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ON CERTAIN CLASSES OF NORMED LINEAR SPACES

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Let E be a (real) normed linear space and for $x,y \in E$ let us denote by $\tau(x,y) = \lim_{t \to 0^+} t^{-1}(\|x+ty\| - \|x\|)$. It is known that

$$(1) \qquad 0 \leq \tau(x,y) + \tau(x,-y) \leq 2\|y\|$$

If for all $x,y \in S_E$, S_E the unit sphere of E, we have equality in the first inequality of (1), then the space E is called smooth. The upper bound in (1) is a rough one, in the sense that if we ask that equality holds in the second inequality of (1) for all $x,y \in S_E$, then no space $E(\neq \{0\})$ has this property. In [1], Remark 2 we improved this bound showing that for all $x,y \in E$ we have:

$$(2) \qquad \tau(x,y) + \tau(x,-y) \leq 2\,\mathrm{dist}(y, N_E(x))$$

where $N_E(x)$ is the closed linear subspace of E defined by

$$N_E(x) = \{z \in E \mid \tau(x,z) + \tau(x,-z) = 0\}$$

In [1] we studied the spaces E with the property - which we called (Λ) - that for all $x,y \in E$ equality holds in (2). Clearly, the smooth spaces have property (Λ) but they are not the only ones. In the meantime we have observed that the upper bound in (2) can be improved, and correspondingly we have obtained other classes of normed linear spaces. The purpose of this note is to raise some questions concerning these classes of normed linear spaces.

We recall ([1] , Theorems 4-6, Remark 10) that the following spaces have property (Λ): $L^1(T,\mu)$, $C^1(Q,\nu)$, c_0, c, l^∞, R^n endowed with the max norm. Let $C(Q)$ be the Banach space of all real-valued continuous functions over the compact Hausdorff space Q, endowed with the max norm.

Question 1. Has $C(Q)$ property (Λ) ? If the answer is affirmative, then $L^\infty(T,\mu)$ has property (Λ) since this property is invariant under linear isometries. If the answer is negative, then characterize those Q for which $C(Q)$ have (Λ), and the same questions for $L^\infty(T,\mu)$.

Let us consider E as a subspace of its second dual E^{**}. Then by (2) the following relation holds for all $x,y \in E$:

$$(3) \qquad \tau(x,y) + \tau(x,-y) \leq 2\, \mathrm{dist}(y, N_{E^{**}}(x))$$

Since for each $x \in E$ we have $N_E(x) \subset N_{E^{**}}(x)$, the upper bound in (3) is better than the upper bound in (2). Then we can introduce a class of normed linear spaces E with the property - which we call (Λ_{**}) - that for all $x,y \in E$, equality holds in (3). Clearly, if E has property (Λ) then E has property (Λ_{**}), but the class of normed linear spaces with property (Λ_{**}) is larger than the class of spaces with property (Λ). Indeed, we gave in [1] an example of a space E with property (Λ), having a dense subspace E_1 without (Λ); it is easy to show that a dense subspace E_1 of a space E with property (Λ_{**}) has also property (Λ_{**}).

If the space $C(Q)$ has not property (Λ), then we have :

Question 2. The same as Question 1, but for property (Λ_{**}).

If the Banach space $C(Q)$ has property (Λ_{**}) if and only if $C(Q)$ has property (Λ), then we have:

Question 3. Give an example of a Banach space E which has property (Λ_{**}) but not (Λ).

We cannot improve the upper bound in (3) using the higher even duals $E^{(2n)}$ of E, $n \geq 2$. This is a consequence of the fact that the existence of a norm-one linear projection P of E^{**} onto the Banach space E, implies that $\operatorname{dist}(y,N_E(x)) = \operatorname{dist}(y,N_{E^{**}}(x))$ for all $x,y \in E$. This also shows that the example required in Question 3 can not be a dual space (since then property (Λ) is equivalent with (Λ_{**})).

It is not difficult to show that if $\dim E \geq 3$ and all closed hyperplanes of E have property (Λ) ((Λ_{**})) then E has property (Λ) ((Λ_{**})). This result is insignificant if someone answers affirmatively the following question:

Question 4. Let E be a normed linear space with $\dim E \geq 3$. If all closed hyperplanes of E have property (Λ) (resp. (Λ_{**})) is then E smooth? For $\dim E = 3$ the answer is affirmative [1], and an affirmative answer in the finite dimensional case for $n > 3$ will give the following characterization of a (not necessarily finite dimensional) smooth space: If $\dim E \geq n+1$, $n \geq 3$, then E is smooth if and only if each of its n-dimensional subspaces has property (Λ). For $n = 2$ this is true ([1], Theorem

In order to give another improvement of the upper bound in (2), let us denote for $x \in E$

$$A_E(x) = \{ f \in E^* \mid \|f\| = 1,\ f(x) = \|x\| \}$$

and for a set $A \subset E$, $\operatorname{diam} A = \sup \{ \|a_1 - a_2\| \mid a_1, a_2 \in A \}$.

One can show that for all $x,y \in E$, the following relation holds:

$$(4) \qquad \tau(x,y) + \tau(x,-y) \leq \operatorname{diam} A_E(x) \operatorname{dist}(y, N_E(x))$$

Since $\operatorname{diam} A_E(x) \leq 2$, the upper bound in (4) is better than the upper bound in (2). We call a space E to have property $(\tilde{\Lambda})$ if for all $x,y \in E$ equality holds in (4). If E has property (Λ), then E has $(\tilde{\Lambda})$, this being a consequence of [1], Remark 5. The class of normed linear spaces with property $(\tilde{\Lambda})$ is larger than that with property (Λ). Indeed, all 2-dimensional spaces have property $(\tilde{\Lambda})$, while ([1], Proposition 1) a 2-dimensional space has property (Λ) if and only if its unit ball is either smooth or a parallelogram. No all spaces have property $(\tilde{\Lambda})$. Indeed, we gave in [1] an example of a space E without (Λ) but with the property that $\operatorname{diam} A_E(x) = 0$ or 2 for all $x \in E$; clearly, this space has not property $(\tilde{\Lambda})$.

Among the spaces with property $(\tilde{\Lambda})$ there are the spaces with the property - which we call (Λ_α), $0 \leq \alpha \leq 2$ - that for all $x,y \in E$, $x \neq 0$, the following relation holds:

$$(5) \qquad \tau(x,y) + \tau(x,-y) = \alpha \operatorname{dist}(y, N_E(x))$$

If E has property (Λ_α) for some $0 \leq \alpha \leq 2$, then $\operatorname{diam} A_E(x) = 0$ or α for all $x \in E \setminus \{0\}$, hence E has also property $(\tilde{\Lambda})$. Now, for each $\alpha \in (0,2)$ there exist spaces with property (Λ_α). Indeed, the 2-dimensional spaces having the unit ball a suitable "lens" (depending on α) have this property. The additional requirement (in the class of 2-dimensional spaces) that the set of the extreme points of the unit ball, ex S_E, to be finite changes the above conclusion. It is obvious that for

$\alpha \in (1,2)$ there exists no 2-dimensional space E with property (Λ_α) and ex S_E finite (though for some $\alpha \in (0,1]$ such spaces exist).

<u>Question</u> 5. Do there exist an $\alpha \in (0,2)$ and a 3-dimensional space E such that ex S_E is finite and E has property (Λ_α) ? If the answer is affirmative, is this true for each $\alpha \in (0,2)$? If not, the same questions for dim E = 4,5,.... , or at least to answer the following question: is it true that for each $\alpha \in (0,2)$ there exists an n-dimensional space E such that ex S_E is finite and E has property (Λ_α) ?

If we regard again E as a subspace of E^{**}, then by (4) we have for all $x,y \in E$:

$$\tau(x,y) + \tau(x,-y) \leq \text{diam } A_{E^{**}}(x) \text{ dist}(y,N_{E^{**}}(x)) \tag{6}$$

We call E with property $(\tilde{\Lambda}_{**})$ if equality holds in (6) for all $x,y \in E$. We have $\text{dist}(y,N_{E^{**}}(x)) \leq \text{dist}(y,N_E(x))$ and $\text{diam } A_E(x) \leq$ $\leq \text{diam } A_{E^{**}}(x)$, $x,y \in E$, so the following question makes sense:

<u>Question</u> 6. Characterize those E for which we have $\text{diam } A_{E^{**}}(x) \text{ dist}(y,N_{E^{**}}(x)) \sim \text{diam } A_E(x) \text{ dist}(y,N_E(x))$ for all $x,y \in E$, where $\sim$ stands for $\leq$, = , or $\geq$. Give an example of a space with property $(\tilde{\Lambda}_{**})$ but without both $(\tilde{\Lambda})$ and (Λ_{**}). Give also an example of a space with property $(\tilde{\Lambda})$ but without $(\tilde{\Lambda}_{**})$. Note that if E is Hahn-Banach smooth ([1]) then obviously $A_E(x) = A_{E^{**}}(x)$ for all $x \in E$.

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