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THE TANGENT AND THE COTANGENT VECTOR SQUARE OF A POINTED DIFFERENTIAL SQUARE

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Let S, M, N, P be differential spaces (see [2]) such that M, N and P are non-empty differential subspaces of S, $M \cup N = S$ and $M \cap N = P$. Let us take $p \in P$. We say that these data define a pointed differential square $\sigma = (S,M,N,P,p)$. One can ask what are relations between tangent vector spaces $\mathcal{T}(M,p)$, $\mathcal{T}(N,p)$, $\mathcal{T}(P,p)$ and $\mathcal{T}(S,p)$ as well as between cotangent vector spaces $\mathcal{T}^*(M,p)$, $\mathcal{T}^*(N,p)$, $\mathcal{T}^*(P,p)$ and $\mathcal{T}^*(S,p)$. In particular, if $\dim \mathcal{T}(S,p) < \infty$, for what σ is the equality

(*) $\quad \dim \mathcal{T}(S,p) = \dim \mathcal{T}(M,p) + \dim \mathcal{T}(N,p) - \dim \mathcal{T}(P,p)$

satisfied? To get partial answers to these questions, we associate with σ the tangent (cotangent) vector commutative square diagram which is shortly called the tangent (cotangent) square of σ. It turns out that if σ is $\mathcal{T}$-couniversal ($\mathcal{T}^*$-couniversal), that is, the tangent (cotangent) square of σ is couniversal, then we get partial (full) information about the above mentioned relations (Theorems 2.2, 2.6 and 2.9). In particular, if $\dim \mathcal{T}(S,p) < \infty$ and σ is $\mathcal{T}^*$-couniversal, then equality (*) is satisfied (Corollary 2.12).

In this paper we are especially interested in glued pointed differential squares, that is, pointed differential squares for which the corresponding commutative square diagram of algebras of real smooth functions is couniversal. Such squares arise as the square gluings of overlapping pointed

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This paper is in final form and no version of it will be submitted for publication elsewhere.

differential spaces. We show that every glued pointed differential square is $\mathcal{T}$-couniversal (Corollary 2.4) but need not be $\mathcal{T}^*$-couniversal (Proposition 3.14). On the other hand, each glued pointed differential square (S,M,N,P,p), such that P is a local smooth retract of M or N at p, is $\mathcal{T}^*$-couniversal (Corollary 3.6). . In particular, we obtain that the square gluing of pointed differential manifolds, which are overlapping relative to a regular differential submanifold, is $\mathcal{T}^*$-couniversal (Corollary 3.10).

It is seen that our notion of a pointed differential square is not a general one in the sense of the theory of categories. Moreover, we do not study arbitrary pointed differential squares, but we are rather interested in finding necessary and sufficient conditions for such squares to be $\mathcal{T}$-couniversal, $\mathcal{T}^*(0)$-couniversal and $\mathcal{T}^*$-couniversal. Therefore the results of this paper are far from being conclusive. It seems that a more complete theory can be obtained by using deeper homological algebra methods.

1. Preliminaries. For any differential space M, denote by $\mathcal{E}(M)$ the *differential structure* of M, i.e. the family of all real smooth functions on M which will be regarded as a *real algebra* under the pointwise operations. If f: M $\longrightarrow$ N is a smooth map of differential spaces, then by f^*: $\mathcal{E}(N) \longrightarrow \mathcal{E}(M)$ we denote the homomorphism of algebras given by $f^*(\alpha) = \alpha\circ f$.

By a *pointed differential space* (M,p) we mean a differential space M together with a *base point* p. We say that f: (M,p) $\longrightarrow$ (N,q) is a smooth map of pointed differential spaces if f: M $\longrightarrow$ N is a smooth map of differential spaces and f(p) = q. The resulting category is called the category of *pointed differential spaces*.

For any pointed differential space (M,p), denote by $\mathcal{G}(M,p)$ the *algebra* of all *germs* of $\mathcal{E}(M)$ at p. If $\alpha \in \mathcal{E}(M)$, then the symbol $\hat{\alpha}_p$ stands for the germ of α at p. Let f: (M,p) $\longrightarrow$ (N,q) be a smooth map of pointed differential spaces. Denote by $f^{\#}$: $\mathcal{G}(N,q) \longrightarrow \mathcal{G}(M,p)$ the homomorphism of algebras defined by $f^{\#}(\hat{\alpha}_q) = (f^*\alpha)\hat{}_p$ where $f^*\alpha = \alpha\circ f$.

Let $\mathcal{E}$ ($\mathcal{G}$) denote the assignment which sends every differential space M (pointed differential space (M,p)) to the algebra $\mathcal{E}(M)$ ($\mathcal{G}(M,p)$) and every smooth map f of differential spaces (pointed differential spaces) to the homomorphism f^* ($f^\#$) of algebras. Clearly, we have

1.1. Lemma. *The assignment $\mathcal{E}$ ($\mathcal{G}$) is a contravariant functor from the category of differential spaces (pointed differential spaces) to the category of commutative real algebras.* ∎

If (M,p) is a pointed differential space, then we set

$$\mathcal{E}(M,p) = \{\alpha \in \mathcal{E}(M): \alpha(p) = 0\}$$

and note that $\mathcal{E}(M,p)$ is a maximal ideal of $\mathcal{E}(M)$. The *cotangent vector space* of (M,p) is defined to be the quotient space

$$\mathcal{T}^*(M,p) = \mathcal{E}(M,p)/\mathcal{E}(M,p)^2.$$

By the *differential* of $\alpha \in \mathcal{E}(M)$ at p we mean the element

$$d_{M,p}\alpha = (\alpha-\alpha(p)) + \mathcal{E}(M,p)^2 \in \mathcal{T}^*(M,p).$$

If f: (M,p) $\longrightarrow$ (N,q) is a smooth map of pointed differential spaces, then the *codifferential* of f (at p) is a linear map $\mathcal{T}^*(f)$: $\mathcal{T}^*(N,q) \longrightarrow \mathcal{T}^*(M,p)$ defined by

$$\mathcal{T}^*(f)(\alpha + \mathcal{E}(N,q)^2) = f^*\alpha + \mathcal{E}(M,p)^2$$

where $\alpha \in \mathcal{E}(N,q)$ and $f^*\alpha = \alpha\circ f$.

The *tangent vector space* of (M,p) is defined to be the vector space $\mathcal{T}(M,p)$ dual to $\mathcal{T}^*(M,p)$ or, equivalently, the vector space $\mathrm{der}_p\mathcal{E}(M)$ of all derivations of the algebra $\mathcal{E}(M)$ at p. In fact, we accept that $v(\alpha) = v(d_{M,p}\alpha)$ for $\alpha \in \mathcal{E}(M)$, where v on the left is meant as a derivation of $\mathcal{E}(M)$ at p. Moreover, if f is the above map, then the *differential* of f (at p) is defined to be the linear map $\mathcal{T}(f)$: $\mathcal{T}(M,p) \longrightarrow \mathcal{T}(N,q)$ dual to $\mathcal{T}^*(f)$ or, equivalently, the linear map given by $\mathcal{T}(f)(v)(\alpha) = v(f^*\alpha)$ where $\alpha \in \mathcal{E}(N)$.

Denote by $\mathcal{T}$ ($\mathcal{T}^*$) the assignment which sends every pointed differential space (M,p) to the vector space $\mathcal{T}(M,p)$ ($\mathcal{T}^*(M,p)$) and every smooth map f of pointed differential spaces to the linear map $\mathcal{T}(f)$ ($\mathcal{T}^*(f)$). We have

1.2. Lemma. *The assignment $\mathcal{T}$ ($\mathcal{T}^*$) is a covariant (contravariant) functor from the category of pointed differential spaces to the category of real vector spaces.* ∎

A *differential pair* (M,A) consists of differential spaces M and A where A is a non-empty differential subspace of M. If (M,A) is a differential pair, then the inclusion map $i\colon A \hookrightarrow M$ is called the *canonical inclusion map* of (M,A). This map determines the *restriction homomorphism* $i^*\colon \mathcal{E}(M) \to \mathcal{E}(A)$ of algebras which is defined by the assignment map $\alpha \mapsto \alpha|A$. Set

$$\mathcal{E}(M,A) = \ker i^*$$

and note that $\mathcal{E}(M,A)$ is an ideal of $\mathcal{E}(M)$.

A *pointed differential pair* (M,A,p) consists of differential pair (M,A) together with a *base point* $p \in A$. If (M,A,p) is a pointed differential pair, then by $i_\cdot\colon (A,p) \hookrightarrow (M,p)$ we denote the inclusion map of pointed differential spaces which will be called the *canonical inclusion map* of (M,A,p). This map determines the algebra epimorphism $i_\cdot^{\#}\colon \mathcal{G}(M,p) \to \mathcal{G}(A,p)$ which is defined by the assignment

$$\alpha^\wedge_p \mapsto \alpha^\wedge_p|A \text{ where } \alpha^\wedge_p|A = (\alpha|A)^\wedge_p.$$

In turn, notice that $i_\cdot$ determines the linear epimorphism $\mathcal{T}^*(i_\cdot)\colon \mathcal{T}^*(M,p) \to \mathcal{T}^*(A,p)$ given by the assignment

$$d_{M,p}\alpha \mapsto d_{A,p}\alpha \text{ where } d_{A,p}\alpha = d_{A,p}(\alpha|A).$$

Let us set

$$\mathcal{T}^*(M,A,p) = \ker \mathcal{T}^*(i_\cdot).$$

We get the following exact sequence of linear maps of vector spaces

$$(1.1)\quad 0 \to \mathcal{T}^*(M,A,p) \xrightarrow{\tau_\cdot^*} \mathcal{T}^*(M,p) \xrightarrow{\mathcal{T}^*(i_\cdot)} \mathcal{T}^*(A,p) \to 0$$

where $\tau_\cdot^*$ denotes the inclusion map. One can prove

1.3. Lemma. *If (M,A,p) is a pointed differential pair, then $\mathcal{T}^*(M,A,p) = \{d_{M,p}\alpha\colon \alpha \in \mathcal{E}(M,A)\}$.* ■

Let (M,A,p) be a pointed differential pair. Since sequence (1.1) is exact, it follows the exactness of the following sequence dual to (1.1)

$$0 \to \mathcal{T}(A,p) \xrightarrow{\mathcal{T}(i_\cdot)} \mathcal{T}(M,p) \xrightarrow{\tau_\cdot} \mathcal{T}(M,A,p) \to 0.$$

Let us set $\mathcal{T}(M|A,p) = \operatorname{im} \mathcal{T}(i_\cdot) = \ker \tau_\cdot$. Obviously, $\mathcal{T}(i_\cdot)$ defines a linear isomorphism from $\mathcal{T}(A,p)$ onto $\mathcal{T}(M|A,p)$. It is easily seen that Lemma 1.3 involves

1.4. Lemma. *If (M,A,p) is a pointed differential pair, then $\mathcal{T}(M|A,p) = \{v \in \mathcal{T}(M,p)\colon \mathcal{T}^*(M,A,p) \subset \ker v\}$.* ■

2. Pointed differential squares. A *differential square* $\sigma = (S,M,N,P)$ consists of differential spaces S, M, N, P where M, N, P are differential subspaces of S such that $M \cup N = S$ and $M \cap N = P \neq \emptyset$. To any such square we assign the following commutative square diagram of inclusion maps of differential spaces:

$$\begin{tikzcd} P \arrow[r, "i'"] \arrow[d, "j'"] & N \arrow[d, "j"] \\ M \arrow[r, "i"] & S \end{tikzcd}$$

Applying the functor $\mathscr{C}$ (Lemma 1.1) to this diagram we get the following commutative square diagram of homomorphisms of real algebras:

$$\begin{tikzcd} \mathscr{C}(S) \arrow[r, "j^*"] \arrow[d, "i^*"] & \mathscr{C}(N) \arrow[d, "i'^*"] \\ \mathscr{C}(M) \arrow[r, "j'^*"] & \mathscr{C}(P) \end{tikzcd}$$

The last diagram will be denoted by $\mathscr{SC}(\sigma)$.

A *pointed differential square* $\sigma = (S,M,N,P,p)$ is a differential square (S,M,N,P) together with a *base point* $p \in P$. With any such square we associate the following commutative square diagram of inclusion maps of pointed differential spaces:

(2.1)
$$\begin{tikzcd} (P,p) \arrow[r, "i'_{\cdot}"] \arrow[d, "j'_{\cdot}"] & (N,p) \arrow[d, "j_{\cdot}"] \\ (M,p) \arrow[r, "i_{\cdot}"] & (S,p) \end{tikzcd}$$

Applying the functor $\mathscr{G}$ (Lemma 1.1) to this diagram we get the following one of homomorphisms of real algebras:

$$\begin{tikzcd} \mathscr{G}(S,p) \arrow[r, "j_{\cdot}^{\#}"] \arrow[d, "i_{\cdot}^{\#}"] & \mathscr{G}(N,p) \arrow[d, "i_{\cdot}'^{\#}"] \\ \mathscr{G}(M,p) \arrow[r, "j_{\cdot}'^{\#}"] & \mathscr{G}(P,p) \end{tikzcd}$$

This diagram will be denoted by $\mathcal{S}\mathcal{G}(\sigma)$.

Now, applying the functors $\mathcal{T}$ and $\mathcal{T}^*$ (Lemma 1.2) to diagram (2.1) we obtain the following ones of linear maps of vector spaces:

$$\begin{tikzcd} \mathcal{T}(P,p) \arrow[r, "\mathcal{T}(i'_\cdot)"] \arrow[d, "\mathcal{T}(j'_\cdot)"] & \mathcal{T}(N,p) \arrow[d, "\mathcal{T}(j_\cdot)"] \\ \mathcal{T}(M,p) \arrow[r, "\mathcal{T}(i_\cdot)"] & (S,p) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T}^*(S,p) \arrow[r, "\mathcal{T}^*(j_\cdot)"] \arrow[d, "\mathcal{T}^*(i_\cdot)"] & \mathcal{T}^*(N,p) \arrow[d, "\mathcal{T}^*(i'_\cdot)"] \\ \mathcal{T}^*(M,p) \arrow[r, "\mathcal{T}^*(j'_\cdot)"] & \mathcal{T}^*(P,p) \end{tikzcd}$$

Denote by $\mathcal{S}\mathcal{T}(\sigma)$ and $\mathcal{S}\mathcal{T}^*(\sigma)$ the square diagrams on the left and the right hand side, respectively.

Consider now an arbitrary commutative square diagram Δ of linear maps of real vector spaces:

$$\begin{tikzcd} D \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"] & C \end{tikzcd}$$

Let us set $A \oplus_\Delta B = \{(a,b) \in A \oplus B\colon \alpha(a) = \beta(b)\}$ and note that $A \oplus_\Delta B$ is a linear subspace of $A \oplus B$. The diagram Δ determines the linear map $\Delta_\#\colon D \longrightarrow A \oplus_\Delta B$ defined as follows $\Delta_\#(d) = (\beta'(d),\alpha'(d))$. This diagram is said to be *couniversal* (*0-couniversal*) if the map $\Delta_\#$ is an isomorphism (monomorphism) (see [1], p. 359, Theorem 1.1). Obviously, Δ is couniversal (0-couniversal) if and only if $\Delta_\#$ is surjective and $\ker \Delta_\# = 0$ ($\ker \Delta_\# = 0$).

From Lemmas 1.3 and 1.4 we get immediately

2.1. Lemma. *Let (S,M,N,P,p) be a pointed differential square. Then*

(1) $\mathcal{T}^*(S,M,p) + \mathcal{T}^*(S,N,p) \subset \mathcal{T}^*(S,P,p)$,

(2) $\mathcal{T}(S|M,p) \cap \mathcal{T}(S|N,p) \supset \mathcal{T}(S|P,p)$. ■

2.2. Theorem. *Let (S,M,N,P,p) be a pointed differential square. Then the following conditions are equivalent:*

(a) $\mathcal{T}^*(S,M,p) + \mathcal{T}^*(S,N,p) = \mathcal{T}^*(S,P,p)$,

(b) $\mathcal{T}(S|M,p) \cap \mathcal{T}(S|N,p) = \mathcal{T}(S|P,p)$,

(c) *Diagram $\mathcal{S}\mathcal{T}(S,M,N,P,p)$ is couniversal,*

Proof. (a) $\Rightarrow$ (b). According to Lemma 2.1 (2) it remains to prove that

$$\mathcal{T}(S|M,p) \cap \mathcal{T}(S|N,p) \subset \mathcal{T}(S|P,p).$$

Indeed, let us take $v \in \mathcal{T}(S|M,p) \cap \mathcal{T}(S|N,p)$. By Lemma 1.4 we get $\mathcal{T}^*(S,M,p) \subset \ker v$ and $\mathcal{T}^*(S,N,p) \subset \ker v$. Hence and from condition (a) it follows that $\mathcal{T}^*(S,P,p) \subset \ker v$. Thus, by Lemma 1.4, we conclude that $v \in \mathcal{T}(S|P,p)$.

(b) $\Rightarrow$ (c). Let $\Delta = \mathcal{S}\mathcal{T}(S,M,N,P,p)$. First, we prove that $\Delta_{\#}$ is surjective. Let us take $(u,v) \in \mathcal{T}(M,p) \oplus_{\Delta} \mathcal{T}(N,p)$, which means that $u \in \mathcal{T}(M,p)$, $v \in \mathcal{T}(N,p)$ and $\mathcal{T}(i.)u = \mathcal{T}(j.)v \in \mathcal{T}(S,p)$. Thus the vector $w = \mathcal{T}(i.)u = \mathcal{T}(j.)v$ belongs to $\mathcal{T}(S|M,p) \cap \mathcal{T}(S|N,p)$, and so, by condition (b), it belongs to $\mathcal{T}(S|P,p)$. Therefore, there is $w^o \in \mathcal{T}(P,p)$ such that $\mathcal{T}(k.)w^o = w$ where $k. = i.\circ j'_. = j.\circ i'_.$. Let us set $u' = \mathcal{T}(j'_.)w^o$ and $v' = \mathcal{T}(i'_.)w^o$. It is seen that $u' \in \mathcal{T}(M,p)$, $v' \in \mathcal{T}(N,p)$ and $\mathcal{T}(i.)u' = \mathcal{T}(i.)u$, $\mathcal{T}(j.)v' = \mathcal{T}(j.)v$. Since $\mathcal{T}(i.)$ and $\mathcal{T}(j.)$ are injective, it follows that $u' = u$ and $v' = v$, whence $u = \mathcal{T}(j'_.)w^o$ and $v = \mathcal{T}(i'_.)w^o$, which means that $\Delta_{\#}w^o = (u,v)$. We have thus proved that $\Delta_{\#}$ is surjective.

To prove that the linear map $\Delta_{\#}$ is an isomorphism, it remains to show that $\ker \Delta_{\#} = 0$. Indeed, if $w^\sim \in \ker\Delta_{\#}$, then $\Delta_{\#}(w^\sim) = (\mathcal{T}(j'_.)w^\sim, \mathcal{T}(i'_.)w^\sim) = (0,0)$. This implies that $w^\sim = 0$ because $\mathcal{T}(j'_.)$ is a monomorphism as well as $\mathcal{T}(i'_.)$.

(c) $\Rightarrow$ (a). Suppose to the contrary that

$$\mathcal{T}^*(S,M,p) + \mathcal{T}^*(S,N,p) \neq \mathcal{T}^*(S,P,p).$$

Hence and from Lemma 2.1 (1) it follows that there is $w \in \mathcal{T}(S,p)$ such that $\mathcal{T}^*(S,M,p) + \mathcal{T}^*(S,N,p) \subset \ker w$ and $\mathcal{T}^*(S,P,p)$ does not contain in $\ker w$. By Lemma 1.4 we conclude that

(2.2) $\qquad w \in \mathcal{T}(S|M,p) \cap \mathcal{T}(S|N,p)$ and $w \notin \mathcal{T}(S|P,p)$.

This implies that there are $u \in \mathcal{T}(M,p)$ and $v \in \mathcal{T}(N,p)$ such that $\mathcal{T}(i.)u = w$ and $\mathcal{T}(j.)v = w$ because $\operatorname{im} \mathcal{T}(i.) = \mathcal{T}(S|M,p)$ and $\operatorname{im} \mathcal{T}(j.) = \mathcal{T}(S|N,p)$. Clearly, we have $(u,v) \in \mathcal{T}(M,p) \oplus_{\Delta} \mathcal{T}(N,p)$. Since diagram Δ is couniversal, there is a vector $w^o \in \mathcal{T}(P,p)$ such that $\Delta_{\#}w^o = (\mathcal{T}(j'_.)w^o, \mathcal{T}(i'_.)w^o) = (u,v)$. Hence $\mathcal{T}(k.)w^o = w$, and so, $w \in \operatorname{im} \mathcal{T}(k.) = \mathcal{T}(S|P,p)$, which contradicts (2.2). ∎

We say that a pointed differential square is *$\mathcal{T}$-couniversal* if it satisfies at least one of the equivalent conditions of

Theorem 2.2.

A differential square $\sigma = (S,M,N,P)$ is said to be *glued* if diagram $\mathcal{S}\mathcal{E}(\sigma)$ is couniversal. This means that the following condition holds:

(G) *If α is a real function on S such that $\alpha|_M \in \mathcal{E}(M)$ and $\alpha|_N \in \mathcal{E}(N)$, then $\alpha \in \mathcal{E}(S)$.*

By an easy verification we get

2.3. Lemma. *If (S,M,N,P) is a glued differential square, then $\mathcal{E}(S,N) + \mathcal{E}(S,M) = \mathcal{E}(S,P)$.* ∎

Note that if (S,M,N,P,p) is a glued pointed differential square, then from Lemma 2.3 it follows that condition (a) of Theorem 2.2 is fulfilled. Thus we have

2.4. Corollary. *Every glued pointed differential square is $\mathcal{T}$-couniversal.* ∎

The following example shows that there are non-glued differential squares (S,M,N,P) such that (S,M,N,P,p) is $\mathcal{T}$-couniversal for each $p \in P$.

2.5. Example. For each $r > 0$ define the function $\phi_r\colon \mathbb{R} \longrightarrow \mathbb{R}$ as follows:

$$\phi_r(x) = \begin{cases} 0 & \text{for } x \le 0, \\ e^{-r/x} & \text{for } x > 0. \end{cases}$$

It is well-known that $\phi_r \in C^\infty(\mathbb{R})$. Let $F_r^+ = \{(x,\phi_r(x))\colon x \in \mathbb{R}\}$ and $F_r^- = \{(x,-\phi_r(x))\colon x \in \mathbb{R}\}$ be the graphs of ϕ_r and $-\phi_r$, respectively.

Let us put $S_r = F_r^+ \cup F_r^-$, $M_r = F_r^+$, $N_r = F_r^-$ and $P_r = F_r^+ \cap F_r^-$, and consider these sets as differential spaces under the natural structure induced from $\mathbb{R}^2$. It is easy to see that for any $p \in P_r$ the pointed differential square (S_r,M_r,N_r,P_r,p) satisfies condition (b) of Theorem 2.2, and so, it is $\mathcal{T}$-couniversal.

Now, we shall prove that the differential squares $\sigma_r = (S_r,M_r,N_r,P_r)$ are non-glued for all $r > 0$. Indeed, suppose to the contrary that σ_r is glued for some $r > 0$. Let us take $r > s > 0$ and consider the real function λ on S_r defined by

$$\lambda(q) = \begin{cases} \phi_s(x) \text{ for } q = (x,\phi_r(x)) \in M_r, \\ -\phi_s(x) \text{ for } q = (x,-\phi_r(x)) \in N_r. \end{cases}$$

Obviously, λ is well-defined and note that $\lambda|M_r \in \mathcal{E}(M_r)$ and $\lambda|N_r \in \mathcal{E}(N_r)$. By our hypothesis, from condition (G) it follows that $\lambda \in \mathcal{E}(S_r)$. Since S_r is a closed subset of $\mathbb{R}^2$, there is a function $\Lambda \in \mathcal{E}(\mathbb{R}^2) = C^\infty(\mathbb{R}^2)$ such that $\Lambda|S_r = \lambda$.

Finally, note that applying the Lagrange Mean-value Theorem it follows that for any $x > 0$ we have

$$2\phi_s(x) = \Lambda(x,\phi_r(x)) - \Lambda(x,-\phi_r(x)) = \frac{\partial\Lambda}{\partial y}(x,\eta_x)\cdot 2\phi_r(x)$$

where $-\phi_r(x) < \eta_x < \phi_r(x)$. Hence we get

$$\frac{\partial\Lambda}{\partial y}(x,\eta_x) = e^{(r-s)/x} \text{ for } x > 0,$$

which is impossible because

$$\lim_{x\to 0+}\frac{\partial\Lambda}{\partial y}(x,\eta_x) = \frac{\partial\Lambda}{\partial y}(0,0) \text{ and } \lim_{x\to 0+} e^{(r-s)/x} = +\infty.$$

This contradiction involves that our hypothesis is false, i.e. all differential squares σ_r are non-glued. ∎

2.6. Theorem. *Let* (S,M,N,P,p) *be a pointed differential square. Then the following conditions are equivalent:*

(a) $\mathcal{T}^*(S,N,p) \cap \mathcal{T}^*(S,M,p) = 0$,

(b) $\mathcal{T}(S|M,p) + \mathcal{T}(S|N,p) = \mathcal{T}(S,p)$,

(c) *Diagram* $\mathcal{S}\mathcal{T}^*(S,M,N,P,p)$ *is 0-couniversal.*

Proof. (a) $\Rightarrow$ (b). Since the inclusion map $k_p: (P,p) \hookrightarrow (S,p)$ determines the epimorphism $\mathcal{T}^*(k_p): \mathcal{T}^*(S,p) \longrightarrow \mathcal{T}^*(P,p)$ of linear spaces, it follows that there is a linear monomorphism $\mathcal{K}_p: \mathcal{T}^*(P,p) \longrightarrow \mathcal{T}^*(S,p)$ such that $\mathcal{T}^*(k_p)\circ\mathcal{K}_p = \mathrm{id}_{\mathcal{T}^*(P,p)}$. We have thus the following decomposition

$$\mathcal{T}^*(S,p) = \mathcal{T}^*(S,P,p) \oplus K_p,$$

where $K_p = \operatorname{im}\mathcal{K}_p$, defined by the identity

$$\alpha = (\alpha - (\mathcal{K}_p\circ\mathcal{T}^*(k_p)\alpha) + (\mathcal{K}_p\circ\mathcal{T}^*(k_p)\alpha.$$

Obviously, condition (a) implies that $\mathcal{T}^*(S,N,p) + \mathcal{T}^*(S,M,p) = \mathcal{T}^*(S,N,p) \oplus \mathcal{T}^*(S,M,p)$. Since this linear subspace of $\mathcal{T}^*(S,P,p)$ has a direct summand L_p, it follows that

$$\mathcal{T}^*(S,P,p) = \mathcal{T}^*(S,N,p) \oplus \mathcal{T}^*(S,M,p) \oplus L_p.$$

To sum up, we get the following decomposition

$$\mathcal{T}^*(S,p) = \mathcal{T}^*(S,N,p) \oplus \mathcal{T}^*(S,M,p) \oplus R_p \tag{2.3}$$

where $R_p = L_p \oplus K_p$.

Clearly, to prove condition (b), it suffices to show the inclusion

$$\mathcal{T}(S,p) \subset \mathcal{T}(S|M,p) + \mathcal{T}(S|N,p).$$

Let us take $u \in \mathcal{T}(S,p)$. From (2.3) it follows that there are vectors $u_1, u_2, u_3 \in \mathcal{T}(S,p)$ such that $u = u_1 + u_2 + u_3$ and the following inclusions are fulfilled:

$$\mathcal{T}^*(S,M,p) \oplus R_p \subset \ker u_1,$$
$$\mathcal{T}^*(S,N,p) \oplus R_p \subset \ker u_2,$$
$$\mathcal{T}^*(S,N,p) \oplus \mathcal{T}^*(S,M,p) \subset \ker u_3.$$

These inclusions imply that $u_1 \in \mathcal{T}(S|M,p)$, $u_2 \in \mathcal{T}(S|N,p)$ and $u_3 \in \mathcal{T}(S|M,p) \cap \mathcal{T}(S|N,p)$, so $u \in \mathcal{T}(S|M,p) + \mathcal{T}(S|N,p)$.

(b) $\Rightarrow$ (a). Let us take a function $\alpha \in \mathscr{C}(S)$ such that $d_{S,p}\alpha \in \mathcal{T}^*(S,N,p) \cap \mathcal{T}^*(S,M,p)$. It remains to show that $d_{S,p}\alpha = 0$ or, equivalently, that $u(d_{S,p}\alpha) = 0$ for each $u \in \mathcal{T}(S,p)$. Indeed, if $u \in \mathcal{T}(S,p)$, then, by condition (b), there are $v \in \mathcal{T}(S|M,p)$ and $w \in \mathcal{T}(S|N,p)$ such that $u = v + w$. Hence, by Lemma 1.4, we get $u(d_{S,p}\alpha) = v(d_{S,p}\alpha) + w(d_{S,p}\alpha) = 0$ because $d_{S,p}\alpha \in \mathcal{T}^*(S,N,p) \cap \mathcal{T}^*(S,M,p)$.

(a) $\Leftrightarrow$ (c). Let $\Delta = \mathscr{P}\mathcal{T}^*(S,M,N,P,p)$. From the definition of $\Delta_{\#}$ it follows that $\ker \Delta_{\#} = \mathcal{T}^*(S,N,p) \cap \mathcal{T}^*(S,M,p)$, which gives an equivalence between conditions (a) and (b). ∎

A pointed differential square is said to be $\mathcal{T}^*(0)$-*couniversal* if it satisfies at least one of the equivalent conditions of Theorem 2.6.

The following example shows that there are $\mathcal{T}^*(0)$-couniversal pointed differential squares which are not $\mathcal{T}$-couniversal.

2.7. Example. Let S be the Euclidean space $\mathbb{R}^n$ $(n \geq 1)$ regarded as a differential space under the natural structure. We define the following differential subspaces of S: $M = \{x \in S\colon x_n \geq 0\}$, $N = \{x \in S\colon x_n \leq 0\}$ and $P = \{x \in S\colon x_n = 0\}$ where $x = (x_1,\dots,x_n)$. Let us consider the pointed differential square $\sigma = (S,M,N,P,o)$ where $o = (0,\dots,0) \in \mathbb{R}^n$. It is seen that σ satisfies condition (b) of Theorem 2.6, which means that it is $\mathcal{T}^*(0)$-couniversal. Moreover, note that σ is not $\mathcal{T}$-couniversal because $\mathcal{T}(S|M,o) = \mathcal{T}(S|N,o) = \mathcal{T}(S,o)$

but $\mathcal{T}(S|P,o)$ is a proper linear subspace of $\mathcal{T}(S,o)$ of codimension 1, and so, condition (b) of Theorem 2.2 evidently does not satisfy. ∎

The next example shows that there are $\mathcal{T}$-couniversal pointed differential squares which are not $\mathcal{T}^*(0)$-couniversal.

2.8. Example. Let $\sigma_{ro} = (S_r,M_r,N_r,P_r,o)$ be the pointed differential square defined in Example 2.5, where $o = (0,0)$. As we know σ_{ro} is $\mathcal{T}$-couniversal. On the other hand, note that $\mathcal{T}(S|M,o) + \mathcal{T}(S|N,o) = \mathcal{T}(S|P,o) \neq \mathcal{T}(S,o)$. Hence and from condition (b) of Theorem 2.6 it follows that σ_{ro} is not $\mathcal{T}^*(0)$-couniversal. ∎

2.9. Theorem. *Let σ be a pointed differential square. Then the following conditions are equivalent:*

(a) *σ is $\mathcal{T}$-couniversal and $\mathcal{T}^*(0)$-couniversal,*

(b) *Diagram $\mathcal{PT}^*(\sigma)$ is couniversal.*

Proof. (a) ⇒ (b). Let us set $\sigma = (S,M,N,P,p)$ and $\Delta = \mathcal{PT}^*(\sigma)$. Since σ is $\mathcal{T}^*(0)$-couniversal, we have $\ker \Delta_{\#} = 0$. Thus, it remains to show that $\Delta_{\#}$ is surjective. Suppose to the contrary that $\Delta_{\#}(\mathcal{T}^*(S,p))$ is a proper linear subspace of $\mathcal{T}^*(M,p) \oplus_{\Delta} \mathcal{T}^*(N,p)$. This implies that there is a non-zero linear functional ℓ on $\mathcal{T}^*(M,p) \oplus_{\Delta} \mathcal{T}^*(N,p)$ such that

(2.4) $\qquad \Delta_{\#}(\mathcal{T}^*(S,P)) \subset \ker \ell$ and $\ell(d_{M,p}\alpha, d_{N,p}\beta) \neq 0$

for some $\alpha \in \mathcal{E}(M)$ and $\beta \in \mathcal{E}(N)$ such that $d_{P,p_*}\alpha = d_{P,p}\beta$.

On the other hand, since $\mathcal{T}^*(M,p) \oplus_{\Delta} \mathcal{T}^*(N,p)$ is a linear subspace of $\mathcal{T}^*(M,p) \oplus \mathcal{T}^*(N,p)$, there is a linear extension $\bar{\ell}$ of ℓ defined on $\mathcal{T}^*(M,p) \oplus \mathcal{T}^*(N,p)$. This implies that there are $v \in \mathcal{T}(M,p)$ and $w \in \mathcal{T}(N,p)$ such that

(2.5) $\qquad \bar{\ell}(d_{M,p}\lambda, d_{N,p}\mu) = v(d_{M,p}\lambda) + w(d_{N,p}\mu)$

for any $\lambda \in \mathcal{E}(M)$ and $\mu \in \mathcal{E}(N)$. In particular, for any $\gamma \in \mathcal{E}(S)$ we have

$$\ell(d_{M,p}\gamma, d_{N,p}\gamma) = v(d_{M,p}\gamma) + w(d_{N,p}\gamma) = 0$$

because $\Delta_{\#}(\mathcal{T}^*(S,p)) \subset \ker \ell$. Thus, if we set $v^\sim = \mathcal{T}(i.)v$ and $w^\sim = \mathcal{T}(j.)w$, then $(v^\sim + w^\sim)(d_{S,p}\gamma) = v(d_{M,p}\gamma) + w(d_{N,p}\gamma) = 0$, and so, $v^\sim + w^\sim = 0$. Hence $(v,-w) \in \mathcal{T}(M,p) \oplus_{\Gamma} \mathcal{T}(N,p)$ where $\Gamma = \mathcal{PT}(\sigma)$. Since, by condition (a), diagram Γ is couniversal, it follows that there is $u \in \mathcal{T}(P,p)$ such that $\mathcal{T}(j'_.)u = v$ and $\mathcal{T}(i'_.)u = -w$. Hence and from (2.5) we get

$$l(d_{M,p}\alpha, d_{N,p}\beta) = v(d_{M,p}\alpha) + w(d_{N,p}\beta) =$$
$$u(d_{P,p}\alpha) - u(d_{P,p}\beta) = u(d_{P,p}\alpha - d_{P,p}\beta) = 0$$

because $d_{P,p}\alpha = d_{P,p}\beta$, which contradicts (2.4). Consequently, our hypothesis is false, and so, diagram Δ is couniversal.

(b) $\Rightarrow$ (a). Clearly, σ is $\mathcal{T}^*(0)$-couniversal. To prove that σ is $\mathcal{T}$-couniversal, observe that by Lemma 2.1 (2) and Theorem 2.2 (b) it remains to show the inclusion

(2.6) $$\mathcal{T}(S|M,p) \cap \mathcal{T}(S|N,p) \subset \mathcal{T}(S|P,p).$$

Let us take $u \in \mathcal{T}(S|M,p) \cap \mathcal{T}(S|N,p)$. This implies that there are $v \in \mathcal{T}(M,p)$ and $w \in \mathcal{T}(N,p)$ such that $u = \mathcal{T}(i.)v = \mathcal{T}(j.)w$. Hence, for any $\gamma \in \mathcal{E}(S)$, we have

(2.7) $$u(d_{S,p}\gamma) = v(d_{M,p}\gamma) = w(d_{N,p}\gamma).$$

Now, if $\delta \in \mathcal{E}(S,P)$, then $(d_{M,p}\delta, 0) \in \mathcal{T}^*(M,p) \oplus_\Delta \mathcal{T}^*(N,p)$. Since the map $\Delta_\#$ is surjective, it follows that there is $\lambda \in \mathcal{E}(S)$ such that $\Delta_\#(d_{S,p}\lambda) = (d_{M,p}\lambda, d_{N,p}\lambda) = (d_{M,p}\delta, 0)$, i.e. $d_{M,p}\lambda = d_{M,p}\delta$ and $d_{N,p}\lambda = 0$. Hence and from (2.7) we obtain

$$u(d_{S,p}\delta) = v(d_{M,p}\delta) = v(d_{M,p}\lambda) =$$
$$u(d_{S,p}\lambda) = w(d_{N,p}\lambda) = w(0) = 0.$$

This means that $\mathcal{T}^*(S,P,p) \subset \ker u$, and so, $u \in \mathcal{T}(S|P,p)$ by Lemma 1.4. We have thus proved inclusion (2.6). ∎

A pointed differential square is said to be $\mathcal{T}^*$-*couniversal* if it satisfies at least one of the equivalent conditions of Theorem 2.9. This theorem implies immediately

2.10. Corollary. *A pointed differential square is $\mathcal{T}^*$-couniversal if and only if it is $\mathcal{T}$-couniversal and $\mathcal{T}^*(0)$-couniversal.* ∎

Clearly, the pointed differential squares from Examples 2.7 and 2.8 are not $\mathcal{T}^*$-couniversal.

2.11. Example. Let $m \geq 0$, $l > 0$ and $n \geq 0$ be integers. Clearly, the Cartesian product $\mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^n$ can be canonically identified with $\mathbb{R}^{m+l+n}$. Let $\theta_s = (0,\dots,0)$ stands for the zero element of $\mathbb{R}^s$. Define the following subsets of $\mathbb{R}^{m+l+n}$: $M = \{\theta_m\} \times \mathbb{R}^l \times \mathbb{R}^n$, $N = \mathbb{R}^m \times \mathbb{R}^l \times \{\theta_n\}$, $S = M \cup N$ and $P = M \cap N$. These sets we shall regard as differential spaces under the natural structures induced from $\mathbb{R}^{m+l+n}$. Note that for any $p \in P$ we get a pointed differential square $\sigma = (S,M,N,P,p)$.

The vector space $\mathcal{T}(S,p)$ can be regarded as a vector subspace of $\mathcal{T}(\mathbb{R}^{m+l+n},p)$ via the differential $\mathcal{T}(i)$ where $i\colon S \hookrightarrow \mathbb{R}^{m+l+n}$ is the inclusion map. In turn, there is the canonical isomorphism between $\mathcal{T}(\mathbb{R}^{m+l+n},p)$ and the vector space $\mathbb{R}^{m+l+n}$. Consequently, $\mathcal{T}(S,p)$ can be regarded as a vector subspace of $\mathbb{R}^{m+l+n}$. It is easy to see that $\mathcal{T}(S,p) = \mathbb{R}^{m+l+n}$, $\mathcal{T}(S|M,p) = \{\theta_m\} \times \mathbb{R}^l \times \mathbb{R}^n$, $\mathcal{T}(S|N,p) = \mathbb{R}^m \times \mathbb{R}^l \times \{\theta_n\}$ and $\mathcal{T}(S|P,p) = \{\theta_m\} \times \mathbb{R}^l \times \{\theta_n\}$. This implies that condition (b) of Theorem 2.2 and condition (b) of Theorem 2.6 are fulfilled for σ. Thus, by the definitions, we infer that σ is $\mathcal{T}$-couniversal and $\mathcal{T}^*(0)$-couniversal, and so, it is $\mathcal{T}^*$-couniversal by Corollary 2.10. ∎

Let $\sigma = (S,M,N,P,p)$ be a pointed differential square such that $\dim \mathcal{T}(S,p) < \infty$. In this case we define the *index* of σ to be the integer $\operatorname{ind} \sigma = \dim \mathcal{T}(S,p) - \dim \mathcal{T}(M,p) - \dim \mathcal{T}(N,p) + \dim \mathcal{T}(P,p)$. From Theorems 2.2 and 2.6 and from Corollary 2.10 it follows

2.12. Corollary. *Let $\sigma = (S,M,N,P,p)$ be a pointed differential square such that* $\dim \mathcal{T}(S,p) < \infty$.

(1) *If σ is $\mathcal{T}$-couniversal, then* $\operatorname{ind} \sigma \geq 0$,

(2) *If σ is $\mathcal{T}^*(0)$-couniversal, then* $\operatorname{ind} \sigma \leq 0$,

(3) *If σ is $\mathcal{T}^*$-couniversal, then* $\operatorname{ind} \sigma = 0$. ∎

The following example shows that there are pointed differential squares σ such that $\operatorname{ind} \sigma = 0$ and they are not $\mathcal{T}$-couniversal and not $\mathcal{T}^*(0)$-couniversal, simultaneously.

2.13. Example. Let us take $(a_r,\dots,a_n) \in S^{n-r}$ and define the following subsets of $\mathbb{R}^{n+r}$ $(n \geq r)$:

$$M = S^n,\ N = \{(1,ta_r,\dots,ta_n)\colon t \in \mathbb{R}\},$$
$$S = M \cup N,\ P = M \cap N = \{(1,0,\dots,0)\}.$$

Consider the pointed differential square $\sigma = (S,M,N,P,p)$ where S, M, N, P are differential spaces under the natural structure induced from $\mathbb{R}^{n+r}$ and $p = (1,0,\dots,0)$. One can see that

$$\dim \mathcal{T}(S,p) = \dim \mathcal{T}(M,p) + \dim \mathcal{T}(N,p) - \dim \mathcal{T}(P,p),$$

that is, $\operatorname{ind} \sigma = 0$. On the other hand, note that we have the following inclusions:

$$\mathcal{T}(S|P,p) \subset \mathcal{T}(S|N,p) \subset \mathcal{T}(S|M,p) \subset \mathcal{T}(S,p)$$

where the first and the last inclusion are essential. This implies that condition (b) of Theorem 2.2 and condition (b) of Theorem 2.6 do not satisfy. Thus, by the definitions, σ is not $\mathcal{T}$-couniversal and not $\mathcal{T}^*(0)$-couniversal. ∎

3. Gluings of pointed differential pairs. Let (M,A) and (N,B) be differential pairs such that there is a diffeomorphism $h\colon A \longrightarrow B$. Denote by $M \amalg_h N$ the gluing of sets M and N via h. Moreover, let $m_h\colon M \longrightarrow M \amalg_h N$ and $n_h\colon N \longrightarrow M \amalg_h N$ be the canonical injections. By the *differential space gluing* of M and N *via* h we mean the set $M \amalg_h N$ regarded as a differential space under the structure coinduced via maps m_h and n_h, that is, $\mathcal{E}(M \amalg_h N)$ consists of all real functions γ on $M \amalg_h N$ such that $m_h^*\gamma \in \mathcal{E}(M)$ and $n_h^*\gamma \in \mathcal{E}(N)$. Obviously, the canonical injections m_h and n_h are smooth maps.

The following example shows that the canonical injections m_h and n_h need not be smooth embeddings.

3.1. Example. Let $M = [0,2\pi]$ and $N = S^1$ be the closed interval and the unit circle, respectively, regarded as differential spaces under natural structures. Let $A = (0,2\pi)$ be the differential subspace of M. We set $h(t) = (\cos t, \sin t)$ for $t \in A$. Let $B = h(A) = S^1 \setminus \{(1,0)\}$ be the differential subspace of N. It is seen that $h\colon A \longrightarrow B$ is a diffeomorphism. Consider the differential space $M \amalg_h N$. One can see that if $\alpha \in \mathcal{E}(M \amalg_h N)$, then $\alpha(m_h(0)) = \alpha(m_h(2\pi)) = \alpha(n_h(1,0))$. This implies that the differential space $M \amalg_h N$ is non-Hausdorff as well as the differential subspace $m_h(M)$ of it. But the differential space M is Hausdorff, and so, m_h is not a smooth (topological) embedding. ∎

If (M,A) and (N,B) are differential pairs and $h\colon A \longrightarrow B$ is a diffeomorphism, we define the *differential square gluing* of (M,A) and (N,B) *via* h to be the differential square

$$(M \amalg_h N,\ M_h,\ N_h,\ M_h \cap N_h)$$

also denoted by $(M,A) \sqcap_h (N,B)$, where $M_h = m_h(M)$, $N_h = n_h(N)$ and $M_h \cap N_h$ are regarded as differential subspaces of $M \amalg_h N$. Obviously, this differential square is glued.

Now, let (M,A,p) and (N,B,q) be pointed differential pairs such that there is a diffeomorphism h: $(A,p) \longrightarrow (B,q)$. The *pointed differential space gluing* of (M,A,p) and (N,B,q) *via* h is defined to be the pointed differential space $(M \amalg_h N, p \amalg_h q)$ where $p \amalg_h q = m_h(p) = n_h(q)$. By the *pointed differential square gluing* of (M,A,p) and (N,B,q) *via* h we mean the pointed differential square

$$(M \amalg_h N, M_h, N_h, M_h \cap N_h, p \amalg_h q)$$

which will be also denoted by $(M,A,p) \sqcap_h (N,B,q)$.

Let (S,M,N,P) and (R,K,L,Q) be differential squares. We say that f: $(S,M,N,P) \longrightarrow (R,K,L,Q)$ is a *smooth map of differential squares* if f: $S \longrightarrow R$ is a smooth map of differential spaces such that $f(M) \subset K$, $f(N) \subset L$ and $f(P) \subset Q$. If now (S,M,N,P,p) and (R,K,L,Q,q) are pointed differential squares, then by a *smooth map* f: $(S,M,N,P,p) \longrightarrow (R,K,L,Q,q)$ of *pointed differential squares* we mean a smooth map f: $(S,M,N,P) \longrightarrow (R,K,L,Q)$ of differential squares such that $f(p) = q$. We have thus defined the *category of differential squares* and the *category of pointed differential squares*. By a *diffeomorphism of differential squares* or *pointed differential squares* we shall mean an isomorphism in the corresponding category.

3.2. Example. Let $\sigma_r = (S_r,M_r,N_r,P_r,o)$ be the pointed differential square defined in Example 2.5 where $o = (0,0)$. For any $c > 0$ consider the linear diffeomorphism $\mathfrak{m}_c$ of $\mathbb{R}^2$ defined by $\mathfrak{m}_c(x,y) = (cx,y)$. It is seen that $\mathfrak{m}_c$ defines diffeomorphisms from F_r^+ onto F_{rc}^+ and from F_r^- onto F_{rc}^-, for any $r > 0$. From the definitions of σ_r and σ_{rc} it follows that $\mathfrak{m}_r$ determines a diffeomorphism from σ_r onto σ_{rc}. In particular, $\mathfrak{m}_r$ determines a diffeomorphism m_r: $\sigma_1 \longrightarrow \sigma_r$ defined by the assignments (see, Example 2.5)

$$(x,\phi_1(x)) \mapsto (rx,\phi_r(rx)) \text{ and } (x,-\phi_1(x)) \mapsto (rx,-\phi_r(rx))$$

where $x \in \mathbb{R}$. Thus, all pointed differential squares σ_r are diffeomorphic. ∎

Differential spaces M and N are called *overlapping* if $P = M \cap N \neq \emptyset$ and the differential structures on P induced from M and N are coincided. Let M and N are such spaces and $h = id_P$. Then (M,P) and (N,P) are differential pairs and h is a diffeomorphism of P. The *differential space gluing* of M and N *over* P is defined to be the differential space $M \amalg_P N = M \amalg_h N$. We define the *differential square gluing* of M and N *over* P to be the gluing $(M,P) \sqcup_P (N,P) = (M,P) \sqcup_h (N,P)$. One can see that if $\sigma = (S,M,N,P)$ is a glued differential square, then M and N are overlapping differential spaces, $M \cap N = P$ and σ is diffeomorphic to $(M,P) \sqcup_P (N,P)$.

Let $\sigma = (S,M,N,P,p)$ be a pointed differential square. By a *(differential) subsquare* of σ we mean a pointed differential square $\rho = (R,K,L,Q,q)$ such that R, K, L, Q are differential subspaces of S, M, N, P, respectively, and q = p. We say that ρ is an *open subsquare* of σ if it is a subsquare of σ such that R, K, L and Q are open subspaces of S, M, N and P, respectively. A pointed differential square σ is called O(*)-*glued* if there is an open subsquare of σ which is glued. Obviously, every glued pointed differential square is O(*)-glued. We say that a pointed differential square $\sigma = (S,M,N,P,p)$ is G(*)-*glued* if diagram $\mathcal{PG}(\sigma)$ is couniversal. This means that the following condition holds:

(G(*)) *If* $\alpha^\wedge_p \in \mathcal{G}(M,p)$, $\beta^\wedge_p \in \mathcal{G}(N,p)$ *and* $\alpha^\wedge_p|P = \beta^\wedge_p|P$, *then there is a germ* $\gamma^\wedge_p \in \mathcal{G}(S,p)$ *such that* $\gamma^\wedge_p|M = \alpha^\wedge_p$ *and* $\gamma^\wedge_p|N = \beta^\wedge_p$.

One can prove

3.3. Proposition. *Every* O(*)-*glued pointed differential square is* G(*)-*glued.* ∎

Since every glued pointed differential square is O(*)-glued, this proposition implies

3.4. Corollary. *Every glued pointed differential square is* G(*)-*glued.* ∎

Let (M,A,p) be a pointed differential pair. Call A a *local smooth retract* of M at p if there are an open neighbourhood U of p in M and a smooth retraction from U onto $U \cap A$.

3.5. Theorem. *Let* $\sigma = (S,M,N,P,p)$ *be a* G(*)-*glued pointed differential square. If* P *is a local smooth retract of* M *or* N *at* p, *then* σ *is* $\mathcal{T}^*$-*couniversal.*

Proof. First, we prove that σ is $\mathcal{T}$-couniversal. Indeed, let us put $\mathcal{G}(S,M,p) = \{\alpha^\wedge_p \in \mathcal{G}(S,p)\colon \alpha^\wedge_p|M = 0\}$, $\mathcal{G}(S,N,p) = \{\alpha^\wedge_p \in \mathcal{G}(S,p)\colon \alpha^\wedge_p|N = 0\}$ and $\mathcal{G}(S,P,p) = \{\alpha^\wedge_p \in \mathcal{G}(S,p)\colon \alpha^\wedge_p|P = 0\}$.

It is seen that these sets are ideals of the algebra $\mathcal{G}(S,p)$. Clearly, we have $\mathcal{G}(S,M,p) + \mathcal{G}(S,N,p) \subset \mathcal{G}(S,P,p)$. Since σ is $G(*)$-glued, it follows from condition $(G(*))$ that for any $\gamma^{\wedge}_p \in \mathcal{G}(S,P,p)$ there are germs $\alpha^{\wedge}_p, \beta^{\wedge}_p \in \mathcal{G}(S,p)$ such that $\alpha^{\wedge}_p|M = \gamma^{\wedge}_p|M$, $\alpha^{\wedge}_p|N = 0$ and $\beta^{\wedge}_p|M = 0$, $\beta^{\wedge}_p|N = \gamma^{\wedge}_p|N$, which implies that $\gamma^{\wedge}_p = \alpha^{\wedge}_p + \beta^{\wedge}_p$ where $\alpha^{\wedge}_p \in \mathcal{G}(S,M,p)$ and $\beta^{\wedge}_p \in \mathcal{G}(S,N,p)$. Consequently, we get $\mathcal{G}(S,M,p) + \mathcal{G}(S,N,p) = \mathcal{G}(S,P,p)$. This involves that condition (a) of Theorem 2.2 is fulfilled, and so, σ is $\mathcal{T}$-couniversal.

By Corollary 2.10 we conclude that it remains to show that σ is $\mathcal{T}^*(0)$-couniversal. By the definition this is equivalent to the following condition

(3.1) $$\mathcal{T}^*(S,M,p) \cap \mathcal{T}^*(S,N,p) = 0$$

(see, Theorem 2.6 (a)).

We can assume that P is a local smooth retract of M at p. Then there are an open neighbourhood U of p in M and a smooth retraction $r\colon U \longrightarrow U \cap P$. Let us take $\alpha \in \mathcal{E}(S)$ such that $d_{S,p}\alpha \in \mathcal{T}^*(S,M,p) \cap \mathcal{T}^*(S,N,p)$. By Lemma 1.3 we can assume that $\alpha \in \mathcal{E}(S,M)$. Since $d_{S,p}\alpha \in \mathcal{T}^*(S,N,p)$, it follows that $d_{N,p}\alpha = 0$, which means that there are $\beta_i^N, \gamma_i^N \in \mathcal{E}(N,p)$ $(i = 1,\dots,n)$ such that $\alpha|N = \Sigma\, \beta_i^N\gamma_i^N$.

Let us set $\beta_i^U = \beta_i^N \circ r$, $\gamma_i^U = \gamma_i^N \circ r$ and note that $\beta_i^U, \gamma_i^U \in \mathcal{E}(U,p)$. Let us set $\mathring{P} = P \cap U$. It is seen that $\beta_i^U|\mathring{P} = \beta_i^N|\mathring{P}$ and $\gamma_i^U|\mathring{P} = \gamma_i^N|\mathring{P}$, which implies that $(\beta_i^U)^{\wedge}_p|\mathring{P} = (\beta_i^N)^{\wedge}_p|\mathring{P}$ and $(\gamma_i^U)^{\wedge}_p|\mathring{P} = (\gamma_i^N)^{\wedge}_p|\mathring{P}$. On the other hand, there are $\beta_i^M, \gamma_i^M \in \mathcal{E}(M,p)$ such that $(\beta_i^U)^{\wedge}_p = (\beta_i^M)^{\wedge}_p|U$ and $(\gamma_i^U)^{\wedge}_p = (\gamma_i^M)^{\wedge}_p|U$. Consequently, we have

$$(\beta_i^M)^{\wedge}_p|P = (\beta_i^N)^{\wedge}_p|P \text{ and } (\gamma_i^M)^{\wedge}_p|P = (\gamma_i^N)^{\wedge}_p|P.$$

Since σ is $G(*)$-glued, there are $\beta_i, \gamma_i \in \mathcal{E}(S,p)$ such that $(\beta_i)^{\wedge}_p|M = (\beta_i^M)^{\wedge}_p$, $(\beta_i)^{\wedge}_p|N = (\beta_i^N)^{\wedge}_p$ and $(\gamma_i)^{\wedge}_p|M = (\gamma_i^M)^{\wedge}_p$, $(\gamma_i)^{\wedge}_p|N = (\gamma_i^N)^{\wedge}_p$.

Consider the function $\lambda = \Sigma\, \beta_i\gamma_i \in \mathcal{E}(S,p)^2$. We shall prove that $\lambda^{\wedge}_p = \alpha^{\wedge}_p$. Clearly, this is equivalent to the conditions $\lambda^{\wedge}_p|M = \alpha^{\wedge}_p|M$ and $\lambda^{\wedge}_p|N = \alpha^{\wedge}_p|N$. Indeed, we have $\lambda^{\wedge}_p|M = (\Sigma\, (\beta_i)^{\wedge}_p(\gamma_i)^{\wedge}_p)|M = \Sigma\, (\beta_i^M)^{\wedge}_p(\gamma_i^M)^{\wedge}_p$, whence

$$\lambda^{\wedge}_p|U = \Sigma\, (\beta_i^U)^{\wedge}_p(\gamma_i^U)^{\wedge}_p = \Sigma\, (\beta_i^N \circ r)^{\wedge}_p(\gamma_i^N \circ r)^{\wedge}_p = (\alpha \circ r)^{\wedge}_p$$

because $\alpha|N = \Sigma\, \beta_i^N\gamma_i^N$. But $\alpha \in \mathcal{E}(S,M)$, and so, $\alpha|r(U) = 0$

or, equivalently, $\alpha \circ r = 0$. Thus $(\alpha \circ r)^{\wedge}_{p} = 0$ and since $\lambda^{\wedge}_{p}|U = (\alpha \circ r)^{\wedge}_{p}$, we get $\lambda^{\wedge}_{p}|U = 0$, which implies that $\lambda^{\wedge}_{p}|M = 0$. Hence $\lambda^{\wedge}_{p}|M = \alpha^{\wedge}_{p}|M$ because $\alpha|M = 0$. In turn, we have $\lambda^{\wedge}_{p}|N = (\Sigma\, (\beta_i)^{\wedge}_{p}(\gamma_i)^{\wedge}_{p})|N = \Sigma\, (\beta^{N}_{i})^{\wedge}_{p}(\gamma^{N}_{i})^{\wedge}_{p} = \alpha^{\wedge}_{p}|N$. To sum up, we infer that $\lambda^{\wedge}_{p} = \alpha^{\wedge}_{p}$, which implies that $d_{S,p}\lambda = d_{S,p}\alpha$. Since $\lambda \in \mathcal{E}(S,p)^{2}$, it follows that $d_{S,p}\alpha = d_{S,p}\lambda = 0$.

Finally, notice that we have thus proved condition (3.1), which completes the proof of our theorem. ■

Clearly, this theorem and Corollary 3.4 imply

3.6. Corollary. *Let $\sigma = (S,M,N,P,p)$ be a glued pointed differential square. If* P *is a local smooth retract of* M *or* N *at* p, *then σ is $\mathcal{T}^{*}$-couniversal.* ■

Let (M,p) and (N,q) be pointed differential spaces. Consider the pointed differential pairs (M,{p},p) and (N,{q},q). There is a unique map $\bullet\colon \{p\} \longrightarrow \{q\}$. We define the *pointed differential square gluing* of (M,p) and (N,q) to be the gluing $(M,\{p\},p) \sqcup_{\bullet} (N,\{q\},q)$ which will be also denoted by $(M,p) \sqcup_{\bullet} (N,q)$. Thus we have the differential square $(M \amalg_{\bullet} N, M_{\bullet}, N_{\bullet}, \{*\}, *) = (M,p) \sqcup_{\bullet} (N,q)$ where $* = p \amalg_{\bullet} q$. Since the constant maps $M \longrightarrow \{*\}$ and $N \longrightarrow \{*\}$ are smooth retractions, from Corollary 3.6 we get

3.7. Corollary. *Every pointed differential square gluing of pointed differential spaces is $\mathcal{T}^{*}$-couniversal.* ■

By a *pointed differential* E-*pair* we shall mean a differential pair (M,A) such that the restriction homomorphism $i^{*}\colon \mathcal{E}(M) \longrightarrow \mathcal{E}(A)$ is surjective. For example, let M be a paracompact smooth manifold regarded as a differential space. Let A be a non-empty closed subset of M. It is known that every smooth function on A can be extended to a real smooth function on M. Thus the restriction homomorphism $i^{*}\colon \mathcal{E}(M) \longrightarrow \mathcal{E}(A)$ is surjective, so (M,A) is a differential E-pair. It is easy to verify

3.8. Lemma. *Let* (M,A) *and* (N,B) *be differential pairs such that there is a diffeomorphism $h\colon A \longrightarrow B$. If* (M,A) *is a differential* E-*pair, the the canonical injection $n_h\colon N \longrightarrow M \amalg_h M$ is a smooth embedding.* ■

This lemma and Corollary 3.6 imply

3.9. Proposition. *Let* (M,A,p) *and* (N,B,q) *be pointed differential pairs such that there is a diffeomorphism* h: (A,p) $\longrightarrow$ (B,q). *If* (M,A) *is a differential* E-*pair and if* B *is a local smooth retract of* N *at* q, *then the gluing* (M,A,p) $\square_h$ (N,B,q) *is* $\mathcal{T}^*$-*couniversal.* ■

By a differential manifold pair we shall mean a differential pair (M,A) where M is a paracompact smooth manifold (of finite dimension) regarded as a differential space and A is a closed regular submanifold of M. This means that the differential structure on A induced from M is the differential structure associated with the atlas of all charts of A. Clearly, if (M,A,p) is a pointed differential manifold pair, then (M,A) is a differential E-pair and A is a local smooth retract of M at p. Thus Proposition 3.9 and Lemma 3.8 imply

3.10. Corollary. *Let* (M,A,p) *and* (N,B,q) *be pointed differential manifold pairs such that there is a diffeomorphism* h: (A,p) $\longrightarrow$ (B,q). *Then the gluing* (M,A,p) $\square_h$ (N,B,q) *is* $\mathcal{T}^*$-*couniversal. Moreover, the canonical injections* m_h: M $\longrightarrow$ M $\amalg_h$ N *and* n_h: N $\longrightarrow$ M $\amalg_h$ N *are smooth embeddings.* ■

Proposition 3.9, Corollary 2.12 and Lemma 3.8 imply

3.11. Proposition. *Let* (M,A,p) *and* (N,B,q) *be pointed differential* E-*pairs such that there is a diffeomorphism* h: (A,p) $\longrightarrow$ (B,q). *Let* A *be a local smooth retract of* M *at* p. *If* dim $\mathcal{T}$(M,p) *and* dim $\mathcal{T}$(N,q) *are finite, then*

dim $\mathcal{T}$(M $\amalg_h$ N, p $\amalg_h$ q)= dim $\mathcal{T}$(M,p) + dim $\mathcal{T}$(N,q) − dim $\mathcal{T}$(A,p). ■

This proposition implies the following corollaries.

3.12. Corollary. *Let* (M,A,p) *and* (N,B,q) *be pointed differential manifold pairs such that there is a diffeomorphism* h: (A,p) $\longrightarrow$ (B,q). *If* m = dim M, n = dim N *and* k = dim A = dim B, *then* dim $\mathcal{T}$(M $\amalg_h$ N, p $\amalg_h$ q) = $m + n - k$. ■

3.13. Corollary. *Let* (M,p) *and* (N,q) *be pointed differential spaces such that* dim $\mathcal{T}$(M,p) *and* dim $\mathcal{T}$(N,q) *are finite. Then* dim $\mathcal{T}$(M $\amalg_\bullet$ N, p $\amalg_\bullet$ q) = dim $\mathcal{T}$(M,p) + dim $\mathcal{T}$(N,q).■

For any pointed differential pair (M,A,p), we can regard the gluing (M,A,p) $\square_A$ (M,A,p) = (M,A,p) $\square_h$ (M,A,p) where h =

id_A. One can ask the following question. If (M,A,p) is a differential E-pair, is it true that $(M,A,p) \sqcap_A (M,A,p)$ is $\mathcal{T}^*$-couniversal? It turns out that there is a negative answer, even, if M is a smooth paracompact manifold regarded as a differential space and A is a closed differential subspace of M. The corresponding counter-example (Proposition 3.14) is defined as follows. Let $\mathbb{R}^2$ be the differential space under the natural structure $C^\infty(\mathbb{R}^2)$. Let $\mathbb{K} = \{(x,y) \in \mathbb{R}^2: xy = 0\}$ be a closed differential subspace of $\mathbb{R}^2$. We have a pointed differential pair $(\mathbb{R}^2,\mathbb{K},o)$ where $o = (0,0)$. One can prove

3.14. Proposition. *The gluing* $(\mathbb{R}^2,\mathbb{K},o) \sqcap_{\mathbb{K}} (\mathbb{R}^2,\mathbb{K},o)$ *is not* $\mathcal{T}^*$*-couniversal.* ∎

The author knows that the gluing $\mathbb{R}^2 \amalg_{\mathbb{K}} \mathbb{R}^2$ is diffeomorphic to the differential subspace $\{(x,y,z) \in \mathbb{R}^3: xyz - z^2 = 0\}$ of $\mathbb{R}^3$, which implies that $\dim \mathcal{T}(\mathbb{R}^2 \amalg_{\mathbb{K}} \mathbb{R}^2,o) = 3$.

Note that Propositions 3.14 and 3.9 imply

3.15. Corollary. *The set* $\mathbb{K}$ *is not a local smooth retract of* $\mathbb{R}^2$ *at* o. ∎

It turns out that this corollary has no analogy in the topological sense. More precisely, one can see that $\mathbb{K}$ is a topological retract of $\mathbb{R}^2$, and so, it is a local topological retract of $\mathbb{R}^2$ at o.

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