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0. Introduction and notation.

In this note we shall investigate, when certain tensor products of Banach lattices have the uniform approximation property (u.a.p.), provided that both factors have this property. We then use these results to prove that if a superreflective Banach lattice has the u.a.p., then the approximating operators can be chosen with controlle moduli. The results are also used to prove when spaces of absolutely summing operators have the u.a.p. All the results of this note will appear in either [2] or [6], to which we refer for further information and detailed proofs.

In the note we shall use the notation and terminology commonly used in the theory of Banach lattices as it appears in [3] and [4].

Throughout the paper we let E denote a Banach space and X a Banach lattice.

1. The tensor product $E\otimes_m X$.

Let us recall that a linear operator $T : E \to X$ is called order bounded, if there exists a $z \in X$, $z \geq 0$ so that

(1) $|Tx| \leq \|x\| z$ for all $x \in X$

and we define the order bounded norm $\|T\|_m$ of T by

$$\|T\|_m = \inf\{\|z\| \mid z \text{ satisfies } (1)\} .$$

$\|\cdot\|_m$ is a norm on the space $\mathcal{B}(E,X)$ of all order bounded operators from E to X, turning it into a Banach space [7] .

1.1 Definition

The m-tensor product $E \otimes_m X$ is defined to be the closure in $\|\cdot\|_m$ of $E \otimes X$ in $\mathcal{B}(E^*,X)$.

It was proved in [2], theorem 2.2 that if X is an order continuous Köthe function space on a probability space $(\Omega, \mathcal{S}, \mu)$, then $E \otimes_m X$ can be identified in a canonical manner with the space $X(E)$, consisting of all measurable functions $f : \Omega \to E$ with $\|f(\cdot)\|_E \in X$.

We now wish to comment a little on computation of norms in $E \otimes_m X$.

If $e_1, e_2, \ldots, e_n \in E$ then the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(t_1, t_2, \ldots, t_n) = \|\sum_{j=1}^{n} t_j e_j\| \qquad (t_1, \ldots, t_n) \in \mathbb{R}^n$$

is a continuous function, homogeneous of degree one. Therefore the Krivine calculus of 1-homogeneous expressions in Banach lattices (see [4]) gives that $f(x_1, \ldots, x_n)$ can be given a unique meaning as an element in X for all $x_1, \ldots, x_n \in X$, we denote that element by $\|\sum_{j=1}^{n} x_j e_j\|_E$. It can easily be proved that

(2) $$\|\sum_{j=1}^{n} x_j e_j\|_E = \sup\{|\sum_{j=1}^{n} e^*(e_j) x_j| \mid e^* \in E^* , \|e^*\| \leq 1\}$$

Hence if $T = \sum_{j=1}^{n} e_j \otimes x_j \in E \otimes X$, then

$$\|T\|_m = \| \| \sum_{j=1}^{n} x_j e_j \|_E \|_X . \tag{3}$$

2. The u.a.p. in $E \otimes_m X$ and X .

Let us recall the following definition

2.1 Definition

Let $\lambda \geq 1$, $\varphi : \mathbb{N} \to \mathbb{R}_+$ a function. A Banach space. E is said to have the (λ,φ) uniform approximation property $((\lambda,\varphi)$-u.a.p.) if for every n and every n-dimensional subspace $F \subseteq E$ there is a bounded operator T on E , so that $\|T\| \leq \lambda$, $\text{rank}(T) \leq \varphi(n)$ and $Tx = x$ for all $x \in F$.

We shall say that E has the u.a.p., if it has the (λ,φ)-u.a.p. for some $\lambda \geq 1$ and some function φ . Likewise we shall say that E has the λ-u.a.p., if it has the (λ,φ)-u.a.p. for some function φ . The u.a.p. was originally introduced in [8]. For further developments on the subject we refer to [1] and [5].

The following theorem can be found in [2]

2.2 Theorem

Let $\lambda \geq 1$. Then the following statements are equivalent

(i) There is a function $\varphi : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{N}$, so that for every $\varepsilon > 0$ and every n-dimensional subspace $F \subseteq X$ there is an operator T on X , with $\|Tx-x\| \leq \varepsilon\|x\|$ for $x \in F$, $\text{rank}(T) \leq \varphi(n,\varepsilon)$ and

$$\| \bigvee_{j=1}^{n} |Tx_j| \| \leq (\lambda+\varepsilon) \| \bigvee_{j=1}^{n} |x_j| \| \tag{*}$$

for all n-tuples $(x_1,x_2,\ldots,x_n) \subseteq X$.

(ii) For every finite dimensional Banach space E, $E\otimes_m X$ has the $(\lambda+\varepsilon)$-u.a.p. for all $\varepsilon > 0$.

(iii) The same as (i) with the addition that $Tx = x$ for all $x \in F$.

Sketch of proof

(i) ⇒ (ii). Assume (i), let $\varepsilon > 0$ and let E be k-dimensional; we choose an Auerbach basis $\{e_1, e_2, \dots, e_k\}$ for E. If $F \subseteq E\otimes_m X$ is n-dimensional with Auerbach basis $\{u_1, u_2, \dots, u_n\}$ then we can find $\{x_{ij} \mid 1 \le i \le k,\ i \le j \le n\}$ so that

$$(1) \qquad u_j = \sum_{i=1}^{k} e_i \otimes x_{ij} \qquad 1 \le j \le n$$

Put $X_1 = \text{span}\{x_{ij} \mid i \le k,\ j \le n\}$ and let $(f_r^*)_{r=1}^{N}$ be an ε-net in the unit ball B_{E^*} of E^*. It is easy to see that for every $u \in E\otimes_m X$

$$(2) \qquad \|u\|_m \le \|\sup |u(e^*)| \ e^* \in B_{E^*}\}\| \le (1-\varepsilon)^{-1} \| \bigvee_{j=1}^{N} |u(f_r^*)| \|$$

According to (i) we can now find an operator T on X so that (*) holds for N vectors and

$$(3) \qquad \|Tx - x\| \le \varepsilon\, n^{-1} k^{-1} \|x\| \qquad \text{for all} \qquad x \in X_1 .$$

$$\text{rank}(T) \le \varphi(\max((N, kn) n^{-1} k^{-1} \varepsilon)$$

For every $u \in E\otimes_m X$ we now get the following estimate of $(I\otimes T)u$ (I denoting the identity on E)

$$(4) \qquad \|(I\otimes T)u\|_m = \|T\circ u\|_m \le (1-\varepsilon)^{-1} \| \bigvee_{r=1}^{N} |T(uf_r^*)| \|$$

$$\le (\lambda+\varepsilon)(1-\varepsilon)^{-1} \| \bigvee_{r=1}^{N} |u(f_r^*)| \| \le (\lambda+\varepsilon)(1-\varepsilon)^{-1} \|u\|_m .$$

so that $\|I\otimes T\| \le (\lambda+\varepsilon)(1-\varepsilon)^{-1}$. Clearly $\text{rank}(I\otimes T) \le k\, \varphi(\max(N, kn), n^{\varepsilon-1} k^{-1} \varepsilon)$ and using (2) and the fact that

$\{u_1,\ldots,u_n\}$ is an Auerbach basis a computation shows that $\|(I\otimes T)u-u\|_m \leq \varepsilon \|u\|_m$ for all $u \in F$. Hence we have shown that $E\otimes_m X$ has the $(\lambda+\varepsilon)$-u.a.p. for all $\varepsilon > 0$.

(ii) ⇒ (iii). Let $\varepsilon > 0$ and $F \subseteq X$, $\dim F = n$. By assumption $\ell_\infty^n\otimes X$ has the $(\lambda+\varepsilon)$-u.a.p. with some dimension function φ, say, and we can therefore find a bounded operator S on $\ell_\infty^n\otimes_m X$ so that $\|S\| \leq \lambda+\varepsilon$, $\text{rank}(S) \leq \varphi(n^2)$ and $S|_{\ell_\infty^n\otimes F} = I_F$.

Let Γ denote the group of all isometries of ℓ_∞^n onto itself and put

$$S_o = (2^n\ n!)^{-1} \sum_{\gamma\in\Gamma} (\gamma^{-1}\otimes I_X)S(\gamma\otimes I_X) .$$

Clearly $\|S_o\| \leq \lambda+\varepsilon$, $\text{rank}(S_o) \leq 2^n\ n!\ \varphi(n^2)$ and $S_o|_{\ell_\ell^n\otimes F} = I_F$. Since S_o is invariant under all isometries $\gamma\otimes I_X$, $\gamma \in \Gamma$ there is a bounded operator T on X with $S_o = I_{\ell_\infty^n}\otimes T$. Clearly $T|_F = I_F$, $\text{rank}(T) \leq (n-1)!\ 2^n\ \varphi(n^2)$. Further for all $x_1,\ldots,x_n \in X$, we have

$$\|\bigvee_{j=1}^n |Tx_j|\| = \|\sum_{j=1}^n e_j\otimes Tx_j\|_m = \|S_o(\sum_{j=1}^n e_j\otimes x_j)\|_m \leq$$

$$(\lambda+\varepsilon)\ \|\sum_{j=1}^n e_j\otimes x_j\|_m = (\lambda+\varepsilon)\ \|\bigvee_{j=1}^n |x_j|\|$$

and (ii) ⇒ (iii) is proved. (iii) ⇒ (i) is trivial.

q.e.

We now introduce the following definition

2.3 Definition

Let $\lambda \geq 1$. X is said to have the $(\lambda+)$-order u.a.p., if it satisfies one of the equivalent conditions of Theorem 2.2.

It is an open problem, whether the u.a.p. is equivalent to the order u.a.p. for general Banach lattices; all known examples of

Banach lattices with the u.a.p. has in fact the positive u.a.p. and hence trivially satisfy condition (i) of theorem 2.2.

The two concepts are equivalent for superreflexive Banach lattices, as it is seen from

2.3 Theorem [2]

If X is superreflexive with the u.a.p. then X has the (1+)-order u.a.p.

Proof

Let E be k-dimensional and let $\varepsilon > 0$. We can then find a k (depending only on k and X.) so that $\|u\|_m \leq k\|u\|$ for all $u \in E \otimes_m X$. Since X has the $(1+\varepsilon)$-u.a.p., [5], all the arguments of (i) $\Rightarrow$ (ii) in theorem 2.2 can be performed except the estimation of the norm of $(I \otimes T)$ there. Instead we get for all $u \in E \otimes_m X$

$$\|(I \otimes T)u\|_m \leq \|T \circ u\|_m \leq k\|T\|\ \|u\| \leq k(1+\varepsilon)\|u\|_m .$$

Hence $E \otimes_m X$ has the $(k+\varepsilon)$-u.a.p. for all $\varepsilon > 0$. Since E is finite dimensional and X is superreflexive $E \otimes_m X$ is superreflexive as well and therefore it has the $(1+\varepsilon)$-u.a.p. by [5].

2.4 Problem

Is the u.a.p. equivalent to the positive u.a.p. for superreflexive Banach lattices ?

Using theorem 2.2 we can prove the following result on the u.a.p. of m-tensor products. The proof is quite long and will therefore be omitted here.

2.5 Theorem [2]

Let E have the λ-u.a.p. If X has the $(\mu+)$-order u.a.p., then $E\otimes_m X$ has the $(\lambda\mu+\varepsilon)$-u.a.p. for all $\varepsilon > 0$. If X is super-reflexive and has the u.a.p., then $E\otimes_m X$ has the $(\lambda+\varepsilon)$-u.a.p. for all $\varepsilon > 0$.

Theorem 2.4 can be used to provide several new examples of spaces with the u.a.p.

3. Spaces of absolutely summing operators.

In this section we let I_p , respectively Π_p , $1 \le p \le \infty$ denote the class of all p-integral operators, respectively the class of all p-integral operators, respectively the class of all p.summing operators. For Banach spaces E and F $B(E,F)$ denotes the space of all bounded operators from E to F .

In [6] the following theorems are proved

3.1 Theorem

(i) Let $1 < q \le 2$, X q-concave and $B(\ell_1,E^*) = \Pi_q^{\gamma}(\ell_1,E^*)$, then

$$T \in E\otimes_m X \Leftrightarrow T \in I_q(E^*,X) \Leftrightarrow T^* \in \Pi_1(X^*,E)$$

(ii) Let $1 < q < p < 2$ or $1 \le q < \infty$ and $p = 2$ or $p = q = 1$. If X is q-concave and E is isomorphic to a subspace of an L_p-space (and E isomorphic to a dual space with the RNP, if $p = 1$). Then

$$T \in E\otimes_m X \Leftrightarrow T \in \Pi_q(E^*,X) \Leftrightarrow T \in \Pi_1(E^*,X) \quad \text{if} \quad q \le 2 .$$

It is readily verified that if $1 \le q < p \le 2$ and E is a subspace of a quotient of an L_p-space, then E satisfies the conditions in (i) of theorem 3.1.

Theorem 3.1 is quite useful since it relates some classical spaces to the m-tensor products, where it is often easier to compute norms. As an example we can mention

3.2 Corollary

Let $1 \leq q < p \leq 2$, (e_n) the unit vector basis of ℓ_p, (e_n^*) the unit vector basis of $\ell_{p'}$, and let X be q-concave. There is a constant k so that if

$$T = \sum_{j=1}^{n} e_j \otimes x_j \in \ell_p \otimes X \text{ , then}$$

$$k^{-1}\pi_1(T) \leq \| (\sum_{j=1}^{n} |x_j|^p)^{1/p} \| \leq k\, \pi_1(T) .$$

Together with the results in section 2, theorem 3.1 can be used to prove that certain spaces of absolutely summing operators have the u.a.p.

3.3 Theorem

(i) Let $1 < q < p < 2$ or $p = 2$ and $1 < q \leq 2$ and let E be isomorphic to a subspace of an L_p-space, F isomorphic to a complemented subspace of a q-concave Banach lattice X. If E and X have the u.a.p., then $\Pi_1(E^*,F)$ and $\Pi_1(F^*,E)$ have the u.a.p.

(ii) Let E be isomorphic to a subspace of an L_1-space, F isomorphic to a complemented subspace of an L_1-space. If E has the u.a.p., then $\Pi_1(E^*,F)$ has the u.a.p.

As a corollary we get

3.4 Corollary

Let $1 \leq s \leq 2 \leq r \leq \infty$ and $r \neq s'$ unless $s = 1,2$. If E is an $\mathcal{L}_r$-space, F an $\mathcal{L}_s$-space with F complemented in F^{**}, then $\Pi_1(E,F)$ has the u.a.p.

With the methods of [6] we have not been able to determine whether $\Pi_1(\mathcal{L}_r,\mathcal{L}_s)$ has the u.a.p. for the remaining values of r and s .

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