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Jürgen Eichhorn

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In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1987. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 16. pp. [29]--41.

Persistent URL: http://dml.cz/dmlcz/701407

THE EULER CHARACTERISTIC AND SIGNATURE FOR OPEN MANIFOLDS

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1. Introduction

In [5] we studied the following situation. Given an open complete Riemannian manifold (M^n,g), a principal fibre bundle $P(M,G) \longrightarrow M^n$ and a connection ω on P. Chern-Weil construction and taking characteristic n-forms $c(P,\omega)$ defines characteristic numbers

$$c(P,\omega,M) = c(P,\omega)[M] = \int_M c(P,\omega)$$

if the latter integral converges. Thus one has at first to assure the existence of the integral and at second to clarify how $c(P,\omega,M)$ depends on the connection ω. To do this we introduced the completed space ${}^{1}\mathcal{C}^{d}_{P,f,b}$ of connections ω with bounded curvature R^{ω} and finite 1-action $\mathcal{A}(\omega) = \int_M |R^{\omega}|$ dvol and proved the

<u>Theorem.</u> Characteristic numbers exist for and are constant at the components of ${}^{1}\mathcal{C}^{d}_{P,f,b}$.

The metric of M^n did not enter into the characteristic numbers but was used to define and topologize the space ${}^{p}\mathcal{C}^{d}_{P,f,b}$.

Here we consider the case M^n open, oriented, g allowed to vary, $\omega = \omega_g$ the Levi-Civita connection. We denote by $E(g)$ the Euler form corresponding to g and set

$$\chi(M^n,g) = \int_M E(g).$$

In an analogous manner $S(g)$ shall denote the signature form and

$$\sigma(M^n,g) = \int_M S(g).$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

Then there arise the following natural questions.
1. Under which conditions on g is $\chi(M^n,g)$ defined?
2. How does it depend on g?
3. What is the topological meaning of $\chi(M^n,g)$?
4. Under which conditions does there hold $\chi(M^n,g) = \chi(M^n)$, i.e. the Gauß-Bonnet formula?
5. The questions 1. - 4. for $\sigma(M^n,g)$, $\sigma(M^n)$.

These questions are attacked successful by fundamental work of Cheeger and Gromov ([3] , [4]) and Rosenberg ([9]). Cheeger and Gromov made the general assumption $\mathrm{vol}(M,g) < \infty$, $|K| \le 1$ for the sectional curvature and $r_{inj}(\tilde{M}) \ge 1$ for some normal or profinite covering $\tilde{M}$ of M. This altogether they denote by $\mathrm{geo}(\tilde{M}) \le 1$. We here exhibit that the condition $\mathrm{geo}(M) \le 1$ is not necessary for answering the above questions and study in particular the dependence on g. To do this we topologize the space of Riemannian metrics in an appropriate manner as described in section 3. In the 4 th section we present the invariance theorems which come out by our approach (theorem 4.1, 4.3). The proofs essential use L_p-cohomology. The 5 th section is devoted to dimension 4 where some nice results immediately come out from our approach.

2. The attack of the problem and first results

Starting with the Euler characteristic, we remark that for $n = \dim M$ odd the Euler form $E(g)$ vanishes identically. Therefore the answers to questions 1.,2. are trivial. 4. is affirmatively answered if and only if $\chi(M^n) = 0$. For M^n with a finite number of ends, each of them smoothly collared, i.e. compactificable to $i: M \longrightarrow \bar{M}$, $\bar{M}$ compact, this holds if and only if $\chi(\partial M) = 0$: $0 = \chi(\bar{M} \cup \bar{M}) = 2\chi(\bar{M}) - \chi(\partial\bar{M}) = 2\chi(M) - \chi(\partial\bar{M})$. The only interesting case for the Euler characteristic $\chi(M^n,g)$ is the case n even. A simple and in a certain sense complete answer to the above questions can be given in the case $\mathrm{vol}(M^n,g) < \infty$, $-b^2 \le K \le -a^2 < 0$.

<u>Theorem 2.1.</u> Suppose (M^n,g) complete, $\mathrm{vol}(M^n,g) < \infty$, $-b^2 \le K \le -a^2 < 0$. Then there holds $\chi(M^n,g) = \chi(M^n)$.

Proof. From the assumption follows that (M^n,g) posses a finite number of ends $\varepsilon_1, \dots, \varepsilon_k$, each of them with a Riemannian collar, i.e. each end has a collared neighbourhood $U \cong \partial U \times [0,\infty[$ such that $ds^2|_U \cong dr^2 + ds^2|_{\partial U \times \{r\}}$. Taking a chart $(U', u^1, \dots, u^{n-1})$ in ∂U, $ds^2|_{U' \times [0,\infty[} = dr^2 + \sum_{i,j=1}^{n-1} h_{ij}(u,r)\,du^i du^j$ with

$$h_{ij}(u,0)\ e^{-2br} \le h_{ij}(u,r) \le h_{ij}(u,0)\ e^{-2ar}.$$

This implies $\lim_{r\to\infty} \mathrm{vol}(\partial U_k \times \{r\}, ds^2|_{\partial U_k \times \{r\}}) = 0,\ k=1,\dots,k.$

Further the second fundamental form of $\partial U_k \times \{r\}$ is bounded.
For M^n we can write $M^n = M'^n \cup \bigcup_{k=1}^{k} \partial U_k \times [0,\infty[$, M'^n compact with boundary $\partial M'^n = \bigcup \partial U_k$.
For any compact manifold $M_i^n \subset M^n$ with boundary ∂M_i^n there holds

$$\chi(M_i^n, g|_{M_i^n}) + II_\chi(\partial M_i^n, g|_{\partial M_i^n}) = \chi(M_i^n), \tag{2.1}$$

where $II_\chi(\partial M_i^n, g|_{M_i^n}) = \int_{\partial M_i^n} II_E$ and II_E is an (n-1)-form directed by the second fundamental form. If one has an exhaustion $M_1^n \subset M_2^n \subset \cdots$ of M^n such that

$$\mathrm{vol}(\partial M_i^n) \xrightarrow[i\to\infty]{} 0 \ , \ II_E(\partial M_i^n) \text{ bounded} \tag{2.2}$$

then, taking in (2.1) the limit $i \longrightarrow \infty$, one obtains

$$\chi(M^n, g) = \chi(M^n), \tag{2.3}$$

since $\lim_{i\to\infty} |II_\chi(\partial M_i^n)| = \lim_{i\to\infty} |\int_{\partial M_i^n} II_E(\partial M_i^n)| \le$

$$\le \lim_{i\to\infty} |II_E(\partial M_i^m)|\cdot \mathrm{vol}(\partial M_i^n) = 0,\ \lim_{i\to\infty} \chi(M_i^n) = \chi(M^n).$$

In our case we set $M_i^n = M'^n \cup \bigcup_k \partial U_k \times [0,i]$ and (2.2) is satified. □

Examples are certain cusp manifolds

$$(M^n, g) = (M'^n \cup N_1 \times [0,\infty[\cup \cdots \cup N_k \times [0,\infty[\ ,g),$$

$$ds^2|_{N_k \times [0,\infty[} = dr^2 + (e^{-r})^2\, d\sigma^2_{N_k}.$$

The curvature formulas at $N\times[0,\infty[$ are well known. These cusp manifolds arise as Riemannian manifolds which are locally symmetric at infinity (generated ba rank 1 lattices $\Gamma \subset G$).
A modified situation is settled by

<u>Theorem 2.2.</u> Suppose (M^n, g) open, complete with a finite number of Riemannian collared ends. Assume for each end ε there exists a neighbourhood $U(\varepsilon) \cong N_1 \times \cdots \times N_k \times [0,\infty[$ with

$$ds^2\Big|_{U(\varepsilon)} = f_1(r)^2\, d\sigma^2_{N_1} + \cdots + f_k(r)^2\, d\sigma^2_{N_k} + dr^2.$$

If $\lim_{r\to\infty} f_\kappa(r) = \lim_{r\to\infty} f'_\kappa(r) = 0$, $\kappa = 1,\dots,k$, then

$$\chi(M^n,g) = \chi(M^n).$$

Proof. [9]. □

As a special case we consider surfaces and start with a famous theorem of Cohn-Vossen.

Theorem 2.3. (Gauß-Bonnet inequality) Suppose (M^2,g) open, complete, oriented, $\pi_1(M^2)$ finitely generated. It the Gaußian curvature K is absolutely integrable, then

$$\chi(M) \geq \frac{1}{2\pi}\int_M K\, dvol. \ \square$$

Theorem 2.4. Suppose M^2 open, complete, oriented, $\pi_1(M^2)$ finitely generated, $vol(M^2) < \infty$, K absolutely integrable. Then

$$\chi(M) = \frac{1}{2\pi}\int_M K\, dvol.$$

Proof. [7]. □

Remark 2.5. The curvature K is allowed to be unbounded.

Remark 2.6. The condition $vol(M) < \infty$ is far of being necessary.

Example. Let (M^2,g) be the surface of revolution $z = f(x^2+y^2)$ for $f \in C^\infty([0,\infty[)$, $f(0) = f'(0) = 0$, the metric induced from $\mathbb{R}^3$. Then $\chi(M) = (2\pi)^{-1}\int K\, dvol$ if and only if $t^{1/2}f'(t) \xrightarrow[t\to\infty]{} \pm\infty$. Taking $f(t) = t^{2n}$, $n \geq 1$, supplies examples with $vol(M,g) = \infty$ and positive curvature.

As a conclusion we see that one has to give up the assumption $|K| \leq 1$, $vol(M) < \infty$ and to consider the more general case. This does not contradict to the matter of fact that in the case $vol(M) < \infty$, $|K| \leq 1$ Gromov and Cheeger were very successful in attacking the problem. Their methods work at the first instance only under their restrictive conditions.

In a similar manner one treats the signature $\sigma(M)$. The starting point is the analogous equation to (2.1) for $(M^n, \partial M^n)$ compact with boundary,

$$\sigma(M) = \sigma(M,g) + \eta(\partial M,g) + II_\sigma(\partial M,g), \qquad (2.4)$$

where $\eta(\partial M,g)$ is the η-invariant of [1] and $II_\sigma(\partial M,g)$ is the integral of an (n-1)-form directed by the second fundamental form. The natural way to attack the problem is seeking for an exhaustion $M_1^n \subset M_2^n \subset \cdots$ of M^n and to assure

$$\eta(\partial M_i,g) \xrightarrow[i\to\infty]{} 0, \quad II_\sigma(\partial M_i,g) \xrightarrow[i\to\infty]{} 0. \tag{2.5}$$

The second limit condition is satisfied if the second fundamental form of ∂M_i is bounded (independent of i) and $\mathrm{vol}(\partial M_i) \longrightarrow 0$, or the second fundamental form tends to 0 and $\mathrm{vol}(\partial M_i)$ is bounded. Then the equation $\sigma(M) = \sigma(M,g)$ holds if and only if $\eta(\partial M_i,g) \xrightarrow[i\to\infty]{} 0$. For this it would be sufficient

$$|\eta(\partial M_i)| \leq C\cdot \mathrm{vol}(\partial M_i) \text{ and } \mathrm{vol}(\partial M_i) \xrightarrow[i\to\infty]{} 0.$$

Along this line Cheeger and Gromov attacked the problem under the assumption $\mathrm{geo}(\tilde{M}) \leq 1$. We return to their solution in the 4 th section.

Remark 2.7. $\sigma(M,g)$ can be (if it exists) an arbitrary irrational number. But $\sigma(M)$ defined as the signature of a certain intersection form is an integer (if it exists in the open case). Therefore the equation $\sigma(M) = \sigma(M,g)$ holds "very rarely". One has to give $\sigma(M,g)$ a new topological meaning as done in [3] . .

3. The space of Riemannian metrics on a noncompact manifold

For the preparation of the invariance theorem we have to introduce an appropriate and natural topology into the set of Riemannian metrics on a noncompact manifold. This shall be done now. Suppose M^n being open, connected, oriented, TM the tangent bundle, g a complete metric on M. The pointwise norm of a tensor $t \in C^\infty(\bigotimes_1^r TM \otimes \bigotimes_1^s T^*M)$ of type (r,s) with respect to g is defined by

$$|t|^2_{g,x} = \frac{1}{r!\,s!}\, t^{j_1\ldots j_s}_{i_1\ldots i_r}\, t^{i_1\ldots i_r}_{j_1\ldots j_s}, \tag{3.1}$$

where we apply the Einstein summation convention. If $e_1,\ldots,e_n$ T_xM is an orthonormal base and $e^1,\ldots,e^n$ the dual base, then we can (3.1) write as

$$|t|^2_{g,x} = \frac{1}{r!\,s!} \sum_{\substack{i_1,\ldots,i_r\\ j_1,\ldots,j_s}} t(e^{i_1},\ldots,e^{i_r},e_{j_1},\ldots,e_{j_s}).$$

As uniform structure ${}^bU(g)$ of g we define the set of all Riemannian metrics g' such that $|g-g'|_{g,x}$ and $|g-g'|_{g',x}$ are bounded on M.

Lemma 3.1. ${}^bU(g)$ coincides with the quasi isometry class of g, in particular are the conditions $g' \in {}^bU(g)$ and $g \in {}^bU(g')$ equivalent.

Proof. Assume $\quad C_1\, g \leq g' \leq C_2\, g, \; C_i = C_i(g,g')$ (3.2) in the sense of positively definite forms. Let $e_1,\dots,e_n \in T_xM$ be an orthonormal base with respect to g. Then (3.2) implies $g'(e_i,e_i) \le C_2\, g(e_i,e_i) = C_2$. Squaring and summing up gives $|g'|^2_{g,x} \le C_2^2\cdot n$, and we obtain together with $|g-g'|_{g,x} \le |g|_{g,x} + |g'|_{g,x} \le \sqrt{n} + C_2\sqrt{n} = (C_2+1)\sqrt{n}$, i.e. $|g-g'|_{g,x}$ is bounded on M. In the same manner one shows $|g-g'|_{g',x}$ bounded on M. Suppose now $|g-g'|_{g,x}$, $|g-g'|_{g',x}$ bounded on M. Then again $|g'|_{g,x} \le |g-g'|_{g,x} + |g|_{g,x} \le C_2$. If $e_1,\dots,e_n \in T_xM$ is an orthonormal base with respect to g, then $\sum g'(e_j,e_j)^2 \le C_2^2 = C_2^2\cdot g(e_i,e_i)^2$, in particular $g'(e_i,e_i) \le C_2 g(e_i,e_i)$, $i = 1,\dots,n$, i.e. $g' \le C_2\cdot g$. The second inequality follows in the same way. □

Let $f\colon R_+ \longrightarrow R_+$ be a nonnegative function. As growth type of f we define the equivalence class of f with respect to the equivalence relation $f_1 \sim f_2$: There exist constants $a,b,c,d > 0$ such that $f_1(t) \le a\, f_2(bt)$, $f_2(t) \le c\, f_1(dt)$. The growth type of a Riemannian manifold (M^n,g) shall be defined by the growth type of $f(t) = \mathrm{vol}(B_t(x_0))$, where $B_t(x_0)$ denotes the metric ball of radius t centered at $x_0 \in M$. The growth type is independent of x_0 and an invariant of ${}^bU(g)$. In particular each metric $g' \in {}^bU(g)$ is complete, since every g'-bounded set N' is contained in a g-bounded set N (lemma 3.1). N is relatively compact, thus N' too. (M^n,g) has the growth type of a bounded function if and only if $\mathrm{vol}(M^n,g)<\infty$. The same then also holds for all $\mathrm{vol}(M^n,g')$, $g' \in {}^bU(g)$. $\mathrm{vol}(M^n,g) = \infty$ if and only if (M^n,g) has the growth type of an unbounded function. This is equivalent to $\mathrm{vol}(M^n,g') = \infty$ for all $g' \in {}^bU(g)$.

If $\sup_{x\in M} |t|_{g,x}$ exists we define the sup-norm ${}^b\|t\|_g$ of t with respect to g by ${}^b\|t\|_g = \sup_{x\in M} |t|_{g,x}$. From $g' \in {}^bU(g)$ follows the existence of bounds $A_k(g,g')$, $B_k(g,g') > 0$ such that

$$A_k|t|_{g,x} \le |t|_{g',x} \le B_k|t|_{g,x}, \tag{3.3}$$

$$A_k\|t\|_g \le \|t\|_{g'} \le B_k\|t\|_g \tag{3.4}$$

for every (r,s) tensor field t with $r+s = k$ and $\|t\| = {}^b\|t\|$.

In what follows we still need norms of higher derivatives. For metrics g,g' we set $B = g' - g$, $D = \nabla' - \nabla = \nabla^{g'} - \nabla^{g}$.

<u>Lemma 3.2.</u> Suppose $g' \in {}^bU(g)$. The boundness on M of one of the the following terms implies the boundness on M of all others, $|\nabla g'|_g$, $|\nabla g'|_{g'}$, $|\nabla' g|_g$, $|\nabla' g|_{g'}$, $|\nabla B|_g$, $|\nabla B|_{g'}$, $|\nabla' B|_g$, $|\nabla' B|_{g'}$, $|D|_g$, $|D|_{g'}$, where $\nabla = \nabla^g$, $\nabla' = \nabla^{g'}$ and we omitted the index x.

Proof. We use

$$|\nabla g'| = |\nabla B|, \ |\nabla' g| = |\nabla' B|, \qquad (3.5)$$

always taken with respect to the same metric,

$$g'(D(X,Y),Z) = \tfrac{1}{2} \ \nabla_X B(Y,Z) + \nabla_Y B(X,Z) - \nabla_Z B(X,Y), \qquad (3.6)$$

$$\nabla'_X B(Y,Z) = g(D(X,Y),Z) + g(Y,D(X,Z)). \qquad (3.7)$$

Then, omitting "bounded on M", we have the following implications: $|\nabla g'|_g \Leftrightarrow |\nabla g'|_{g'} \Leftrightarrow |\nabla B|_{g'}$ by (3.3) and (3.5), $|\nabla B|_{g'} \Rightarrow$ $\Rightarrow |\nabla D|_{g'}$ by (3.6), $|D|_{g'} \Leftrightarrow |D|_g$ by (3.3), $|D|_g \Rightarrow |\nabla' B|_g$ by (3.7), $|\nabla' B|_g \Leftrightarrow |\nabla' g|_g \Leftrightarrow |\nabla' g|_{g'}$ by (3.5) and (3.3), $|\nabla' B|_g \Leftrightarrow |D|_g$ follows from (3.6), replacing g' by g, ∇ by ∇' ($g' \in {}^bU(g)$ if and only if $g \in {}^bU(g')$!), by the same procedure for (3.7) we obtain $|D|_{g'} \Leftrightarrow |\nabla B|_{g'}$, by (3.5) $|\nabla B|_{g'} \Leftrightarrow |\nabla g'|_{g'}$, and the circle is closed. □

Now we set

$${}^{b,1}U(g) = \{g' \in {}^bU(g) \mid {}^b\|D\|_g < \infty, \ \|\nabla^g D\|_g < \infty\}.$$

Lemma 3.3. $g' \in {}^{b,1}U(g)$ if and only if $g \in {}^{b,1}U(g')$.

Proof. According to lemma 3.2 it remains only to show ${}^b\|D\|_g < \infty$, ${}^b\|\nabla D\|_g < \infty$ imply ${}^b\|\nabla' D\|_{g'} < \infty$ (the other direction one gets by changing g, ∇ with g', ∇'). Now $\nabla' D = \nabla' D - \nabla D + \nabla D$, ${}^b\|\nabla' D\|_{g'} \le {}^b\|DD\|_{g'} + {}^b\|\nabla D\|_{g'} = B_3({}^b\|DD\|_g + {}^b\|\nabla D\|_g) <$.

Remark 3.4. We now ${}^b\|D\|_g < \infty$ is equivalent to ${}^b\|\nabla g'\|_g < \infty$. Therefore ${}^b\|\nabla D\|_g < \infty$ means a condition for the second derivatives of the metric, and it would be also reasonable to write ${}^{b,2}U(g)$ instead of ${}^{b,1}U(g)$. We decided to write ${}^{b,1}U(g)$ since we consider the conditions on the second derivatives of the metric as conditions on the first derivatives of D.

Assume $p \ge 1$. We set

$${}^bU^p(g) = \{g' \in {}^bU(g) \mid \int |g-g'|^p_{g,x} \, dvol(g)_x < \infty\},$$

$${}^bU^{p,1}(g) = \{g' \in {}^bU^p(g) \mid \int |D|^p_{g,x} \, dvol(g)_x < \infty, \ |\nabla D|^p_{g,x} \, dvol_x < \infty\},$$

$${}^{b,1}U^{p,1}(g) = {}^{b,1}U(g) \cap {}^bU^{p,1}(g).$$

Lemma 3.5. a. $g' \in {}^bU^p(g)$ if and only if $g \in {}^bU^p(g')$.

b. $g' \in {}^bU^{p,1}(g)$ if and only if $g \in {}^bU^{p,1}(g')$.

c. $g' \in {}^{b,1}U^{p,1}(g)$ if and only if $g \in {}^{b,1}U^{p,1}(g')$.

Proof. This follows immediately from lemma 3.1, 3.2 and the derivation of (3.6), (3.7). □

Now we consider

$$^b\mathcal{M} = \{g \mid g \text{ complete metric with } |R^g|_{g,x} \text{ bounded on } M\},$$

$$\mathcal{M}^p_f = \{g \mid g \text{ complete metric with } \int |R^g|^p_{g,x}\, dvol(g)_x < \infty\},$$

$$^b\mathcal{M}^p_f = {}^b\mathcal{M} \cap \mathcal{M}^p_f.$$

Lemma 3.6. If $g \in {}^b\mathcal{M}$, $g' \in {}^{b,1}U(g)$, then $g' \in {}^b\mathcal{M}$.
Proof. With $R = R^g$, $R' = R^{g'}$ there holds

$$R'(U,V)W = R(U,V)W + D(U,D(V,W)) - D(V,D(U,W)) - D(D(U,V),W) + D(D(V,U),W) + \nabla_U D(V,W) - \nabla_V D(U,W), \tag{3.8}$$

i.e. $R, D, \nabla D$ bounded imply R' bounded. □
Now we are able to introduce a natural topology for $^b\mathcal{M}$.
If $g \in {}^b\mathcal{M}$, $\varepsilon > 0$, then we set

$$^{b,1}U_\varepsilon(g) = \{g' \in {}^{b,1}U(g) \mid {}^b\|g-g'\|_g < \varepsilon,\ {}^b\|\nabla^i D\|_g < \infty,\ i=0,1\}.$$

According to (3.4) and lemma 3.3 there exists a $\delta > 0$ such that $^{b,1}U_\varepsilon(g)$ is a neighbourhood for all $g' \in {}^{b,1}U_\delta(g)$. Altogether this means that the system of all $^{b,1}U_\varepsilon(g)$, $g \in {}^b\mathcal{M}$, $\varepsilon > 0$, defines a locally metrizable topology for $^b\mathcal{M}$ with $\{^{b,1}U_\varepsilon(g)\}_{\varepsilon>0}$ as neighbourhood base for $g \in {}^b\mathcal{M}$. Let $^b\overline{\mathcal{M}}$ be the completion of $^b\mathcal{M}$ with respect to this topology.
In similar manner we treat $\mathcal{M}^p_f$. This shall be prepared by
Lemma 3.7. If $g \in \mathcal{M}^p_f$ and $g' \in {}^{b,1}U^{p,1}(g)$ then $g' \in \mathcal{M}^p_f$.
Proof. This follows from (3.8) and by use of $|(s,t)_x|^p \le |t|^p_x \cdot |s|^p_x$ and $|t|^p_x\,|s|^p_x$ is an element of L_1 if $|t|^p_x$ L_1 and $|s|_x$ is bounded.
We denote $^p\|g-g'\| \equiv {}^p\|g-g'\|_g \equiv (\int |g-g'|^p_{g,x}\, dvol(g)_x)^{1/p}$, analogous $^p\|\nabla^i D\| = {}^p\|\nabla^i D\|_g$ and set for $g \in \mathcal{M}^p_f$, $\varepsilon > 0$

$$^{b,1}U^{p,1}_\varepsilon(g) = \{g' \in {}^{b,1}U^{p,1}(g) \mid {}^p\|g-g'\|_g < \varepsilon,\ {}^p\|\nabla^i D\|_g < \infty,\ i = 0,1\}.$$

Again according to (3.4) and lemma 3.3 the system of all $^{b,1}U^{p,1}(g)$, $g \in \mathcal{M}^p_f$, $\varepsilon > 0$, defines a locally metrizable topology for $\mathcal{M}^p_f$ whose completion we denote by $\overline{\mathcal{M}}^p_f$.
Lemma 3.8. $^b\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}^p_f$ are locally arcwise connected.
Proof. it is sufficient to show the locally arcwise connectness of $^b\mathcal{M}$ and $\mathcal{M}^p_f$. We show the local contractability which implies the locally arcwise connectness. This is done if for $0 < t < 1$, $g' \in {}^{b,1}U_\varepsilon(g)$ $tg'+(1-t)g \in {}^{b,1}U_\varepsilon(g)$. But $^b\|tg'+(1-t)g-g\|_g = {}^b\|t(g'-g)\|_g = t({}^b\|g'-g\|_g) < \varepsilon$, the first condition is satisfied. Now $\|D\|_g < \infty$ is equivalent to $\|\nabla g'\|_g < \infty$, thus $^b\|\nabla(tg'+(1-t)g\|_g = t\|\nabla g'\| < \infty$. In analogous manner $^b\|\nabla^2(tg'+(1-t)g)\|_g = t \cdot {}^b\|\nabla^2 g'\|_g < \infty$, altogether we have pro-

ven ${}^{b}\|\nabla^{tg'+(1-t)g} - \nabla^{g}\|_{g} < \infty$, ${}^{b}\|\nabla(\nabla^{tg'+(1-t)g} - \nabla^{g}\|_{g} < \infty$. The proof for $\mathcal{M}_f^p$ is completely parallel, replacing ${}^{b}\|\ \|_g$ by ${}^{p}\|\ \|_g$.

Corollary 3.9. In ${}^{b}\mathcal{M}$, ${}^{b}\overline{\mathcal{M}}$, $\mathcal{M}_f^p$, $\overline{\mathcal{M}}_f^p$ coincide components and arc components. □

Now we are able to prove our first main theorem.

Theorem 3.10. a. Suppose $g \in {}^{b}\mathcal{M}$. Then the component of g in ${}^{b}\mathcal{M}$ resp. ${}^{b}\overline{\mathcal{M}}$ coinsides with ${}^{b,1}U(g)$ resp. ${}^{b,1}\overline{U(g)}$.

b. Suppose $g \in \mathcal{M}_f^p$. Then the component of g in $\mathcal{M}_f^p$ resp. $\overline{\mathcal{M}}_f^p$ coincides with ${}^{b,1}U^{p,1}(g)$ resp. ${}^{b,1}\overline{U^{p,1}(g)}$.

Proof. We start with ${}^{b}\mathcal{M}$. According to corollary 3.9 we have to consider arc components. Assume g' to be an element of the arc component of g in ${}^{b}\mathcal{M}$, and let $\{g_t\}_{0 \le t \le 1}$ be an arc between g and g', $g_0 = g$, $g_1 = g'$. The arc can be covered by a finite number of open neighbourhoods ${}^{b,1}U_\varepsilon(g_0), {}^{b,1}U_\varepsilon(g_{t_1}), \dots, {}^{b,1}U_\varepsilon(g_{t_r}) = {}^{b,1}U_\varepsilon(g')$, ${}^{b,1}U_\varepsilon(g_{t_{i-1}}) \cap {}^{b,1}U_\varepsilon(g_{t_i}) = \emptyset$.

If $g_{i-1,i} \in {}^{b,1}U_\varepsilon(g_{t_{i-1}}) \cap {}^{b,1}U_\varepsilon(g_{t_i})$, then we have

$$ {}^{b,1}U(g_{t_{i-1}}) \supset {}^{b,1}U_\varepsilon(g_{t_{i-1}}) \ni g_{i-1,i} \in {}^{b,1}U_\varepsilon(g_{t_i}) \subset {}^{b,1}U(g_{t_i}), $$

i.e. according to lemma 3.5 ${}^{b,1}U(g_{t_{i-1}}) = {}^{b,1}U(g_{t_i})$, which implies ${}^{b,1}U(g) = {}^{b,1}U(g')$, $g' \in {}^{b,1}U(g)$. Suppose now $g' \in {}^{b,1}U(g)$. We will show that g' is an element of the arc component of g. This is done if there exists an arc in ${}^{b}\mathcal{M}$ lying in ${}^{b,1}U(g)$ between g' and g. Set $g_t = tg' + (1-t)g$. This is in fact an arc in ${}^{b}\mathcal{M}$ since

$$ |R^{g_t}|_{g_t,x} \le |R^{g}|_{g,x} + |R^{g'}|_{g',x} + (|D|_{g,x} + |D|_{g',x})^2 \qquad (3.9) $$

([3]). Further we conclude as in the proof of lemma 3.8 that ${}^{b}\|g-g_t\|_g < \infty$, ${}^{b}\|\nabla^{g_t} - \nabla^{g}\|_g < \infty$, ${}^{b}\|\ (\nabla^{g_t} - \nabla^{g})\|_g < \infty$. Since g,g' ly in the same isometry class g and g_t ly in the same isometry class too. Thus we obtain ${}^{b}\|\ \|_{g_t} < \infty$ for the above expressions. For ${}^{b}\mathcal{M}$ a. is proven, and the extension to ${}^{b}\overline{\mathcal{M}}$ is trivial. The proof for b. is performed completely parallel, replacing ${}^{b}\|\ \|$ by ${}^{p}\|\ \|$, using $|R^{g}|_{g,x}$, $|R^{g'}|_{g',x}$, $|D|_{g,x}$, $|D|_{g',x} \in L_p$, $|D|_{g,x}$, $|D|_{g',x}$ bounded, (3,9) and the translated arguments in the proof of lemma 3.8. □

Finally we consider ${}^{b}\mathcal{M}_f^p = {}^{b}\mathcal{M} \cap \mathcal{M}_f^p$ with the weakest topology such that both inclusions ${}^{b}\mathcal{M}_f^p \hookrightarrow {}^{b}\mathcal{M}$, $\mathcal{M}_f^p$ are continuous. This is just the topology generated by the ${}^{b,1}U^{p,1}(g)$, $g \in {}^{b}\mathcal{M}_f^p$, $\varepsilon > 0$. Then immediately follows

Theorem 3.11. Suppose $g \in {}^b\mathcal{M}^p_f$. Then the component of g in ${}^b\mathcal{M}^p_f$ resp. ${}^b\overline{\mathcal{M}}^p_f$ coinsides with ${}^{b,1}U^{p,1}(g)$ resp. ${}^{b,1}\overline{U}^{p,1}(g)$. □

4. The existence and invariance of the Euler characteristic and signature

For the proof of the invariance theorem we still need L_p-cohomology which we now shortly define. By Ω^q we denote the vector space of all smooth q-forms on M. Given some metric g on M, then for $\varphi \in \Omega^q$ ${}^p\|\varphi\| = (\int |\varphi|^p dvol)^{1/p}$ is defined, if the latter integral converges. Denote

$$ {}^p\Omega^q_d = {}^p\Omega^q_d(g) = \{\varphi \in \Omega^q \mid {}^p\|\varphi\| < \infty ,\ {}^p\|d\varphi\| < \infty\} $$

and

$$ {}^p\Omega^{q,d} = \text{completion of } {}^p\Omega^q_d \text{ with respect to } {}^p\|\ \|_d, $$
$$ {}^p\|\varphi\|_d = {}^p\|\varphi\| + {}^p\|d\varphi\| . $$

The cohomology of the complex

$$ 0 \longrightarrow {}^p\Omega^{0,d} \longrightarrow {}^p\Omega^{1,d} \longrightarrow \cdots \longrightarrow {}^p\Omega^{q,d} \longrightarrow \cdots \longrightarrow {}^p\Omega^{n,d} \longrightarrow 0 $$

is called the analytical L_p-cohomology ${}^pH^*(M,\overline{d})$ of (M^n,g),

$$ {}^pH^q(M,d) := \ker(d: {}^p\Omega^{q,d} \longrightarrow {}^p\Omega^{q+1,d})/\operatorname{im}(d: {}^p\Omega^{q-1,d} \longrightarrow {}^p\Omega^{q,d}) = $$
$$ = {}^pZ^q(M,\overline{d})/{}^pB^q(M,\overline{d}). $$

The complex

$$ 0 \longrightarrow {}^p\Omega^0_d \longrightarrow {}^p\Omega^1_d \longrightarrow \cdots \longrightarrow {}^p\Omega^q_d \longrightarrow \cdots \longrightarrow {}^p\Omega^n_d \longrightarrow 0 $$

defines the cohomology ${}^pH^*(M,d)$. According to a result of Cheeger ([2]) the inclusion ${}^p\Omega^*_d \hookrightarrow {}^p\Omega^{*,d}$ induces an isomorphism ${}^pH^*(M,d) \longrightarrow {}^pH^*(M,\overline{d})$. For this reason we identify these spaces and write simply ${}^pH^q(M)$.

Now we are able to prove the invariance

Theorem 4.1. a. If $g \in \mathcal{M}^1_f$ and $\chi(M,g)$ exists, then $\chi(M,g')$ exists for all g' of the component of g and $\chi(M,g) = \chi(M,g')$.
b. If $g \in \mathcal{M}^1_f$ and $\sigma(M,g)$ exists, then $\sigma(M,g')$ exists for all g' of the component of g and $\sigma(M,g) = \sigma(M,g')$.

Proof. Suppose $\chi(M,g) = \int_M E(g)$ exists. If g' is an element of the component of g in $\mathcal{M}^1_f$, then $g' \in {}^{b,1}U^{1,1}(g)$. Since g,g' are quasi isometric $({}^1\Omega^{*,d}(g),d)$, $({}^1\Omega^{*,d}(g'),d)$ are equivalent L_1-complexes and ${}^1H^*((M,g),d)$, ${}^1H^*((M,g'),d)$ coincide. There exists an arc between g and g' in ${}^{b,1}U^{1,1}(g) \subset \mathcal{M}^1_f$ which generates an (n-1)- form φ , φ and $d\varphi$ absolutely integrable, such that $E(g') = E(g) + d\varphi$, i.e. E(g) and E(g') are cohomological cocycles in ${}^1H^n(M,d)$ ([5]). According to a fundamental theorem of Gaffney ([6]) $\int_M d\varphi = 0$, i.e.

$$\chi(M^n,g') = \int_M E(g') = \int_M E(g) + \int_M d\varphi = \int_M E(g) = \chi(M^n,g).$$

The proof of b. for n=4k is completely analogous using $\sigma(M^n,g) = \int_M L$, $L = L(p(g))$ the Hirzebruch polynomial. □

Remark 4.2. The theorem extends immediately to the components in ${}^b\bar{\mathcal{M}}^1_f$.

Assume $\tilde{M} \longrightarrow M$ a normal covering with Deck($\tilde{M}$) = Γ , geo($\tilde{M}$) $\leq$ 1. Let $\Pi^q : {}^2\Omega^q(M) \longrightarrow \tilde{\mathcal{H}}^q$ be the orthogonal projection onto the L_2-harmonic forms $\tilde{\mathcal{H}}^q$, $\Pi^q(\varphi) = \int \tilde{h}^q(x,y)\varphi(y)dvol_y$ with a C^∞ symmetric kernel $\tilde{h}^q(x,y)$. The pointwise trace, $tr(h^q(x,x))$, is invariant under Γ and thus can be considered as a function on M. We set

$$\tilde{b}^q_{(2)}(M) = \int_M tr(\tilde{h}^q(x,x))dvol_x \text{ and}$$

$$\tilde{\chi}_{(2)}(M) = \sum_{q=0}^{n} (-1)^q \tilde{b}^q_{(2)}(M),$$

$$\tilde{\sigma}_{(2)}(M^{4k}) = \int_M tr(* \tilde{h}^{2k}(x,x))dvol_x.$$

Corollary 4.2. a. Suppose geo($\tilde{M}$,g) $\leq$ 1, $\tilde{M}$ a normal or profinite covering of M. If M has finite topological type (i.e. M has a finite number of ends, each of them smoothly collared), then

$$\chi(M^n) = \chi(M^n,g) = \chi(M^n,g'), \tag{4.1}$$

$$\sigma(M^n) = \sigma(M^n,g) = \sigma(M^n,g') \tag{4.2}$$

for all g' of the component of g in $\mathcal{M}^1_f$.

b. Suppose geo($\tilde{M}$,g) $\leq$ 1 for some normal covering $\tilde{M}$ of M. Then

$$\tilde{\chi}_{(2)}(M^n) = \chi(M^n,g) = \chi(M^n,g') \tag{4.3}$$

$$\tilde{\sigma}_{(2)}(M^n) = \sigma(M^n,g) = \sigma(M^n,g') \tag{4.4}$$

for all g' of the component of g in $\mathcal{M}^1_f$.

Proof. The first equation in (4.1)-(4.4) is contained in [3] , the second comes from theorem 4.1. □

Theorem 4.3. If $g \in {}^b\mathcal{M}^1_f$, then $\chi(M^n,g)$ resp. $\sigma(M^n,g)$ exists and $\chi(M^n,g) = \chi(M^n,g')$ resp. $\sigma(M^n,g) = \sigma(M^n,g')$ for all g' in the component of g.

Proof. The existence follows immediately from lemma 3.8, corollary 3.9 of [5] , the invariance from 4.1 above. □

5. Applications to 4-manifolds

M^4 shall denote an open oriented 4-manifold. The special orthogonal group acts on the space $\mathcal{C}_b^2$ of algebraic curvature tensors on M. Let $\mathcal{C}_b^2 = \mathcal{U} + \mathcal{S} + \mathcal{W}$ the corresponding decomposition into irreducible subspaces. Then this induces for the curvature tensor $R = R^g$ a decomposition $R = U+S+W$. For $R = R^g = R_+ + R_-$ we denote by $Ric = Ric^g$ the Ricci tensor, by $\tau = \tau^g$ the scalar curvature, by K the sectional curvature and by $W = W^g = W_+ + W_-$ the Weyl tensor. The sign $_+$ resp. $_-$ refers to the decomposition of $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ into self dual and anti self dual components.

Theorem 5.1. If $g \in \mathcal{M}_f^2$, then $\chi(M,g)$, $\sigma(M,g)$ exist and are constant on the component of g in $\mathcal{M}_f^2$.

Proof. For $|R|^2 = |R|_x^2$ there holds

$$|R|^2 = |U|^2 + |S|^2 + |W|^2, \tag{5.1}$$

$$|Ric|^2 = 6\,|U|^2 + 2\,|S|^2, \tag{5.2}$$

$$\tau^2 = 24\,|U|^2, \tag{5.3}$$

$$|R|^2 = 4|W_+|^2 + 4|W_-|^2 + 2|Ric|^2 - \tfrac{1}{3}\tau^2. \tag{5.4}$$

Therefore $\int |R|^2\, dvol < \infty$ implies the integrability of $|Ric|^2$, τ^2, $|W_+|^2$, $|W_-|^2$. The equations

$$E(g) = \frac{1}{32\pi}2(|R|^2 - 4|Ric|^2 + \tau^2)dvol \tag{5.5}$$

$$\frac{3}{2}\, S(g) = \frac{1}{8\pi}2\,(|W_+|^2 - |W_-|^2)\, dvol$$

finish the proof. □

Corollary 5.2. Suppose $g \in \mathcal{M}_f^2$. If there exists an $A > 0$ such that $-A\cdot g \le Ric \le -\frac{2}{3}A\cdot g$ or $-A \le K \le -\frac{1}{4}A$, then

$$|\sigma(M,g)| \le \frac{2}{3}\,\chi(M,g),$$

and the inequality holds for all g' in the component of g.

Proof. The pinching conditions imply the corresponding inequalities for the integrands ([8]). □

We conclude with

Theorem 5.3. Suppose $g \in {}^b\mathcal{M}_f^1$. Then $\chi(M,g)$, $\sigma(M,g)$ exist and $\chi(M,g') = \chi(M,g)$, $\sigma(M,g') = \sigma(M,g)$ for all g' in the component of g in ${}^b\mathcal{M}_f^1$. This in particular holds if $vol(M,g) < \infty$, K bounded.

Proof. $g \in {}^b\mathcal{M}_f^1$ implies $g \in \mathcal{M}_f^2$. □

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SEKTION MATHEMATIK DER EMAU
JAHNSTRASSE 15 a
DDR-2200 GREIFSWALD