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ON THE NATURAL OPERATORS TRANSFORMING VECTOR FIELDS TO THE r–TH TENSOR POWER

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We determine explicitely all natural operators transforming vector fields on a manifold M into vector fields on the r-th tensor power $\overset{r}{\otimes} TM$ of its tangent bundle and we outline how the result is modified when replacing $\overset{r}{\otimes} TM$ by a natural subbundle.

All manifolds and maps are assumed to be infinitely differentiable.

1. In [5] we determined all natural operators transforming vector fields on m-dimensional manifolds into vector fields on an arbitrary Weil bundle over m-manifolds. According to [3] or [6], the Weil bundles coincide with the product preserving bundle functors on the category $\mathcal{M}f$ of all smooth manifolds and all smooth maps. The problem of determining all natural operators transforming vector fields to the vector fields on the bundle functor of r-th order tangent vectors, which does not preserve products for $r > 1$, was solved by Mikulski, [7]. In the present paper we study the same problem for another non-product-preserving functor $\overset{r}{\otimes} T$ of the r-th tensor power of the tangent bundle. The list of all such operators is deduced in item 7. Then we outline how this result can be modified to the natural subbundles of $\overset{r}{\otimes} T$.

2. Denote by $C^\infty TM$ the set of all vector fields on a manifold M. Let F be an arbitrary natural bundle over m-manifolds, [8], and $A : T \to TF$ be a natural operator transforming vector fields on m-manifolds into vector fields on the natural bundle F, [5]. In other words, A is a system of maps $A_M : C^\infty TM \to C^\infty T(FM)$ commuting with diffeomorphisms and satisfying two simple additional conditions of locality and regularity, [5]. The following lemma is a direct consequence of the well-known fact that for every vector field $X \in C^\infty TM$ with

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$X(a) \neq 0$, $a \in M$, there exists a neighbourhood U of a and a coordinate system on U such that $X|U$ coincides with the vector field $\partial_1 = \partial x^1$, [5], [7].

Lemma 1. *If two natural operators $A, \bar{A} : T \to TF$ satisfy $A_{\mathbf{R}^m}(\partial_1)|F_0\mathbf{R}^m = \bar{A}_{\mathbf{R}^m}(\partial_1)|F_0\mathbf{R}^m$, then $A = \bar{A}$.*

One general example of a natural operator $A : T \to TF$ is the *flow operator* $\mathcal{F}$ defined by

$$\mathcal{F}_M(X) = \frac{\partial}{\partial t}\bigg|_0 F(\exp tX)$$

where $\exp tX$ means the flow of a vector field $X \in C^\infty TM$.

A natural operator $A : T \to TF$ will be called *vertical*, if $A_M X$ is a vertical vector field on FM for every m-manifold M and all $X \in C^\infty TM$.

3. The r-th tensor power $\overset{r}{\bigotimes} T$ is a bundle functor defined on the category of all smooth manifolds and all smooth maps. Its restriction to the subcategory of all m-dimensional manifolds and their local diffeomorphisms is a natural bundle over m-manifolds. Consider an arbitrary natural operator $A : T \to T\overset{r}{\bigotimes} T$. For every vector field X on M, we take the restriction of the vector field $A_M X$ to the zero section $O_M : M \to \overset{r}{\bigotimes} TM$ and project it by Tp_M into TM, where $p_M : \overset{r}{\bigotimes} TM \to M$ is the bundle projection. This yields a natural operator $A^0 : T \to T$ transforming vector fields on m-manifolds into vector fields on m-manifolds, $A^0_M(X) = Tp_M \circ (A_M X) \circ O_M$. By [5], all natural operators $T \to T$ are of the form $X \mapsto kX$, $k \in \mathbf{R}$. Hence A defines a real number λ_A by $A^0_M(X) = \lambda_A X$. Then the constant multiple $\lambda_A \overset{r}{\bigotimes} \mathcal{T}$ of the flow operator $\overset{r}{\bigotimes} \mathcal{T}$ is naturally determined by A.

The first step to solving our main problem is the following assertion.

Lemma 2. *For every natural operator $A : T \to T\overset{r}{\bigotimes} T$, the difference $A - \lambda_A \overset{r}{\bigotimes} \mathcal{T}$ is a vertical operator.*

Since a similar phenomenon also appears in some other cases (e.g. for all Weil bundles, [5], and for the bundle of r-th order tangent vectors, [7]), we shall not prove Lemma 2 directly, but we deduce a general result of such a type.

4. Consider an r-th order natural bundle F over m-manifolds determined by a left action ϱ of the group G^r_m of all invertible r-jets from $\mathbf{R}^m$ into itself with source and target zero on a manifold S. Hence FM is the fiber bundle associated with the r-th order frame bundle $P^r M$ of M with standard fiber S, [6]. Let $q \in S$ be a fixed point of ϱ. Then $(u, q) \in P^r_x M \times S$ determines the same equivalence class for all $u \in P^r_x M$, $x \in M$, i.e. the same point of $F_x M$. This yields a natural section of $F_x M$, which will be denoted by $q_M : M \to FM$ and called the q-section of FM. In particular, if FM is a natural vector bundle and q is the zero vector of its standard fiber, then q_M is the zero section of FM.

In such a situation, every natural operator $A : T \to TF$ determines a natural operator $A^q : T \to T$ by $A^q_M(X) = Tp^F_M \circ (A_M X) \circ q_M$, where $p^F_M : FM \to M$ is the bundle projection. This yields a real number λ_A such that $A^q_M(X) = \lambda_A X$. We are going to deduce a sufficient condition for $A - \lambda_A \mathcal{F}$ to be a vertical operator, where $\mathcal{F}$ is the flow operator of F. Consider the canonical injection $i : GL(m, \mathbf{R}) \to G^r_m$ transforming each matrix into the r-jet at 0 of the corresponding linear map $\mathbf{R}^m \to \mathbf{R}^m$ and write $\tilde{\varrho}(k,w) = \varrho(i(kE),w)$, $0 \neq k \in \mathbf{R}$, $w \in S$, provided E means the unit matrix.

Definition 1. A G^r_m-space S is said to be naturally contractible to a fixed point q, if

(i) $\lim_{k\to 0} \tilde{\varrho}(k,w) = q$ for all $w \in S$,

(ii) each curve $\gamma_w : \mathbf{R} \to S$, $\gamma_w(t) = \tilde{\varrho}(t,w)$ for $t \neq 0$ and $\gamma_w(0) = q$, $w \in S$, is smooth and the induced map $\sigma : S \to T_qS$, $\sigma(w) = \frac{d}{dt}|_0 \gamma_w(t)$ is also smooth.

Proposition 1. *If the standard fiber S of a natural bundle F over m-manifolds is naturally contractible to q, then, for every natural operator $A : T \to TF$, the difference $A - \lambda_A \mathcal{F}$ is a vertical operator.*

Proof. Write $B = A - \lambda_A \mathcal{F}$ and define a map $h : \mathbf{R} \times S \to T_0\mathbf{R}^m = \mathbf{R}^m$ by

$$h(t,w) = Tp^F_{\mathbf{R}^m}(B_{\mathbf{R}^m}(t\partial_1)(w)), \quad t \in \mathbf{R}, \quad w \in S .$$

Since B is natural, h is equivariant with respect to the homotheties in $GL(m,\mathbf{R})$. This yields the relation

$$kh(t,w) = h(kt, \tilde{\varrho}(k,w)) , \quad 0 \neq k \in \mathbf{R} .$$

Differentiating with respect to k and letting $k \to 0$, the conditions from Definition 1 imply

$$h(t,w) = vt + H(\sigma(w)) , \quad v \in \mathbf{R}^m , \quad H \in Lin(T_qS, \mathbf{R}^m) ,$$

where Lin denotes the space of all linear maps. By definition, $h(0,w)$ is the restriction of the value of $B_{\mathbf{R}^m}$ on the zero vector field on $\mathbf{R}^m$. This corresponds to the so-called absolute operator determined by B, [5], and every absolute operator is vertical by Lemma 4 from [5]. Consequently, $H \circ \sigma = 0$. Furthermore, for $t = 1$ we obtain $h(1,w) = h(1,0) = v$, which vanishes by the definition of λ_A. Hence B is vertical by Lemma 1 and by naturality.

Clearly, the standard fiber of $\overset{r}{\bigotimes} T$, which is a vector space, is naturally contractible to the zero vector. Thus, Lemma 2 follows from Proposition 1.

We remark that the standard fiber of every Weil bundle is naturally contractible to its canonical zero point, that corresponds to the generalized jet of the constant map, [5]. Hence Proposition 1 holds for every Weil bundle. This could be observed from the list of all natural operators transforming vector fields to the Weil bundles in [5], but now we have proved it geometrically. On the other hand, the standard fiber of the cotangent bundle T^* is not naturally contractible to its zero vector, since $\tilde{\varrho}(k,w) = k^{-1}w$ in this case. The list of all natural operators $T \to TT^*$ from [3] shows that Proposition 1 does not hold in the case of the cotangent bundle.

5. Consider a vertical natural operator $B : T \to T\overset{r}{\bigotimes} T$. Since $\overset{r}{\bigotimes} TM$ is a vector bundle, its vertical tangent bundle $V(\overset{r}{\bigotimes} TM)$ coincides with the Whitney sum $\overset{r}{\bigotimes} TM \oplus \overset{r}{\bigotimes} TM$ and the second projection $pr_2 \circ B_M X \circ O_M$ of the restriction of $B_M X$ to the zero section O_M of $\overset{r}{\bigotimes} TM$ defines a natural operator transforming vector fields on M into sections of $\overset{r}{\bigotimes} TM$. Hence we can apply the following general result by Mikulski, [7]. Let G be any bundle functor on $\mathcal{M}f$.

Lemma 3. *The natural operators $T \to G$ transforming vector fields on manifolds into the sections of bundle functor G are in bijection with the elements of the fiber $G_0\mathbf{R}$.*

In our case, we have $\overset{r}{\bigotimes} T_0\mathbf{R} = \mathbf{R}$, so that the natural operators $T \to \overset{r}{\bigotimes} T$ form a one-parameter family. Obviously, the rule $X \mapsto k(X \otimes \cdots \otimes X)$, $k \in \mathbf{R}$, $X \in C^\infty TM$, is a one-paremeter family of such operators. Thus, Lemma 3 implies that every natural operator $T \to \overset{r}{\bigotimes} T$ belongs into this family.

The section $X \otimes \cdots \otimes X$ of $\overset{r}{\bigotimes} TM$ induces, by means of the translations into the individual fibers of $\overset{r}{\bigotimes} TM$, a vertical vector field $V_M X$ on $\overset{r}{\bigotimes} TM$. This yields a natural operator $V : T \to T\overset{r}{\bigotimes} T$.

6. If we take the value $B_M(O_{TM})$ of a vertical operator $B : T \to T\overset{r}{\bigotimes} T$ at the zero vector field O_{TM} on M, then the second projection of $\overset{r}{\bigotimes} TM \oplus \overset{r}{\bigotimes} TM$ defines a map $\overset{r}{\bigotimes} TM \overset{r}{\bigotimes} TM$. This yields a natural transformation $\overset{r}{\bigotimes} T \to \overset{r}{\bigotimes} T$. All natural transformations $\overset{r}{\bigotimes} T$ are determined in Section 24 of [6]. Let S_r denote the permutation group of r letters. Every $s \in S_r$ defines a permutation of indices $P^s_M : \overset{r}{\bigotimes} TM \to \overset{r}{\bigotimes} TM$. Item 24.7 of [6] implies directly

Lemma 4. *All natural transformations $\overset{r}{\bigotimes} T \to \overset{r}{\bigotimes} T$ are linearly generated by the permutations of indices, i.e. they are of the form*

$$\sum_{s \in S_r} k_s P^s \,, \quad k_s \in \mathbf{R} \,.$$

On every $\overset{r}{\bigotimes} TM$, the vector field L^s_M tangent to the curve $(\mathrm{id}(\overset{r}{\bigotimes} TM) + kP^s_M)$, $k \in \mathbf{R}$, at $k = 0$ will be called the *Liouville vector field of type s*, $s \in S_r$.

7. Now we can prove the main result of the paper.

Theorem. *All natural operators $T \to T\overset{r}{\bigotimes} T$ form the following $(2 + r!)$ parameter family*

$$k_1 \overset{r}{\bigotimes} T + k_2 V + \sum k_s L^s \,, \quad s \in S_r \,,$$

with arbitrary $k_1, k_2, k_s \in \mathbf{R}$.

Proof. Having a natural operator $A : T \to T\overset{r}{\bigotimes} T$, we first construct the vertical operator $B = A - \lambda_A \overset{r}{\bigotimes} T$. By item 5, B determines a natural operator $T \to \overset{r}{\bigotimes} T$ of the form $X \mapsto \mu_A \overset{r}{\bigotimes} X$, $\mu_A \in \mathbf{R}$. By item 6, B defines a natural transformation $\overset{r}{\bigotimes} T \to \overset{r}{\bigotimes} T$ of the form $\sum_{s \in S_r} \kappa^A_s P^s$, $\kappa^A_s \in \mathbf{R}$. Write

$$\bar{A} = A - \lambda_A \overset{r}{\bigotimes} T - \mu_A V - \sum_{s \in \sigma_r} \kappa^A_s L^s$$

By Lemma 2 and by construction, $\bar{A}$ is a vertical operator. We are going to prove $\bar{A} = 0$. By Lemma 1 it suffices to deduce $\bar{A}_{\mathbf{R}^m}(\partial_1)|\overset{r}{\bigotimes} T_0\mathbf{R}^m = 0$. Taking into account $\overset{r}{\bigotimes} T_0\mathbf{R}^m = \overset{r}{\bigotimes}\mathbf{R}^m$, we define $g : \mathbf{R} \times \overset{r}{\bigotimes}\mathbf{R}^m \to \overset{r}{\bigotimes}\mathbf{R}^m$ by

$$g(t, w) = \bar{A}_{\mathbf{R}^m}(t\partial_1)(w) \ , \quad t \in \mathbf{R} \ , \ w \in \overset{r}{\bigotimes}\mathbf{R}^m \ .$$

Since $\bar{A}$ is natural, the equivariancy with respect to the homotheties in $GL(m, \mathbf{R})$ yields a homogeneity condition

$$k^r g(t, w) = g(kt, k^r w) \ , \ 0 \neq k \in \mathbf{R} \ .$$

By the homogeneous function theorem, [6], g is a polynomial of degree r in t and is linear in w, i.e.

$$g(t, w) = vt^r + C(w) \ , \ v \in \overset{r}{\bigotimes}\mathbf{R}^m \ , \ C \in Lin(\overset{r}{\bigotimes}\mathbf{R}^m, \overset{r}{\bigotimes}\mathbf{R}^m) \ .$$

For $t = 1$ and $w = 0$, we obtain the value of the operator $A - \lambda_A \overset{r}{\bigotimes} T - \mu_A V$, which is zero by the construction of μ_A. Hence $v = 0$. After that, for $t = 0$ we obtain the value of the absolute operator associated with $\bar{A}$. By the construction of κ_s^A, this is zero. Hence $C = 0$, which completes the proof.

8. Finally we remark that one can study the natural subbundles of $\overset{r}{\bigotimes} T$ in the same way. Let us start with the r-th symmetric tensor power $S^r TM$. Since the values of the map $X \mapsto X \otimes \cdots \otimes X$, $X \in C^\infty TM$, are sections of $S^r TM$, the only modification of the previous procedure should be done in item 6. By [6], all natural transformations $S^r T \to S^r T$ are the vector bundle homotheties, so that we only have one Liouville vector field L_M on each $S^r TM$. This implies

Proposition 2. *All natural operators transforming vector fields on manifolds into vector fields on the r-th symmetric tensor power of the tangent bundles form the following 3-parameter family*

$$k_1 S^r T + k_2 V + k_3 L \ , \quad k_1, k_2, k_3 \in \mathbf{R} \ .$$

From the theory of Young diagrams, [2], and from the fact that the action of $GL(m, \mathrm{R})$ on $\overset{r}{\bigotimes}\mathbf{R}^m$ is completely reducible, [1], it follows that we have a quite similar situation for an arbitrary natural subbundle of $\overset{r}{\bigotimes} T$. By the complete reducibility, the section $X \otimes \cdots \otimes X$, $X \in C^\infty TM$, can be naturally projected into the natural subbundle in question, which gives a single operator V in each case. On the other hand, the number of the Liouville vector fields depends on the natural subbundle in question.

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