B \in A\neg \underline{C}$ there is a $\iota : A \longrightarrow C$ with $U\iota = 1_{UA}$, $C \in A\neg\underline{C}$, and $\overline{\varphi} : C \longrightarrow B$ such that $\overline{\varphi}\circ\iota = \varphi$.

Using Theorem **2** we can find the following list of weakly subdirectly irreducibles :

A. directed graphs :



B. symmetric graphs :



C. symmetric graphs with loops :

D. symmetric graphs without loops :

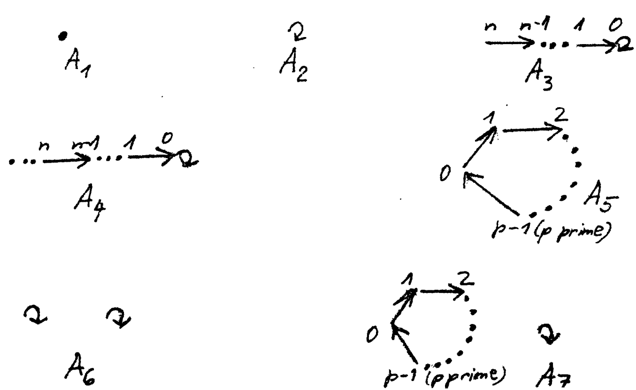
every complete symmetric graph without loops

E. reflexive posets :

F. antireflexive posets :

every antireflexive linearly ordered set

G. partial unary algebras :



H. topological spaces :

topological spaces with cardinality of an underlying set less or equal to 2

All the weakly subdirectly irreducibles in categories from examples A.-H. are subdirectly irreducibles.

Another situation is e.g. in the category Top_1 of T_1-spaces or in the category $Top_{3,5}$ of completely regular T_1-spaces :

Theorem 3. An object A of Top_1 ($Top_{3,5}$ resp.) is SI iff the cardinality of its underlying set is less or equal to 2.

Theorem 4. An object A of Top_1 is WSI iff it is either one- or two-point space, or an infinite space with the maximal T_1-topology.

Theorem 5. An object A of $Top_{3,5}$ is WSI iff it is either one- or two-point space, or it is homeomorphic to a one-dimensional subspace of the unit interval.

Remark. For proving Theorem 5 one have to use the Tychonoff theorem and some characterizations of locally connected metrizable continua.

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