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DOMAINS OF BIREGULARITY IN CLIFFORD ANALYSIS

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1. Introduction

1.1. In the theory of the Clifford algebra-valued monogenic functions it is proved (see [5]) that, such as in the case of the holomorphic functions in the complex plane, every domain in Euclidean space is a domain of monogenicity. This result depends heavily upon the existence of pointwise singularities. So when considering the biregular functions ([1,2,3]) where, due to the Hartogs Extension Theorem, pointwise singularities do not occur, one is lead, such as in the several complex variables case, to the problem of characterizing the so-called domains of biregularity; these are, roughly speaking, domains for which there exists a biregular function which cannot be extended biregularly beyond the boundary of the domain.

In this paper a chain of necessary conditions for a domain in Euclidean space to be a domain of biregularity is established; some of these conditions are of a more functiontheoretic nature (§2), others of a more geometric nature (§3). If those conditions are also sufficient conditions is still an open problem. Nevertheless it is proved (§4) that a special class of domains are so-called biregular extension domains, a notion which is even stronger than that of a domain of biregularity.

For notations , definitions and properties of Clifford algebra and monogenic functions we refer the reader to [4]. For the biregular functions, which in some sense , form a generalization to two Clifford variables of the monogenic functions, some definitions and results are repeated here in order to make this paper more readable.

1.2. Let Ω be an open subset of $R^{m+1} x R^{k+1}$ and let A_n be the universal Clifford algebra constructed over R^n ($1<m,k<n$) with orthonormal

basis $(e_1,\ldots,e_n)$. Then functions $f:\Omega\to A_n$ of the variables $(x,y)=(x_0,\ldots,x_m,y_0,\ldots,y_k)$, $x\in R^{m+1}$, $y\in R^{k+1}$, are considered.

Denoting the intersections of Ω parallel to the x- and y-spaces by $U_y=\{x\in R^{m+1}:(x,y)\in\Omega\}$, y fixed in R^{k+1} and $V_x=\{y\in R^{k+1}:(x,y)\}\in\Omega\}$, x fixed in R^{m+1}, respectively, the notion of a biregular function is introduced as follows :

DEFINITION 1.1. A function $f:\Omega\to A_n$ is called biregular in Ω if

(i) for each fixed $y\in R^{k+1}$, f is C^1 in $x\in U_y$ and satisfies $D_x f=0$;

(ii) for each fixed $x\in R^{m+1}$, f is C^1 in $y\in V_x$ and satisfies $fD_y=0$,

where D_x and D_y are the generalized Cauchy-Riemann operators given by $D_x=\sum_{i=0}^{m} e_i\partial_{x_i}$ and $D_y=\sum_{j=0}^{k} e_j\partial_{y_j}$.

Notice that a biregular function in Ω is at the same time left monogenic in x and right monogenic in y, and hence real-analytic in the variables x and y separately.
The Hartogs Theorem ([1], Theorem 2.4) then ensures us of the global real-analyticity of a biregular function in all the variables $(x_0,\ldots,x_m,y_0,\ldots,y_k)$ together.

The Cauchy kernels of the above mentioned generalized Cauchy-Riemann operators are given by

$$E_m(x)=\frac{1}{\omega_{m+1}}\frac{\bar{x}}{|x|^{m+1}} \text{ and } E_k(y)=\frac{1}{\omega_{k+1}}\frac{\bar{y}}{|y|^{k+1}}$$

where $x=\sum_{i=0}^{m} e_i x_i$, $|x|$ is the Euclidean norm of x and ω_{m+1} is the area of the (m+1)-dimensional unit sphere. Those kernels satisfy $D_xE_m=E_mD_x=\delta$ and $D_yE_k=E_kD_y=\delta$, δ being the Dirac measure at the origin.

If the function f is biregular in an open biball $\mathring{B}_m(0,R_m)\times\mathring{B}_k(0,R_k)$ then in this region f may be expanded uniquely into the normally convergent Taylor series

$$f(x,y)=\sum_{t=0}^{\infty}\sum_{s=0}^{\infty} P_{r,s}f(x,y)$$

([2], Theorem 4.1), where the polynomials $P_{r,s}f$ are given by

$$P_{r,s}f(x,y)=\sum_{(1)_r}\sum_{(h)_s} V_{(1)_r} \cdots {}_{r(h)_s} W_{(h)_s}(y) \, .$$

the sums running over all possible combinations of r elements out of $\{1,\ldots,m\}$ (respectively s elements out of $\{1,\ldots,k\}$) repetitions being allowed, containing the basic polynomials

$$V_{(1)_r}(x)=\frac{1}{r!}\sum_{\pi(1)_r}\xi_{1_1}\ldots\xi_{1_r} \text{ and } W_{(h)_s}(y)=\frac{1}{s!}\sum_{\pi(h)_s}\eta_{h_1}\ldots\eta_{h_s},$$

the summation running over all distinguishable permutations, where appear the hypercomplex variables

$$\xi_1=x_1e_0-x_0e_1,\ 1=1,\ldots,m \text{ and } \eta_h=y_he_0-y_0e_h,\ h=1,\ldots,k.$$

1.3. The domains of biregularity are now precisely circumscribed as follows.

<u>DEFINITIONS 1.2</u>. Let Ω be a domain in R^{m+k+2}, i.e. an open connected set.

(i) Ω is said to be a domain of biregularity (DB) if it is impossible to find two domains U_1 and U_2 in R^{m+k+2} satisfying the conditions

(a) $\phi\neq U_2\subset\Omega\cap U_1\subsetneq U_1$;

(b) for each biregular function f in Ω there exists a biregular function $\tilde{f}$ in U_1 which coincides with f on U_2.

(ii) Ω is said to be a weak domain of biregularity (WDB) if for each domain $\tilde{\Omega}$ containing Ω there exists a biregular function in Ω which is not the restriction to Ω of a biregular function in $\tilde{\Omega}$.

It is clear that each DB is also a WDB and moreover it may be shown that both notions coincide on locally connected domains.

2. Analytic properties of domains of biregularity

2.1. First we state without proof some estimates of the basic polynomials appearing in the Taylor series of a biregular function and of the derivatives of the Cauchy kernels.

LEMMA 2.1. The basic polynomials $V_{(l)_r}$ satisfy the estimate

$$|V_{(l)_r}(x)|_0 \leq 2^{n/2} \frac{\|x\|^r}{\nu_1!\ldots\nu_m!} \leq 2^{n/2} \frac{|x|^r}{\nu_1!\ldots\nu_m!}$$

where $\|x\| = \sup_{i=1,\ldots,m} (x_0^2+x_i^2)^{1/2}$, and ν_i $(i=1,\ldots,m)$ represents the number of times the index i is appearing in the combination $(l)_r=(l_1,\ldots,l_r)\in\{1,\ldots,m\}^r$.

REMARK 2.2. As $\sum_{(l)_r} \frac{r!}{\nu_1!\ldots\nu_m!} = m^r$, we also have the estimate

$$\sum_{(l)_r} |V_{(l)_r}(x)|_0 \leq 2^{n/2} \frac{(m|x|)^r}{r!}.$$

LEMMA 2.3. The derivatives of the Cauchy kernel $X_{(l)_r}(x)=\partial_{x_{(l)_r}} E_m(x)$ satisfy the estimate

$$|X_{(l)_r}(x)|_0 \leq \frac{2^{n/2}}{\omega_{m+1}} C(m,r)\,|x|^{-m-r}$$

where $C(m,r)=(m+1)[(2m+2)(3m+5)\ldots((m+3)r+(m-1))]$.

We also need the following technical lemma on the convergence of a power series in $|x|$.

LEMMA 2.4. The series $\sum_{r=0}^{\infty} C(m,r)\frac{m^r}{r!}\left(\frac{|x|}{R}\right)^r$, $R>0$, is normally convergent in $\mathring{B}_m(0,R_m)$, where $R_m=\frac{R}{m(m+3)}$.

Proof. Let K be an arbitrary compact subset of $\mathring{B}_m(0,R_m)$ and choose $0<R'<R$ such that $K\subset\overline{B}_m(0,R'_m)$. The numerical series $\sum_{r=0}^{\infty} C(m,r)\frac{m^r}{r!}\left(\frac{R'_m}{R}\right)^r$ clearly is convergent, which implies that the considered power series is normally convergent in $|x|\leq R'_m$ and a fortiori on K. ∎

In the sequel we shall also make use of the following distance function. Let (x,y) be a point in Ω and let A be a subset of Ω. Then we define

(i) $\delta_\Omega(x,y)=\sup\{r\in R^+ : \overset{\circ}{B}_m(x,r)\times\overset{\circ}{B}_k(y,r)\subset\Omega\}$;

(ii) $\delta_\Omega(A)=\inf\{\delta_\Omega(x,y):(x,y)\in A\}$.

It is clear that $\delta_\Omega(x,y)\in]0,+\infty]$, $\delta_\Omega(A)\in[0,+\infty]$ and that for a proper subset Ω of R^{m+k+2} the function $\delta_\Omega:\Omega\to R$ is continuous. Moreover for $A\subset\overline{A}\subset\Omega$ holds $\delta_\Omega(A)=\delta_\Omega(\overline{A})$.

Indispensable in the description of the properties of a DB is the notion of convex hull of a compact subset K of Ω; these hulls are defined by means of families F of biregular functions in Ω, which have to fulfil the condition that they always contain the hypercomplex variables ξ_i $(i=1,\ldots,m)$ and η_j $(j=1,\ldots,k)$.

DEFINITION 2.5. The F-convex hull of K is given by

$$\hat{K}(F)=\{(x,y)\in\Omega:|f(x,y)|_0\leqslant\sup_{(u,v)\in K}|f(u,v)|_0,\ \text{for all } f\in F\}.$$

If $F=B(\Omega)$, the family of all biregular functions in Ω, then we speak of the biregularly-convex hull and denote it simply by $\hat{K}_\Omega$.

The following properties of F-convex hulls are immediate.

PROPOSITION 2.6. The F-convex hull $\hat{K}(F)$ of a compact subset K of Ω satisfies the following properties :
(i) $\hat{K}(F)$ is relatively closed in Ω;
(ii) $\hat{K}(F)$ is bounded;
(iii) $K\subset\hat{K}(F)$;
(iv) if $\hat{K}(F)$ is compact then $(\hat{K}(F))^\wedge(F)=\hat{K}(F)$;
(v) if $F_1\subset F_2$ then $\hat{K}(F_2)\subset\hat{K}(F_1)$.

2.2. Now we come to the first characteristic property of a DB.

DEFINITION 2.7. A domain $\Omega\subset R^{m+k+2}$ is called metrically convex w.r.t. δ_Ω if any compact subset $K\subset\Omega$ satisfies

$$\delta_\Omega(\hat{K}_\Omega)\geqslant c_{m,k}\delta_\Omega(K)$$

where $\quad c_{m,k}=\min\left(\frac{1}{m(m+3)}, \frac{1}{k(k+3)}\right).$

In order to prove that each DB is metrically convex w.r.t. δ_Ω, we first establish a key result on an estimate for the radius of convergence of the Taylor series about a point of $\hat{K}_\Omega$; it turns out that this estimate is independent of the point considered, and hence independent of $\hat{K}_\Omega$.

PROPOSITION 2.8. Let Ω be a domain in R^{m+k+2}, let K be a compact subset of Ω and let f be biregular in Ω. Then the Taylor series of f about $(x^*,y^*)\in\hat{K}_\Omega$ is normally convergent in $\mathring{B}_m(x^*,\delta_{\Omega,m}(K))$ $\times\mathring{B}_k(y^*,\delta_{\Omega,k}(K))$, where $\delta_{\Omega,m}(K)=\frac{1}{m(m+3)}\delta_\Omega(K)$ and $\delta_{\Omega,k}(K)=\frac{1}{k(k+3)}\delta_\Omega(K)$.

Proof. The Taylor series of f about (x^*,y^*),

$$f(x,y)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} P_{r,s}f(x,y)$$

converges normally in $\mathring{B}_m(x^*,\delta_\Omega(x^*,y^*))\times\mathring{B}_k(y^*,\delta_\Omega(x^*,y^*))$, and in this region

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}|P_{r,s}f(x,y)|_0\leq\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}2^{n/2}\frac{m^r}{r!}|x-x^*|^r|\partial_{x_{(1)_r}}\partial_{y_{(h)_s}}f(x^*,y^*)|_0\, 2^{n/2}\frac{k^s}{s!}|y-y^*|^s$$

As all the functions $\partial_{x_{(1)_r}}\partial_{y_{(h)_s}}f$ are biregular in Ω, we have by the definition of $\hat{K}_\Omega$ that

$$|\partial_{x_{(1)_r}}\partial_{y_{(h)_s}}f(x^*,y^*)|_0\leq\sup_{(x',y')\in K}|\partial_{x_{(1)_r}}\partial_{y_{(h)_s}}f(x',y')|_0.$$

Choose $0<R<\delta_\Omega(K)$; then $K_R=\bigcup_{(x',y')\in K}\mathring{B}_m(x',R)\times\mathring{B}_k(y',R)$ is relatively compact in Ω and we put $M=\sup_{(x,y)\in K_R}|f(x,y)|_0$.

Let (x',y') be an arbitrary point of K, then by Cauchy's Integral Formula (see [1], Theorem 2.6),

$$|\partial_{x_{(1)_r}} \partial_{y_{(h)_s}} f(x',y')|_0 \le C_n . M \int_{(u,v)\in\partial B_m(x',R)\times\partial B_k(y',R)} |\partial_{x_{(1)_r}} E_m(u-x')|_0 |\partial_{y_{(h)_s}} E_k(v-y')|_0 \, dS_m dS_k$$

where dS_m, respectively dS_k, is the ordinary surface-area element in R^{m+1}, resp. R^{k+1},

or $|\partial_{x_{(1)_r}} \partial_{y_{(h)_s}} f(x',y')|_0 \le C_n' M C(m,r) C(k,s) R^{-r-s}$

and hence

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} |P_{r,s} f(x,y)|_0 \le C_n'' M \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} C(m,r) C(k,s) \frac{m^r k^s}{r!s!} \left(\frac{|x-x^*|}{R}\right)^r \left(\frac{|y-y^*|}{R}\right)^s$$

As it can be shown that the product of two normally convergent series of A_n-valued functions is again normally convergent in the cartesian product of the convergence domains (see [1], Lemma 2.1), it follows that the considered Taylor series converges normally in $\mathring{B}_m(x^*,R_m) \times \mathring{B}_k(y^*,R_k)$. Letting $R \to \delta_\Omega(K)$ yields the desired result.■

THEOREM 2.9. Every domain of biregularity is metrically convex w.r.t. δ_Ω .

Proof. Assume that Ω is a DB but is not metrically convex w.r.t. δ_Ω; then there exists a compact subset K for which $\delta_\Omega(\hat{K}_\Omega) < c_{m,k}\delta_\Omega(K)$. So there exists a point $(x^*,y^*) \in \hat{K}_\Omega$ for which

$\delta_\Omega(x^*,y^*) < c_{m,k}\delta_\Omega(K)$. Call $U_1 = \mathring{B}_m(x^*,\delta_{\Omega,m}(K)) \times \mathring{B}_k(y^*,\delta_{\Omega,k}(K))$

and $U_2 = \mathring{B}_m(x^*,\delta_\Omega(x^*,y^*)) \times \mathring{B}_k(y^*,\delta_\Omega(x^*,y^*))$;then U_1 and U_2 are

two domains satisfying condition (a) of Definition 1.2(i). Now given a biregular function f in Ω, the function defined by the Taylor series of Proposition 2.8 is biregular in U_1 and coincides with f on U_2. This contradicts condition (b) of the same definition. ■

Metrical convexity is closely related to the so-called strong continuity principle, which is defined as follows.

DEFINITION 2.10. The domain Ω satisfies the strong continuity principle w.r.t. δ_Ω (SCP) if for each couple (S,T) of subsets of Ω for which

(i) $\sup_{(x,y)\in T} |f(x,y)|_0 = \sup_{(x,y)\in S\cup T} |f(x,y)|_0$, for all $f\in B(\Omega)$, and

(ii) $S\cup T$ is relatively compact in Ω,

holds that $\delta_\Omega(S) \geq c_{m,k}\delta_\Omega(T)$.

Now it is easily shown that every DB satisfies SCP.

THEOREM 2.11. A domain $\Omega\subset R^{m+k+2}$ which is metrically convex w.r.t. δ_Ω satisfies the strong continuity principle w.r.t. δ_Ω.

2.3. By handling the intrinsic properties of the F-convex hulls introduced in subsection 2.1, a new necessary condition for a DB may be obtained.

DEFINITION 2.12. A domain $\Omega\subset R^{m+k+2}$ is called F-convex if for each compact subset K, the F-convex hull $\hat{K}(F)$ is again a compact set. If $F=B(\Omega)$ we call Ω biregularly convex.

About the behaviour of this notion of convexity when several families F are involved, the following results, the proof of which is rather straightforward, may be stated.

PROPOSITION 2.13. Let Ω be an F-convex domain.

(i) If $F'\supset F$ then Ω is also F'-convex.

(ii) If F' is dense in F, i.e. if for each $f\in F$, for each compact $K\subset\Omega$ and for each $\varepsilon>0$ there exists $f'\in F'$ for which $|f'(x,y)-f(x,y)|_0<\varepsilon$ for all $(x,y)\in K$, then Ω is also F'-convex.

Now we show that every DB is biregularly convex.

THEOREM 2.14. A domain Ω which satisfies the strong continuity principle w.r.t. δ_Ω is biregularly convex.

Proof. Assume that Ω is not biregularly convex; then there exists a compact subset K for which $\hat{K}_\Omega$ is not compact, although it is bounded and relatively closed in Ω. So there exists a sequence $(x^{(\nu)},y^{(\nu)})_{\nu=1}^{\infty}$ in $\hat{K}_\Omega$ for which $\lim_{\nu\to\infty}(x^{(\nu)},y^{(\nu)})=(x,y)\in\partial\Omega$.

Putting $S_\nu=\{(x^{(\nu)},y^{(\nu)}\}$ and $T_\nu=K$ for all $\nu\in\mathbb{N}$, it is clear that all (S_ν,T_ν) satisfy the conditions (i) and (ii) of Definition 2.10. Hence, by the SCP, $\delta_\Omega(S_\nu)\geq c_{m,k}\delta_\Omega(T_\nu)$ or $0<\delta_\Omega(K)\leq c_{m,k}^{-1}\delta_\Omega(x^{(\nu)},y^{(\nu)})$

for all $\nu\in\mathbb{N}$. Letting $\nu\to+\infty$ yields $0<\delta_\Omega(K)\leq 0$, clearly a contradiction.∎

3. Geometric properties of domains of biregularity

3.1. The necessary conditions of a DB in this section are of a more geometric nature and rely on the notion of a (pluri) subconstant function. A subconstant function behaves w.r.t. a constant such as a subharmonic function does w.r.t. a harmonic function. More precisely:

DEFINITION 3.1. A function $g:\omega\subset\mathbb{R}^{m+1}\to\mathbb{R}\cup\{-\infty\}$ is called subconstant in ω if

(i) g is upper semi-continuous in ω;

(ii) for each compact subset $K\subset\omega$ and for each real constant M, $f\leq M$ on ∂K implies that $f\leq M$ on K.

REMARK 3.2. Clearly any subharmonic function in ω is also subconstant in ω. For a monogenic function g in ω it can be shown that $|g|^p$ $(p\geq1)$ is subharmonic and hence subconstant in ω. Those subconstant functions may be characterized as follows.

PROPOSITION 3.3. A function g is subconstant in ω if and only if g is the pointwise limit of a decreasing sequence $(g_\nu)_{\nu=1}^{\infty}$ of subconstant functions in ω.

PROPOSITION 3.4. An upper semi-continuous function g in ω is subconstant if and only if for each compact subset $K\subset\omega$,
$\sup_{x\in K} g(x)=\sup_{x\in\partial K} g(x)$.

The notion of a subconstant function is now extended to the case of the two variables $x\in\mathbb{R}^{m+1}$ and $y\in\mathbb{R}^{k+1}$.

DEFINITION 3.5. A function $f:\Omega\subset\mathbb{R}^{m+k+2}\to\mathbb{R}\cup\{-\infty\}$ is called plurisubconstant if

(i) f is upper semi-continuous in Ω;

(ii) for each y fixed, $f(.,y)$ is subconstant in $x\in U_y$;

(iii) for each x fixed, $f(x,.)$ is subconstant in $y\in V_x$.

The family of all (positive) plurisubconstant functions in Ω is denoted by $PSC(\Omega)$, respectively $PSC^+(\Omega)$.

REMARK 3.6. If f is biregular in Ω then it may be shown that $|f|^p \in PSC^+(\Omega)$ for all $p \geq 1$.

3.2. In 2.1 we introduced the notion of F-convex hull of a compact subset of Ω; this notion may be generalized to the so-called G-convex hull of a compact $K \subset \Omega$, where G is now an arbitrary family of functions defined in Ω and with values in R , C or A_n, e.g. the family $PSC^+(\Omega)$, and we call a domain Ω G-convex if for any compact $K \subset \Omega$, $\hat{K}(G)$ is relatively compact in Ω.

The first geometric condition on a DB reads as follows.

THEOREM 3.7. If the domain $\Omega \subset R^{m+k+2}$ is biregularly convex then it is also $PSC^+(\Omega)$-convex.

Proof. Let K be a compact subset of Ω. Then, by Remark 3.6, $\hat{K}(PSC^+(\Omega)) \subset \hat{K}_\Omega$, the latter being a compact subset of Ω. ∎

PROPOSITION 3.8. If the domain $\Omega \subset R^{m+k+2}$ is geometrically convex then it is also $PSC^+(\Omega)$-convex.

Proof. As is well known, geometrical convexity is equivalent with $\mathcal{L}$-convexity, where $\mathcal{L}$ stands for the real linear functions on R^{m+k+2} . Now take $f \in \mathcal{L}$, then f and -f are harmonic, and so subharmonic , in the variables x and y separately, and hence plurisubconstant in Ω. This means that $|f| = \max(f,-f) \in PSC^+(\Omega)$, from which it follows that $\hat{K}(PSC^+(\Omega)) \subset \hat{K}(\mathcal{L})$, the latter being a relatively compact subset of Ω. ∎

DEFINITION 3.9. Given a domain $\Omega \subset R^{m+k+2}$, consider subdomains S_α, $\alpha \in I$ with boundaries $T_\alpha = \partial S_\alpha$, $\alpha \in I$, such that for all $\alpha \in I$ there exist y_α (resp. x_α) for which $S_\alpha \cup T_\alpha \subset U_y$ (resp. $S_\alpha \cup T_\alpha \subset V_x$), the $S_\alpha \cup T_\alpha$ moreover being compact subsets of Ω. Put $S_0 = \bigcup_\alpha S_\alpha$ and $T_0 = \bigcup_\alpha T_\alpha$ and assume that T_0 is relatively compact in Ω. Then Ω is said to satisfy the weak continuity principle (WCP) if S_0 is relatively compact in Ω.

The second geometric condition on a DB may then be stated as follows.

THEOREM 3.10. If the domain $\Omega\subset R^{m+k+2}$ is $PSC^+(\Omega)$-convex then Ω satisfies the weak continuity principle.

Proof. Consider subdomains $S_\alpha,\alpha\in I$ as described in Definition 3.7 and suppose that they all are lying in U_{y_α} parallel to the x-space. Take $f\in PSC^+(\Omega)$ and put for all $\alpha\in I$, $M_\alpha=\sup_{(u,v)\in T_\alpha} f(u,v)$. So $f\leq M_\alpha$ on T_α, and hence $f\leq M_\alpha$ on S_α for all $\alpha\in I$. This yields $S_\alpha\subset\hat{T}_\alpha(PSC^+(\Omega))$ for all $\alpha\in I$, and hence

$S_0\subset\bigcup_{\alpha\in I}\hat{T}_\alpha\subset\hat{T}_0(PSG^+(\Omega))$, the latter being relatively compact in Ω.∎

Now we come to the condition which may be viewed upon as the analogue of pseudoconvexity in classic complex analysis.

DEFINITION 3.11. A domain $\Omega\subset R^{m+k+2}$ is called plurisubconstant if $(\delta_\Omega(x,y))^{-1}$ is plurisubconstant in Ω.

THEOREM 3.12. If the domain $\Omega\subset R^{m+k+2}$ satisfies the weak continuity principle, then Ω is plurisubconstant.

Proof Assume Ω not to be plurisubconstant; then the function $(\delta_\Omega(x,y))^{-1}$ is not plurisubconstant, which means that e.g. there exists an x^* for which $(\delta_\Omega(x^*,y))^{-1}$ is not subconstant in V_{x^*}. So

an open relatively compact Λ in V_{x^*}, a constant $M\in R$ and a point

$(x^*,y^*)\in\Lambda$ may be found for which $0<(\partial_\Omega(x^*,y))^{-1}\leq M$ for all $(x^*,y)\in\partial\Lambda$ and at the same time $(\delta_\Omega(x^*,y^*))^{-1}>M$. Hence $\inf_{\partial\Lambda} M\delta_\Omega(x^*,y)\geq 1$.

As we shall show that $\inf_{\partial\Lambda}\delta_\Omega(x^*,y)=\inf_{\overline{\Lambda}}\delta_\Omega(x^*,y)$, it is obtained

that $\inf_{\Lambda} M\delta_\Omega(x^*,y)\geq 1$. But in $(x^*,y^*)\in\Lambda$ we have $1>M\delta_\Omega(x^*,y^*)$, clearly a contradiction, implying the plurisubconstantness of Ω.

Now let us come back to the equality of the infima mentioned above. Clearly it is sufficient to prove that $\inf_{\overline{\Lambda}}\delta_\Omega(x^*,y)\geq\inf_{\partial\Lambda}\delta_\Omega(x^*,y)$.

Call the last right hand side ρ; then $\rho>0$ since Λ is relatively compact in Ω, and we have to show that $\delta_\Omega(x^*,y)\geq\rho$ for all $(x^*,y)\in\Lambda$.

Without loss of generality we may suppose that Λ is connected. For $\alpha,\beta>0$ we construct the following regions :

$$S_{\alpha,\beta}=\{(x,y)\in R^{m+k+2}:x=x^*+\alpha\nu_m,\ y=y'+\beta\nu_k,(x^*,y')\in\Lambda\}$$

$$T_{\alpha,\beta}=\{(x,y)\in R^{m+k+2}:x=x^*+\alpha\nu_m;\ y=y'+\beta\nu_k,(x^*,y')\in\partial\Lambda\}$$

where ν_m and ν_k are arbitrary but for the time being fixed unit vectors in R^{m+1} and R^{k+1} respectively. Clearly the $S_{\alpha,\beta}$ are domains with boundaries $T_{\alpha,\beta}$ and $S_{0,0}=\Lambda$ while $T_{0,0}=\partial\Lambda$.

First we show that $T_{\alpha,\beta}$ is relatively compact in Ω, for all $(\alpha,\beta)\in[0,\rho[\times[0,\rho[$. For all $(x,y)\in T_{\alpha,\beta}$ we have $|x-x^*|=\alpha$, $|y-y'|=\beta$, and putting $\rho'=\min(\rho-\alpha,\rho-\beta)$ we get that $\mathring{B}_m(x,\rho')\times\mathring{B}_k(y,\rho')\subset\mathring{B}_m(x^*,\rho)\times\mathring{B}_k(y',\rho)\subset\Omega$. Hence $\delta_\Omega(T_{\alpha,\beta})\geqslant\rho'>0$ and each $T_{\alpha,\beta}$, being bounded, turns out to be relatively compact in Ω.

Next we show that $S_{\alpha,\beta}$ is relatively compact in Ω for all $(\alpha,\beta)\in[0,\rho[\times[0,\rho[$. Start with $S_{0,0}\subset\subset\Omega$; then by the continuity of the transformations it follows that $S_{\alpha,\beta}\subset\subset\Omega$ for all $(\alpha,\beta)\in[0,a[\times[0,a[$, a being chosen sufficiently small in $]0,\rho[$. Now by exploiting the WCP it is shown that $S_{\alpha,\beta}\subset\subset\Omega$ for all $(\alpha,\beta)\in[0,a]\times[0,a]$ and by a same reasoning as before we find at last the $S_{\alpha,\beta}$ to be relatively compact in Ω for all $(\alpha,\beta)\in[0,\rho[\times[0,\rho[$.

Finally if ν_m and ν_k are taken to vary through the whole set of unit vectors in R^{m+1} and R^{k+1} respectively, then it is obtained that $\mathring{B}_m(x^*,\rho)\times\mathring{B}_k(y',\rho)\subset\Omega$ for all $(x^*,y')\in\Lambda$. This means that $\delta_\Omega(\Lambda)\geqslant\rho$, which completes the proof. ∎

An important property of a subconstant domain is the possibility to construct a so-called plurisubconstant exhaustion function, which constitutes the last geometric condition on a DB.

<u>THEOREM 3.13</u>. Let the domain $\Omega\subset R^{m+k+2}$ be plurisubconstant. Then there exists a continuous $\varphi\in PSC^+(\Omega)$ such that for each $c\in R$, $\Omega_{c,\varphi}=\{(x,y)\in\Omega:\varphi(x,y)<c\}$ is relatively compact in Ω.

Proof. From the hypothesis made it follows that $f(x,y)=(\delta_\Omega(x,y))^{-1}$ is a continuous $PSC^+(\Omega)$-function. From Remark 3.2 it follows that

$g(x,y)=\sum_{i=1}^{m}|\xi_i|^2+\sum_{j=1}^{k}|\eta_j|^2$ is also a continuous $PSC^+(\Omega)$-function.

The function $\varphi=\max(f,g)$ is the desired one. ■

As a last step the four geometric conditions for a DB are shown to be equivalent.

THEOREM 3.14. If the domain $\Omega\subset R^{m+k+2}$ has a positive, continuous plurisubconstant exhaustion function, then Ω is $PSC^+(\Omega)$-convex.

Proof. Take a compact subset K of Ω. For $f\in PSC^+(\Omega)$ put $c_f=\sup_{(u,v)\in K} f(u,v)$. Then using the exhaustion function $\varphi\in PSC^+(\Omega)$ we have $\hat{K}(PSC^+(\Omega))\subset\{(x,y)\in\Omega:\varphi(x,y)<c_\varphi\}=\overline{\Omega}_{c_\varphi,\varphi}$, the latter being a compact subset of Ω. ■

In conclusion of this section we may thus state :

THEOREM 3.15. If Ω is a domain in R^{m+k+2} then the following conditions are equivalent :
(a) Ω is a plurisubconstant domain;
(b) Ω has a positive, continuous plurisubconstant exhaustion function;
(c) Ω is convex w.r.t. the positive plurisubconstant functions;
(d) Ω satisfies the weak continuity principle.
Moreover a domain of biregularity has all the properties (a) to (d).

4. Biregular existence domains

4.1 First let us introduce the notion of a biregular existence domain.

DEFINITION 4.1. Let Ω be a domain in R^{m+k+2} and let f be biregular in Ω.
(i) Ω is called a weak biregular existence domain (WBED) for f, if for each biregular function F in a domain $\Omega'\supsetneq\Omega$, $F|\Omega\neq f$.
(ii) Ω is called a biregular existence domain (BED) for f, if for each pair (U_1,U_2) of domains in R^{m+k+2} for which $\phi\neq U_2\subset\Omega\cap U_1\subsetneq U_1$

and for each biregular function F in U_1, $F|U_2 \neq f|U_2$.

REMARK 4.2. Obviously a BED is also a WBED and it may be shown that under the additional assumption of the local connectedness of Ω, both notions coincide. Next it is also clear that a (W)BED is also a (W)DB.

4.2. As is well known in classic complex analysis a holomorphically convex domain, or equivalently a pseudoconvex domain, is also a holomorphic existence domain. The analogous problem in the biregular setting is still open; nevertheless it may partially be solved by introducing a stronger notion of convexity.

DEFINITION 4.3. The family ${}_2\mathcal{B}(\Omega)$ consists of the functions f satisfying the following conditions :
(i) f is biregular in Ω;
(ii) $|f.f|=|f|^2$;
(iii) all the functions f^{2^n}, $n\in N$, satisfy (i) and (ii).

In the same way as was done in the subsections 2.1 and 2.3, the notions of a ${}_2\mathcal{B}$-convex hull of a compact subset of Ω, and of ${}_2\mathcal{B}$-convexity may be introduced. It is such that ${}_2\mathcal{B}$-convexity implies the biregular convexity.

The following result can be proved now along analogous lines as in the proof of [5], Theorems 4.5, 4.6 and 4.7.

THEOREM 4.4. If the domain Ω is ${}_2\mathcal{B}(\Omega)$-convex then there exists a biregular function in Ω for which Ω is a biregular existence domain.

EXAMPLES 4.5.
(i) The cartesian product of a left-monogenic and a right-monogenic existence domain in R^{m+1} and R^{k+1} respectively, becomes a biregular existence domain in R^{m+k+2}.
(ii) It is a remarkable fact that the case of holomorphic functions of two complex variables is included in the biregular function theory (see also [1]). Indeed if f is holomorphic in $\omega\subset C^2$ then $F(x,y)=(\mathrm{Re}f)e_1-(\mathrm{Im}f)e_2$ is biregular in $\Omega=\{(x,y)\in R^6 : (x_1+ix_2,y_1+iy_2)\in\omega,\ x_0\in R,\ y_0\in R\}$. So if ω is taken to be pseudo-

convex and hence a holomorphic existence domain for a certain f, then the tube domain Ω is a biregular existence domain for the corresponding F.

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