S. Armas-Gómez; J. Margalef-Roig; Enrique Outerelo-Domínguez; E. Padrón-Fernández

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S.Armas-Gómez J.Margalef-Roig E.Outerelo-Domínguez E.Padrón-Fernández

Abstract

In order to study the transversality on manifolds with corners, we give a definition of boundary-transversality. This is a local definition in which the boundaries on the involved Banach spaces are viewed. This local definition implies the infinitesimal global description. The main results in this paper are: the characterization, for some type of submanifolds, of the boundary-transversality by means of the infinitesimal global description and the construction of submanifolds by inverse image of neat submanifolds by means of a boundary-transversal map.

1 Introduction

As it is well known, transversality theory is an important tool in Differential Topology. By means of these techniques we can, for instance, define algebraic invariants of the manifolds.

In the study of differentiable manifolds without boundary, modeled on real Banach spaces, the usual definition of map transversal to a submanifold is the following:

Definition 1.1 *[1] Let X and X' be C^p-manifolds without boundary, Y' a C^p-submanifold of X' with $\partial Y' = \emptyset$, $f : X \longrightarrow X'$ a C^p-map and $x \in X$. We say f is transversal to Y' at x if $f(x) \notin Y'$ or $f(x) \in Y'$, $T_{f(x)}X' = (T_x f)(T_x X) + (T_{f(x)}j')(T_{f(x)}Y')$, $(T_x f)^{-1}[(T_{f(x)}j')(T_{f(x)}Y')]$ admits a topological supplement in $T_x X$, where $j' : Y' \hookrightarrow X'$ is the inclusion map.*

This infinitesimal description of the transversality is equivalent to the following local description.

Proposition 1.2 *[1] Let X and X' be C^p-manifolds without boundary, Y' a C^p-submanifold of X' with $\partial Y' = \emptyset$, $f : X \longrightarrow X'$ a C^p-map and $x \in f^{-1}(Y')$. Then f is transversal to Y' at x if and only if there are a chart $c' = (U', \varphi', E')$ of X' adapted to Y' at $f(x)$ by means of E'_1, a topological supplement E'_2 of E'_1 in E' and an open neighbourhood U of x in X such that $f(U) \subseteq U'$ and the C^p-map:*

$$h : U \xrightarrow{f|_U} U' \xrightarrow{\varphi'} \varphi'(U') \xrightarrow{(\theta')^{-1}} E'_1 \times E'_2 \xrightarrow{p'_2} E'_2$$

is a submersion at x, where $\theta' : E'_1 \times E'_2 \longrightarrow E'$ is the linear homeomorphism defined by $\theta'(a', b') = a' + b'$ and p'_2 is the second projection from $E'_1 \times E'_2$ onto E'_2.

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Since the inverse image of a submanifold by a submersion is a submanifold, the above proposition allows us to prove that the inverse image of a submanifold by means a transversal map is a submanifold. This property is one of the most important features in the theory of transversality. Therefore when one considers manifolds with corners it seems reasonable to take a definition of transversality that fulfils this property.

Notice that if we take 1.1 as definition of transversality in manifolds with corners, we have not the above property even if f preserves the boundary.

On the other hand we note that in manifolds with corners we give a definition of submersion as a C^p-map that has a local section at every point. In this way one obtains a universal property to characterize C^p-maps. If f preserves the boundary, f is submersion at x if and only if T_xf is surjective and its kernel admits a topological supplement in T_xX . Finally if f is submersion that preserves the boundary, then the inverse image of a submanifold by f is a submanifold.

In orther to make coherent the theory of transversality in manifolds with corners we give the definition 3.1 of boundary-transversality. This is a local definition in which the boundaries are viewed on the involved Banach spaces. In Proposition 3.2 we prove the uniqueness, up to linear homeomorphism, of the quadrant of the target Banach space and we emphasize the intrinsic natural number associated to the boundary-transversality at a point x . In Theorem 3.6 one proves an infinitesimal characterization of the boundary-transversality for a type of submanifolds.

Finally in Theorem 3.8 one proves the main result: If f preserves the boundary and f is boundary-transversal to the neat submanifold Y' at every $x \in f^{-1}(Y')$, then the inverse image of Y' by f is a submanifold without boundary whose tangent space at x is the inverse image of $T_{f(x)}Y'$ by T_xf .

2 Prerequisites

Along this paper manifolds will be Banach manifolds and may have boundary if otherwise is not specified. Terminology and notation can be found in [2] but we explain here some of them.

Let E be real Banach space and Λ is a finite linearly independent system of elements of $L(E, \mathbb{R})$. Then the quadrant $\{x \in E/\lambda(x) \geq 0 \text{ for all } \lambda \in \Lambda\}$ will be denoted by E_Λ^+ and the closed linear subspace $\{x \in E/\lambda(x) = 0 \text{ for all } \lambda \in \Lambda\}$ will be denoted by E_Λ^0 .

If X is a manifold, a chart of X will be denoted by $c = (U, \varphi, (E, \Lambda))$, where U is the domain of the chart, φ is the morphism, E is the model space, $\varphi : U \longrightarrow E_\Lambda^+$ is injective and $\varphi(U)$ is an open set of E_Λ^+. If $\varphi(x) = 0$, we shall say the chart is centred at x. For instance $(E, 1_E, E)$ is the natural chart of E and $(E_\Lambda^+, j, (E, \Lambda))$ is the natural chart of E_Λ^+, where j is the inclusion map.

Let E_Λ^+ be a quadrant, U an open set of E_Λ^+ and $x \in E_\Lambda^+$. Then we call index of x and denote by $ind(x)$, the cardinal of the set $\{\lambda \in \Lambda/\lambda(x) = 0\}$. The set $\{y \in U/ind(y) \geq 1\}$ will be called boundary of U and denoted ∂U . The set $\{y \in U/ind(y) = 0\}$ will be called interior of U and denoted by $int(U)$, the set $\{x \in U/ind(x) = k\}$ will be denoted by $B_k(U)$, where $k \in \mathbb{N} \cup \{0\}$. From the local boundary invariance theorem we can define in a natural way, the index of points and the boundary of manifolds.

If X is a manifold and $a \in X$, we take the set

$$\{(c,v)/c = (U,\varphi,(E,\Lambda)) \text{ is a chart of } X \text{ with } a \in U \text{ and } v \in E\}$$

and we consider the binary relation, $\sim$, on this set defined by:

$$(c,v) \sim (c',v') \Longleftrightarrow D(\varphi' \circ \varphi^{-1})(\varphi(a))(v) = v'.$$

Then this relation is an equivalence relation and the quotient set will be denoted by T_aX .

Let $c = (U,\varphi,(E,\Lambda))$ be a chart of X and $a \in U$. It is clear that the map $\theta_c^a : E \longrightarrow T_aX$ defined by $\theta_c^a(v) = [(c,v)]$ is a bijective map, where $[(c,v)]$ denotes the class of equivalence of (c,v) . Via the map θ_c^a the space T_aX becomes an intrinsic real Banach space that will be called tangent space of X at a and θ_c^a becomes a linear homeomorphism. Moreover if $c = (U,\varphi,(E,\Lambda))$, $c' = (U',\varphi',(E',\Lambda'))$ are charts of X, $a \in U \cap U'$, then $(\theta_{c'}^a)^{-1} \circ \theta_c^a = D(\varphi' \circ \varphi^{-1})(\varphi(a))$.

If $f : X \longrightarrow X'$ is a C^p-map and $a \in X$, it is clear that there is a unique continuous linear map, $T_af : T_aX \longrightarrow T_{f(a)}X'$, such that, for every chart $c = (U,\varphi,(E,\Lambda))$ of X at a and for every chart $c' = (U',\varphi',(E',\Lambda'))$ of X' at $f(a)$, it holds $T_af = \theta_{c'}^{f(a)} \circ D(\varphi' \circ f \circ \varphi^{-1})(\varphi(a)) \circ (\theta_c^a)^{-1}$.

Let X be a C^p-manifold and $x \in X$. A C^r-curve on X with origin x, $0 \leq r \leq p$, is a C^r-map $\alpha : [0,a[\longrightarrow X$ such that $\alpha(0) = x$.

If α is a C^r-curve on X $(1 \leq r \leq p)$ with origin x defined on $[0,a[$, then the element of T_xX defined by $(T_0\alpha)(\theta_{c_0}^0(1))$ is called tangent vector to α at 0 and denoted by $\dot{\alpha}(0)$, where $c_0 = ([0,a[, i, (I\!R, 1_{I\!R}))$ is the natural chart of $[0,a[$. We note that if $c = (U,\varphi,(E,\Lambda))$ is a chart of X at x, then

$$\dot{\alpha}(0) = (T_0\alpha)(\theta_{c_0}^0(1)) = (\theta_c^x \circ D(\varphi \circ \alpha)(0))(1) = \theta_c^x(\lim_{t \to O^+} \frac{(\varphi \circ \alpha)(t) - (\varphi \circ \alpha)(0)}{t}) = \theta_c^x((\varphi \circ \alpha)'(0)),$$

where $\theta_{c_0}^0 : I\!R \longrightarrow T_0[0,a[$ and $\theta_c^x : E \longrightarrow T_xX$ are the natural linear homeomorphisms.

If v is a tangent vector of X at x given by a C^1-curve $\alpha : [0,a[\longrightarrow X$ on X with origin x, i.e. $\dot{\alpha}(0) = v$, then we shall say v is an inner tangent vector at x . The set of the inner tangent vectors at x will be denoted by $(T_xX)^i$. It holds that $T_xX = L((T_xX)^i)$, where L is the linear operator.

If $c = (U,\varphi,(E,\Lambda))$ is a chart of X such that $x \in U$ and $\varphi(x) \in \overset{0}{E_\Lambda}$ and $\Lambda' = \Lambda \circ (\theta_c^x)^{-1}$, then $\theta_c^x(E_\Lambda^+) = (T_xX)^i = (T_xX)_{\Lambda'}^+$.

Let X be a C^p-manifold and X' a subset of X . We say X' is a C^p-submanifold of X if for every $x' \in X'$ there are a chart $c = (U,\varphi,(E,\Lambda))$ of X with $x' \in U$ and $\varphi(x') = 0$, a closed linear subspace E' of E that admits a topological supplement in E and a finite linearly independent system Λ' of elements of $L(E', I\!R)$ such that $\varphi(U \cap X') = \varphi(U) \cap (E')_{\Lambda'}^+$ and this set is open in $(E')_{\Lambda'}^+$. (Usually we shortly say c is adapted to X' at x' by means of (E',Λ')).

We say the submanifold X' is a totally neat submanifold if $ind_{X'}(x') = ind_X(x')$ for every $x' \in X'$. If only $\partial X' = X' \cap \partial X$, we say X' is a neat submanifold.

Let $f : X \longrightarrow X'$ be a C^p-map and $x_0 \in X$. The map f is called submersion at x_0 if there are an open neighbourhood $V^{f(x_0)}$ of $f(x_0)$ in X' and a C^p-map $s : V^{f(x_0)} \longrightarrow X$ such that $s(f(x_0)) = x_0$ and $(f \circ s)(y) = y$ for all $y \in V^{f(x_0)}$.

We take this definition in order to have a universal property: *If f is surjective submersion and g is a map, then $g \circ f$ is a C^p-map if and only if g is a C^p-map.*

Nevertheless if f preserves the boundary, this definition is equivalent to the classical one: *$T_x f$ is surjective and its kernel admits a topological supplement in $T_x X$.* Here, of course, to say f preserves the boundary at x means there exists an open neighbourhood V^x of x in X such that $f(V^x \cap \partial X) \subseteq \partial X'$.

Proposition 2.1 *[3] Let f be a C^p-map from X into X' and Y' a C^p-submanifold of X'. Suppose, for every $x \in f^{-1}(Y')$, f is a submersion at x and f preserves the boundary at x. Then we have:*

1. *$f^{-1}(Y')$ is a C^p-submanifold of X and $\partial f^{-1}(Y') = f^{-1}(\partial Y')$.*
2. *$(T_x j)(T_x f^{-1}(Y')) = (T_x f)^{-1}[(T_{f(x)} j')(T_{f(x)} Y')]$, for all $x \in f^{-1}(Y')$, where $j : f^{-1}(Y') \hookrightarrow X$ and $j' : Y' \hookrightarrow X'$ are the inclusion maps.*
3. *For every $x \in f^{-1}(Y')$, $codim_x(f^{-1}(Y')) = codim_{f(x)}(Y')$.*

3 Boundary-transversality

Definition 3.1 *Let X and X' be C^p-manifolds, Y' a C^p-submanifold of X' and $f : X \longrightarrow X'$ a C^p-map. The map f is said to be boundary-transversal to Y' at $x \in X$, and it will be denoted by $f \pitchfork^{\partial}_x Y'$, if and only if $f(x) \notin Y'$ or $f(x) \in Y'$ and there are a chart $c' = (U', \varphi', (E', \Lambda'))$ of X' adapted to Y' at $f(x)$ by means of (E'_1, Λ'_1), a topological supplement E'_2 of E'_1 in E', a finite linearly independent system Λ'_2 of $L(E'_2, \mathbb{R})$ and an open neighbourhood U of x in X such that $f(U) \subseteq U'$, $(p_2 \circ \theta^{-1} \circ \varphi' \circ f)(U) \subseteq (E'_2)^+_{\Lambda'_2}$, $p_2(\partial(E'_1 \times E'_2)^+_{\Lambda' \circ \theta} \cap (\theta^{-1} \circ \varphi' \circ f)(\partial U)) \subseteq \partial(E'_2)^+_{\Lambda'_2}$ and the C^p-map $h : U \longrightarrow (E'_2)^+_{\Lambda'_2}$, defined by $h(y) = (p_2 \circ \theta^{-1} \circ \varphi' \circ f_{|U})(y)$ for all $y \in U$, is a submersion at x, being $\theta : E'_1 \times E'_2 \longrightarrow E'$ the linear homeomorphism defined by $\theta(a, b) = a + b$ and p_2 the second projection from $E'_1 \times E'_2$ onto E'_2.*

Notice if $j : (E'_2)^+_{\Lambda'_2} \hookrightarrow E'_2$ is the inclusion map, then the following diagram is commutative:

$$U \xrightarrow{f_{|U}} U' \xrightarrow{\varphi'} \varphi'(U') \xrightarrow{\theta^{-1}} (E'_1 \times E'_2)^+_{\Lambda' \circ \theta} \xrightarrow{p_{2|}} E'_2$$

$$U \xrightarrow{\quad j \circ h \quad} E'_2$$

If $A \subseteq X$ and f is boundary-transversal to Y' at each point of A, we will say f is boundary-transversal to Y' along A and it will be denoted by $f \pitchfork^{\partial}_A Y'$. If $A = X$, we will say f is boundary-transversal to Y' and it will be denoted by $f \pitchfork^{\partial} Y'$.

Now we are interested in proving the uniqueness, up to linear homeomorphisms, of $(E'_2)^+_{\Lambda'_2}$.

Proposition 3.2 *Let X, X' be C^p-manifolds, Y' a C^p-submanifold of X', $f : X \longrightarrow X'$ a C^p-map and $x \in f^{-1}(Y')$. Suppose $c' = (U', \varphi', (E', \Lambda'))$ and $c'_1 = (U'_1, \varphi'_1, (F', M'))$ are charts of X' adapted to Y' at $f(x)$ by means of (E'_1, Λ'_1) and (F'_1, M'_1) respectively, E'_2 and F'_2 are topological supplements of E'_1 in E' and F'_1 in F' respectively, Λ'_2 and M'_2 are finite linearly independent systems of $L(E'_2, \mathbb{R})$ and $L(F'_2, \mathbb{R})$ respectively and U is an open neighbourhood of x in X such that $f(U) \subseteq U' \cap U'_1$, $(p_2 \circ (\theta')^{-1} \circ \varphi' \circ f)(U) \subseteq (E'_2)^+_{\Lambda'_2}$, $(p'_2 \circ (\theta'_1)^{-1} \circ \varphi'_1 \circ f)(U) \subseteq (F'_2)^+_{M'_2}$ and the*

C^p-maps $h : U \longrightarrow (E_2')^+_{\Lambda_2'}$ and $h_1 : U \longrightarrow (F_2')^+_{M_2'}$, defined by $h(y) = (p_2 \circ (\theta')^{-1} \circ \varphi' \circ f)(y)$ and $h_1(y) = (\, p_2' \circ (\theta_1')^{-1} \circ \varphi_1' \circ f\,)(y)$ for all $y \in U$, are submersions at x, where $\theta' : E_1' \times E_2' \longrightarrow E'$ and $\theta_1' : F_1' \times F_2' \longrightarrow F'$ are the linear homeomorphisms defined by $\theta'(a,b) = a+b$ and $\theta_1'(a',b') = a'+b'$ and $p_2 : E_1' \times E_2' \longrightarrow E_2'$ and $p_2' : F_1' \times F_2' \longrightarrow F_2'$ are the second projections. Then there is a linear homeomorphism $\delta : E' \longrightarrow F'$ such that $\delta((E_2')^+_{\Lambda_2'}) = (F_2')^+_{M_2'}$.

Moreover $card(\Lambda_2') = card(M_2') \leq ind_X(x)$. Finally if h preserves the boundary in a neighbourhood of x in U, then $card(\Lambda_2') = ind_X(x)$.

Proof: We have that $\mu = D(\varphi_1' \circ (\varphi')^{-1})(0) = (\theta^{f(x)}_{c_1'})^{-1} \circ \theta^{f(x)}_{c'} : E' \longrightarrow F'$ is a linear homeomorphism such that $\mu(E_1') = F_1'$ since c' and c_1' are adapted charts to Y' at $f(x)$.

Since $h : U \longrightarrow (E_2')^+_{\Lambda_2'}$ is a submersion at x, there exist an open neighbourhood V^0 of 0 in $(E_2')^+_{\Lambda_2'}$ and a C^p-map $s : V^0 \longrightarrow U$ such that $s(0) = x$ and $h(s(y)) = y$, for every $y \in V^0$.

Now we take the C^p-map τ defined by the diagram:

$$\tau : V^0 \xrightarrow{s} U \xrightarrow{f} U_1' \xrightarrow{\varphi_1'} \varphi_1'(U_1') \xrightarrow{(\theta_1')^{-1}} (\theta_1')^{-1}(\varphi_1'(U_1')) \xrightarrow{p_2'} F_2'$$

the chart $c_2 = (V^0,\ i,\ (E_2',\ \Lambda_2'))$ of V^0 being i the inclusion map and the natural chart $c_2' = (F_2', 1_{F_2'}, F_2')$ of F_2' . Then we have $\tau(0) = 0$, $\tau = j \circ h_1 \circ s$, where $j : (F_2')^+_{M_2'} \hookrightarrow F_2'$ is the inclusion map, $(D(\tau)(0))((E_2')^+_{\Lambda_2'}) \subseteq (F_2')^+_{M_2'}$, $(D(\tau)(0))((E_2')^0_{\Lambda_2'}) \subseteq (F_2')^0_{M_2'}$ and

$$D(\tau)(0) = (\theta^0_{c_2'})^{-1} \circ T_0\tau \circ \theta^0_{c_2} = (\theta^0_{c_2'})^{-1} \circ T_x(j \circ h_1) \circ T_0 s \circ \theta^0_{c_2} =$$

$$= (\theta^0_{c_2'})^{-1} \circ \theta^0_{c_2'} \circ D(j \circ h_1 \circ \varphi^{-1})(0) \circ (\theta^x_c)^{-1} \circ T_0 s \circ \theta^0_{c_2},$$

where $c = (V, \varphi, (E, \Lambda))$ is a chart of U centred at x . Therefore

$$D(\tau)(0) = p_2' \circ (\theta_1')^{-1} \circ D(\varphi_1' \circ f \circ \varphi^{-1})(0) \circ (\theta^x_c)^{-1} \circ T_0 s \circ \theta^0_{c_2} = p_2' \circ (\theta_1')^{-1} \circ (\theta^{f(x)}_{c_1'})^{-1} \circ T_x f \circ T_0 s \circ \theta^0_{c_2} =$$

$$= p_2' \circ (\theta_1')^{-1} \circ \mu \circ (\theta^{f(x)}_{c'})^{-1} \circ T_x f \circ T_0 s \circ \theta^0_{c_2}.$$

Let us consider the diagram:

$$\begin{array}{ccccc} E' & \xrightarrow{\quad\mu\quad} & F' & \xrightarrow{\quad(\theta_1')^{-1}\quad} & F_1' \times F_2' \\ {\scriptstyle(\theta')^{-1}}\big\downarrow & & & & \big\downarrow{\scriptstyle p_2'} \\ E_1' \times E_2' & \xrightarrow{\quad p_2\quad} & E_2' & \xrightarrow{\quad\alpha\quad} & F_2' \end{array}$$

where α is defined by $\alpha = p_2' \circ (\theta_1')^{-1} \circ \mu_{|E_2'}$. Then, since $\mu(E_1') = F_1'$, this diagram is commutative and $D(\tau)(0) = \alpha \circ p_2 \circ (\theta')^{-1} \circ (\theta^{f(x)}_{c'})^{-1} \circ T_x f \circ T_0 s \circ \theta^0_{c_2} = \alpha \circ (\theta^0_{c_1})^{-1} \circ T_0 k \circ T_x h \circ T_0 s \circ \theta^0_{c_2}$, where $k : (E_2')^+_{\Lambda_2'} \longrightarrow E_2'$ is the inclusion map and $c_1 = (E_2', 1_{E_2'}, E_2')$ is the natural chart of E_2'. Finally $D(\tau)(0) = \alpha \circ (\theta^0_{c_1})^{-1} \circ T_0 k \circ T_0 1_{V^0} \circ \theta^0_{c_2} = \alpha \circ (\theta^0_{c_1})^{-1} \circ T_0 k \circ \theta^0_{c_2} = \alpha$ and, of course, α is a linear homeomorphism since $\mu(E_1') = F_1'$. Thus we have $card(\Lambda_2') \geq card(M_2')$. But if we take a section s_1 of h_1 at x and $\tau_1 = k \circ h \circ s_1$, analogously we obtain $card(M_2') \geq card(\Lambda_2')$, $\tau_1(0) = 0$,

$$(D(\tau_1)(0))((F_2')^+_{M_2'}) \subseteq (E_2')^+_{\Lambda_2'},\quad (D(\tau_1)(0))((F_2')^0_{M_2'}) \subseteq (E_2')^0_{\Lambda_2'},$$

$D(\tau_1)(0) : F_2' \longrightarrow E_2'$ is a linear homeomorphism, $D(\tau_1)(0) = \alpha_1$, where $\alpha_1 : F_2' \longrightarrow E_2'$ is defined by $\alpha_1 = p_2 \circ (\theta')^{-1} \circ \mu_{|F_2'}^{-1}$ and $\alpha = (\alpha_1)^{-1}$. Consequently $\alpha((E_2')^+_{\Lambda_2'}) = (F_2')^+_{M_2'}$ and $card(M_2') = card(\Lambda_2')$ and the desired linear homeomorphism is $\delta = \theta_1' \circ (\mu_{|E_1'} \times \alpha) \circ (\theta')^{-1} : E' \longrightarrow F'$.

To see $card(\Lambda_2') \leq ind_X(x)$, we take a chart $c = (V, \varphi, (E, \Lambda))$ of U centred at x. So that the map $l = D(h \circ \varphi^{-1})(0) : E \longrightarrow E_2'$ is surjective,

$$l(E_\Lambda^+) \subseteq (E_2')^+_{\Lambda_2'}, l(E_\Lambda^0) \subseteq (E_2')^0_{\Lambda_2'}, E = E_\Lambda^0 \oplus_T L\{v_1, \ldots, v_k\},$$

where $card(\Lambda) = k = ind_X(x)$, $E_2' = l(E_\Lambda^0) + l(L\{v_1, \ldots, v_k\})$ and

$$k \geq dim(l(L\{v_1, \ldots, v_k\})) \geq codim(l(E_\Lambda^0)) \geq codim((E_2')^0_{\Lambda_2'}) = card(\Lambda_2').$$

The last stament is a consequence of the following general result: *If g is a submersion at x which preserves the boundary at x, then $ind(x) = ind(g(x))$.* □

The preceding proposition allows to define an intrinsic number associated to the transversality at a point x.

Definition 3.3 *Let X and X' be C^p-manifolds, Y' a C^p-submanifold of X', $f : X \longrightarrow X'$ a C^p-map and $x \in f^{-1}(Y')$ such that $f \pitchfork_x^\partial Y'$. We call transversal index of x respect to f and Y', $ind(x, f, Y')$, to the number $card(\Lambda_2')$, after a localization as in proposition 3.2.*

Proposition 3.4 *Let X and X' be C^p-manifolds, Y' a C^p-submanifold of X', $f : X \longrightarrow X'$ a C^p-map and $x \in f^{-1}(Y')$ such that $f \pitchfork_x^\partial Y'$. Then $ind(x, f, Y') \leq ind_X(x)$, but if f preserves the boundary at x, we have $ind(x, f, Y') = ind_X(x)$.* □

Example: We take $X = I\!R \times (I\!R^+ \cup \{0\})$, $X' = I\!R^3$, $Y' = \{(x, y, z) \in I\!R^3 / y = 0\}$, $f : X \longrightarrow X'$ defined by $f(x, y) = (0, x, 0)$, $g : X \longrightarrow X'$ defined by $g(x, y) = (x, y, 0)$ and $x = (0, 0)$. Then

$$x \in f^{-1}(Y'),\ f \pitchfork_x^\partial Y',\ ind_X(x) = 1 > ind(x, f, Y') = 0,$$

$$x \in g^{-1}(Y'),\ g \pitchfork_x^\partial Y' \text{ and } ind_X(x) = 1 = ind(x, g, Y').$$

Next we shall prove an infinitesimal characterization of the bounday-transversality.

Lemma 3.5 *Let E be a real Banach space, E_1 a closed linear subspace of E which admits a topological supplement in E and Λ a finite linearly independent system of $L(E, I\!R)$.*

If $codim(E_1) \geq card(\Lambda)$, then there exist a topological supplement F^* of E_1 in E and a finite linearly independent system Λ^* of $L(F^*, I\!R)$ such that $(F^*)^+_{\Lambda^*} = F^* \cap E_\Lambda^+$ and $\partial(F^*)^+_{\Lambda^*} = F^* \cap \partial E_\Lambda^+$. Moreover, if $E_1 \subseteq E_\Lambda^0$, then $E_\Lambda^+ = E_1 + (F^*)^+_{\Lambda^*}$ and $\partial E_\Lambda^+ = E_1 + \partial(F^*)^+_{\Lambda^*}$.

Proof:If $card(\Lambda) = 0$, then we can take any topological supplement of E_1 in E as F^* and $\Lambda^* = \emptyset$.

Suppose $card(\Lambda) \geq 1$ and let $E_2 \subseteq E$ be a topological supplement of E_1 in E. If $n = card(\Lambda)$, since $dim(E_2) \geq n \geq 1$, then there is a linearly independent system $\{v_1, \ldots, v_n\}$ of E_2.

Let F_2 be a closed linear subspace of E such that $E_2 = F_2 \oplus_T L\{v_1, \ldots, v_n\}$. Therefore,

$$E = (E_1 + F_2) \oplus_T L\{v_1, \ldots v_n\}.$$

Moreover, since $E \neq E_1 + F_2$ then $Int(E_\Lambda^+) \not\subseteq E_1 + F_2$. Let $w_1 \in Int(E_\Lambda^+) - (E_1 + F_2)$ and $\varepsilon \in I\!R^+$ such that $B_\varepsilon(w_1) \subseteq Int(E_\Lambda^+)$ and $B_\varepsilon(w_1) \cap (E_1 + F_2) = \emptyset$.

We have the vector w_1 verifies the following properties:

1. $E^0_\Lambda \cap L\{w_1\} = \{0\}$, $(E_1 + F_2) \cap L\{w_1\} = \{0\}$.
2. If $n > 1$, $E^0_\Lambda + L\{w_1\} \not\supseteq B_\varepsilon(w_1) - [(E_1 + F_2) + L\{w_1\}]$.

Let $w_2 \in B_\varepsilon(w_1) - [(E^0_\Lambda + L\{w_1\}) \cup ((E_1 + F_2) + L\{w_1\})]$, then:

1. $E^0_\Lambda \cap L\{w_1, w_2\} = \{0\}$, $(E_1 + F_2) \cap L\{w_1, w_2\} = \{0\}$.
2. If $n > 2$, $E^0_\Lambda + L\{w_1, w_2\} \not\supseteq B_\varepsilon(w_1) - [(E_1 + F_2) + L\{w_1, w_2\}]$.

If we continue this process, we obtain a linearly independent system $\{w_1, \cdots, w_n\}$ of E^+_Λ such that $(E_1 + F_2) \oplus_T L\{w_1, \ldots, w_n\} = E$, $E^0_\Lambda \oplus_T L\{w_1, \ldots, w_n\} = E$ and for all $i \in \{1, \ldots, n-1\}$

$$w_{i+1} \notin (E_1 + F_2 + L\{w_1, \ldots, w_i\}) \cup (E^0_\Lambda + L\{w_1, \ldots, w_i\}).$$

Thus we take $F^* = F_2 + L\{w_1, \ldots, w_n\}$ and $\Lambda^* = \{\lambda_{|F^*} / \lambda \in \Lambda\}$. Moreover, if $E_1 \subseteq E^0_\Lambda$, then $codim(E_1) \geq card(\Lambda)$ and we have $E^+_\Lambda = E_1 + (F^*)^+_{\Lambda^*}$ and $\partial E^+_\Lambda = E_1 + \partial(F^*)^+_{\Lambda^*}$. □

Theorem 3.6 *Let X and X' be C^p-manifolds, $f : X \longrightarrow X'$ a C^p-map, Y' a C^p-submanifold of X' and $x \in f^{-1}(Y')$ such that f preserves the boundary at x and $(T_{f(x)}j)(T_{f(x)}Y') \subseteq (T_{f(x)}X')^i$, where $j : Y' \hookrightarrow X'$ is the inclusion map and $(T_{f(x)}X')^i$ is the set of inner tangent vectors of X' at $f(x)$. Then the following statements are equivalent:*

1. *(a) $T_{f(x)}X' = (T_xf)(T_xX) + (T_{f(x)}j)(T_{f(x)}Y')$*

   (b) $(T_xf)^{-1}[(T_{f(x)}j)(T_{f(x)}Y')]$ admits a topological supplement in T_xX .
2. *f is boundary-transversal to Y' at x .*

Proof: 2) ⇒ 1) By definition, there are a chart $c' = (U', \varphi', (E', \Lambda'))$ of X' adapted to Y' at $f(x)$ by means of (E'_1, Λ'_1), a topological supplement E'_2 of E'_1 in E', a finite linearly independent system Λ'_2 of $L(E'_2, I\!R)$ and an open neighbourhood U of x in X such that $f(U) \subseteq U'$,

$$(p_2 \circ \theta^{-1} \circ \varphi' \circ f)(U) \subseteq (E'_2)^+_{\Lambda'_2}, \quad p_2(\partial(E'_1 \times E'_2)^+_{\Lambda' \circ \theta} \cap (\theta^{-1} \circ \varphi' \circ f)(\partial U)) \subseteq \partial(E'_2)^+_{\Lambda'_2}$$

and the C^p-map $h : U \longrightarrow (E'_2)^+_{\Lambda'_2}$, defined by $h(y) = (p_2 \circ \theta^{-1} \circ \varphi' \circ f_{|U})(y)$ for every $y \in U$, is a submersion at x, being $\theta : E'_1 \times E'_2 \longrightarrow E'$ the linear homeomorphism $\theta(a, b) = a + b$ and p_2 the second projection from $E'_1 \times E'_2$ onto E'_2. Then T_xh is a surjective linear continuous map and $Ker(T_xh)$ admits a topological supplement in T_xX.

Let $c = (U_1, \varphi, (E, \Lambda))$ be a chart of X being $x \in U_1 \subseteq U$, $\varphi(x) = 0$ and $c_1 = ((E'_2)^+_{\Lambda'_2}, i, (E'_2, \Lambda'_2))$ the natural chart of $(E'_2)^+_{\Lambda'_2}$. Then $T_xh = \theta^0_{c_1} \circ D(i \circ h \circ \varphi^{-1})(0) \circ (\theta^x_c)^{-1} =$

$$= \theta^0_{c_1} \circ p_2 \circ \theta^{-1} \circ D(\varphi' \circ f \circ \varphi^{-1})(0) \circ (\theta^x_c)^{-1} = \theta^0_{c_1} \circ p_2 \circ \theta^{-1} \circ (\theta^{f(x)}_{c'})^{-1} \circ T_xf$$

and $Ker(T_xh) = (T_xf)^{-1}[(T_{f(x)}j)(T_{f(x)}Y')]$, which proves $b)$.

To prove $a)$, let us consider $v \in T_{f(x)}X'$, then $t = (\theta^0_{c_1} \circ p_2 \circ \theta^{-1} \circ (\theta^{f(x)}_{c'})^{-1})(v) \in T_0(E'_2)^+_{\Lambda'_2}$ and therefore there exists $u \in T_xX$ such that $(T_xh)(u) = t$. On the other hand

$$s = (\theta^{f(x)}_{c'} \circ p_1 \circ \theta^{-1} \circ (\theta^{f(x)}_{c'})^{-1})(v) \in (T_{f(x)}j)(T_{f(x)}Y'),$$

where $p_1 : E'_1 \times E'_2 \longrightarrow E'_1$ is the first projection. Finally

$$v = (T_xf)(u) + s - (\theta^{f(x)}_{c'} \circ p_1 \circ \theta^{-1} \circ (\theta^{f(x)}_{c'})^{-1} \circ T_xf)(u).$$

1) $\Rightarrow$ 2) Since Y' is a C^p-submanifold of X', there is a chart $c' = (U', \varphi', (E', \Lambda'))$ of X' adapted to Y' at $f(x)$ by means of (E'_1, Λ'_1). By the hypothesis $(T_{f(x)}j)(T_{f(x)}Y') \subseteq (T_{f(x)}X')^i$, we have:

$$\theta_{c'}^{f(x)}(E'_1) = (T_{f(x)}j)(T_{f(x)}Y') \subseteq (T_{f(x)}X')^i = \theta_{c'}^{f(x)}((E')^+_{\Lambda'})$$

and therefore $E'_1 \subseteq (E')^0_{\Lambda'}$. Then, using the preceding lemma, there are a topological supplement F^* of E'_1 in E' and a finite linearly independent system Λ^* of $L(F^*, \mathbb{R})$ such that

$$(E')^+_{\Lambda'} = E'_1 + (F^*)^+_{\Lambda^*} \text{ and } \partial(E')^+_{\Lambda'} = E'_1 + \partial(F^*)^+_{\Lambda^*}.$$

Let U be an open neighbourhood of x in X where f preserves the boundary and $f(U) \subseteq U'$. Then the C^p-map $h : U \longrightarrow (F^*)^+_{\Lambda^*}$, defined by $h(y) = (p_2 \circ \theta^{-1} \circ \varphi' \circ f_{|U})(y)$ for every $y \in U$, also preserves the boundary in U, where $\theta : E'_1 \times F^* \longrightarrow E'$ is the linear homeomorphism defined by $\theta(a, b) = a + b$ and $p_2 : E'_1 \times F^* \longrightarrow F^*$ is the second projection. Thus it suffices to prove $T_x h : T_x X \longrightarrow T_0(F^*)^+_{\Lambda^*}$ is a surjective map and $Ker(T_x h)$ admits a topological supplement in $T_x X$ .But we know $Ker(T_x X) = (T_x f)^{-1}[(T_{f(x)}j)(T_{f(x)}Y')]$ and therefore $Ker(T_x h)$ admits topological supplement in $T_x X$.

Now let us consider $v \in T_0(F^*)^+_{\Lambda^*}$ and the natural chart $c^* = ((F^*)^+_{\Lambda^*}, i, (F^*, \Lambda^*))$ of $(F^*)^+_{\Lambda^*}$. We have $(\theta_{c'}^{f(x)} \circ (\theta_{c^*}^0)^{-1})(v) \in T_{f(x)}X'$ and there exist $u \in T_x X$ and $w \in (T_{f(x)}j)(T_{f(x)}Y')$ such that

$$(\theta_{c'}^{f(x)} \circ (\theta_{c^*}^0)^{-1})(v) = (T_x f)(u) + w.$$

Finally $(T_x h)(u) = (\theta_{c^*}^0 \circ p_2 \circ \theta^{-1} \circ (\theta_{c'}^{f(x)})^{-1} \circ T_x f)(u) =$

$$= (\theta_{c^*}^0 \circ p_2 \circ \theta^{-1} \circ (\theta_{c'}^{f(x)})^{-1})((\theta_{c'}^{f(x)} \circ (\theta_{c^*}^0)^{-1})(v) - w) = (\theta_{c^*}^0 \circ p_2 \circ \theta^{-1} \circ (\theta_{c^*}^0)^{-1})(v) = v$$

□

Remark:

1. The hypotheses "*f preserves the boundary at x*" and "$(T_{f(x)}j)(T_{f(x)}Y') \subseteq (T_{f(x)}X')^i$" have not been used in the implication 2) $\Rightarrow$ 1).
2. The condition $(T_{f(x)}j)(T_{f(x)}Y') \subseteq (T_{f(x)}X')^i$ implies there is an open neighbourhood U' of $f(x)$ in X' such that $U' \cap Y' \subseteq B_{k'}X'$, where $k' = ind_{X'}(f(x))$.

Lemma 3.7 *Let X'' be a neat C^p-submanifold of a C^p-manifold X' . Then:*

1. $int(X'') \cap \partial X' = \emptyset$

2. If $(U', \varphi', (E', \Lambda'))$ is a chart of X' adapted to X'' at $x'' \in X''$ by (E'', Λ''), then:

a) $(E'')^+_{\Lambda''} \subseteq (E')^+_{\Lambda'}$, *b)* $\partial(E'')^+_{\Lambda''} \subseteq \partial(E')^+_{\Lambda'}$, *c)* $int(E'')^+_{\Lambda''} \subseteq int(E')^+_{\Lambda'}$,

d) $E'' - (E'')^+_{\Lambda''} \subseteq E' - (E')^+_{\Lambda'}$, *e)* $(\varphi')^{-1}(E'') = U' \cap X''$, *f)* $(E'')^+_{\Lambda''} = E'' \cap (E')^+_{\Lambda'}$.

Proof:

1. By definition $\partial X'' = X'' \cap \partial X'$.Then, since $int(X'') \cap \partial X'' = \emptyset$, we conclude $int(X'') \cap \partial X' = \emptyset$.

2. From the definition of submanifold it is clear $(E'')^+_{\Lambda''} \subseteq (E')^+_{\Lambda'}$.

If $x'' \in int(X'')$, obviously $\partial(E'')^+_{\Lambda''} = \emptyset$. Suppose $x'' \in \partial X''$ and let $z \in \partial(E'')^+_{\Lambda''}$. Then there exists $r > 0$ such that $r \cdot z \in \varphi'(U') \cap \partial(E'')^+_{\Lambda''}$. Thus $(\varphi')^{-1}(r \cdot z) \in \partial X'' \subseteq \partial X'$ and therefore $r \cdot z \in \partial(E')^+_{\Lambda'}$ and $z \in \partial(E')^+_{\Lambda'}$.

Analogously $int(E'')^+_{\Lambda''} \subseteq int(E')^+_{\Lambda'}$.

Now suppose there is $z \in E'' - (E'')^+_{\Lambda''}$, being $z \in (E')^+_{\Lambda'}$, then we take $z_0 \in int(E'')^+_{\Lambda''}$ and the segment $[z, z_0]$ joining z and z_0. Of course, $[z, z_0] \subseteq E''$ and there is $y_0 \in]z, z_0[$ such that $y_0 \in \partial(E'')^+_{\Lambda''}$, but $\partial(E'')^+_{\Lambda''} \subseteq \partial(E')^+_{\Lambda'}$. Thus there is $\lambda' \in \Lambda'$ such that $\lambda'(y_0) = 0$, $\lambda'(z) \geq 0$ and $\lambda'(z_0) > 0$. On the other hand there is $t_0 \in]0,1[$ such that $y_0 = t_0 \cdot z + (1 - t_0) \cdot z_0$. Hence $0 = t_0 \cdot \lambda'(z) + (1 - t_0) \cdot \lambda'(z_0) > 0$, which is a contradiction.

If $x \in int(X'')$, then $\varphi'(U' \cap X'') = \varphi'(U') \cap E''$ and $(\varphi')^{-1}(E'') = U' \cap X''$.

If $x'' \in \partial X''$, then $\varphi'(U') \cap (E'')^+_{\Lambda''} = \varphi'(U' \cap X'')$. Let $x \in U' \cap X''$, then $\varphi'(x) \in E''$ and $x \in (\varphi')^{-1}(E'')$. Hence $U' \cap X'' \subseteq (\varphi')^{-1}(E'')$.

Now let $x \in (\varphi')^{-1}(E'')$, then $\varphi'(x) \in E'' \cap \varphi'(U') \subseteq E'' \cap (E')^+_{\Lambda'}$ and therefore, using 4), $\varphi'(x) \in (E'')^+_{\Lambda''} \cap \varphi'(U')$. Hence $x \in U' \cap X''$ and $(\varphi')^{-1}(E'') \subseteq U' \cap X''$. Finally it is clear that $(E'')^+_{\Lambda''} = E'' \cap (E')^+_{\Lambda'}$. □

Next theorem gives us a method to build submanifolds by means of the inverse image of a submanifold by a boundary-transversal map.

Theorem 3.8 *Let X and X' be C^p-manifolds, $f : X \longrightarrow X'$ a C^p-map and Y' a C^p-submanifold of X' which has empty boundary or is a neat submanifold. Suppose f preserves the boundary at every $x \in f^{-1}(Y')$. Then, if f is boundary-transversal to Y' along $f^{-1}(Y')$, it holds the following statements:*

1. *$f^{-1}(Y')$ is a C^p-submanifold without boundary of X .*
2. *For every $x \in f^{-1}(Y')$, $(T_xj)(T_xf^{-1}(Y')) = (T_xf)^{-1}[(T_{f(x)}j')(T_{f(x)}Y')]$ where $j : f^{-1}(Y') \hookrightarrow X$ and $j' : Y' \hookrightarrow X'$ are the inclusion maps.*
3. *For all $y \in f^{-1}(Y')$, $codim_y(f^{-1}(Y')) = codim_{f(y)}(Y')$*
4. *$f_{|f^{-1}(Y')} : f^{-1}(Y') \longrightarrow Y'$ is a C^p-map and*

$$\{x \in f^{-1}(Y')/T_xf_{|f^{-1}(Y')} : T_xf^{-1}(Y') \to T_{f(x)}Y' \text{ is a surjective map } \} =$$

$$= \{x \in X/T_xf : T_xX \to T_{f(x)}X' \text{is a surjective map } \} \cap f^{-1}(Y')$$

Proof: Let $x \in f^{-1}(Y')$ and U^x an open neighbourhood of x in X such that $f(U^x \cap \partial X) \subseteq \partial X'$, then there are a chart $c' = (U', \varphi', (E', \Lambda'))$ of X' adapted to Y' at $f(x)$ by means of (E'_1, Λ'_1), a topological supplement E'_2 of E'_1 in E', a finite linearly independent system Λ'_2 of $L(E'_2, \mathbb{R})$ and an open neighbourhood U of x in X such that $f(U) \subseteq U'$, $U \subseteq U^x$,

$$(p_2\theta^{-1} \circ \varphi' \circ f_{|U})(U) \subseteq (E'_2)^+_{\Lambda'_2}, \quad p_2(\partial(E'_1 \times E'_2)^+_{\Lambda' \circ \theta} \cap (\theta^{-1} \circ \varphi' \circ f_{|U})(\partial U)) \subseteq \partial(E'_2)^+_{\Lambda'_2}$$

and the C^p-map $h : U \longrightarrow (E'_2)^+_{\Lambda'_2}$, defined by $h(y) = (p_2 \circ \theta^{-1} \circ \varphi' \circ f_{|U})(y)$ for every $y \in U$, is a submersion at x, being $\theta : E'_1 \times E'_2 \longrightarrow E'$ the linear homeomorphism defined by $\theta(a, b) = a + b$ and p_2 the second projection from $E'_1 \times E'_2$ onto E'_2 .

Since $\partial Y' = \emptyset$ or Y' is a neat submanifold we have $(\varphi')^{-1}(E'_1) = U' \cap Y'$. Indeed, if $\partial Y' = \emptyset$ obviously the equality holds and if Y' is a neat submanifold it follows from Lemma 3.7.

Since h preserves the boundary, $V = \{y \in U/h$ is a submersion at y$\}$ is an open set of U being $x \in V$. Thus $H = (h_{|V})^{-1}(\{0\})$ is a closed C^p-submanifold without boundary of V such that for all $y \in H$ $(T_y k)(T_y H) = (T_y h_{|V})^{-1}(\{0\})$, where $k : H \hookrightarrow V$ is the inclusion map.

On the other hand, using the equality $(\varphi')^{-1}(E_1') = U' \cap Y'$, we have $H = f^{-1}(Y') \cap V$ and therefore $f^{-1}(Y')$ is a C^p-submanifold of X without boundary which fulfils the statement 2).

The statement 3) follows from the statement 2) and from the equality:

$$T_{f(x)}X' = (T_x f)(T_x X) + (T_{f(x)}j')(T_{f(x)}Y'), \text{ for every } x \in f^{-1}(Y').$$

Finally 4.) is straightforward to be checked. Indeed, let x be an element of $f^{-1}(Y')$ such that $T_x f : T_x X \longrightarrow T_{f(x)}X'$ is surjective and let v be an element of $T_{f(x)}Y'$. Then there is $u \in T_x X$ such that $(T_x f)(u) = (T_{f(x)}j')(v)$. Hence, by 2), $u \in (T_x j)(T_x f^{-1}(Y'))$ and therefore there is $u_1 \in T_x(f^{-1}(Y'))$ such that $(T_x j)(u_1) = u$. Thus we conclude $(T_x f_{|f^{-1}(Y')})(u_1) = v$.

Conversely, if $x \in f^{-1}(Y')$, $T_x(f_{|f^{-1}(Y')}) : T_x f^{-1}(Y') \rightarrow T_{f(x)}Y'$ is surjective and $v \in T_{f(x)}X'$, using $T_{f(x)}X' = (T_x f)(T_x X) + (T_{f(x)}j')(T_{f(x)}Y')$, we have $v = (T_x f)(u) + (T_{f(x)}j')(u_1)$, where $u \in T_x X$ and $u_1 \in T_{f(x)}Y'$. Moreover, there is $u_2 \in T_x f^{-1}(Y')$ such that $(T_x f_{|f^{-1}(Y')})(u_2) = u_1$, and consequently: $v = (T_x f)(u) + (T_{f(x)}j')(T_x f_{|f^{-1}(Y')})(u_2) = (T_x f)(u + (T_x j)(u_2))$. □

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S.ARMAS-GOMEZ and
E.PADRON-FERNANDEZ
Dpto.MAT. FUNDAMENTAL.
UNIV. de LA LAGUNA.
TENERIFE. CANARIAS.SPAIN

J.MARGALEF-ROIG.
U.DE MATEMATICAS
C.S.I.C. SERRANO 123.
28006.MADRID.SPAIN

E.OUTERELO-DOMINGUEZ.
Dpto.GEOM.-TOPOLOGIA
UNIV. COMPLUTENSE
MADRID.SPAIN