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NATURAL TRANSFORMATIONS OF AFFINORS INTO LINEAR FORMS

Jacek Dębecki

An affinor on a manifold M is a tensor field of type (1,1) on M and a linear form on M is a tensor field of type (0,1) on M.

In this paper we give a characterisation of the natural transformations of affinors into linear forms satisfying the regularity condition. In Section 2 we prove that these natural transformations are of the form

$$T_M(t)=\sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}(a_1(t),...,a_n(t))\cdot d(a_i(t))\circ t^{n-j}$$

where f_{ij} are smooth functions on $\mathbf{R}^n$ and $a_1(t)$,...,$a_n(t)$ denote the coefficients of the characteristic polynomial of the linear endomorphism t.

In the proof of this theorem we will use a classification of the natural transformations of affinors into tensor fields of type (2,2) which we give in Section 1.

All manifolds and maps are assumed to be infinitely differentiable.

1. Natural transformations of affinors into tensor fields of type (2,2).

Let n, p, q, r, s be nonnegative integers. Let M be an n-dimensional manifold. We denote by $\mathcal{X}^p_q M$ the space of tensor fields of type (p,q) on M.

Definition 1.1. *A family of maps $T_M : \mathcal{X}^p_q M \longrightarrow \mathcal{X}^r_s M$ is called a natural transformation of tensor fields of type (p,q) into tensors fields of type (r,s) if for any n-dimensional manifolds M, N, for $t \in \mathcal{X}^p_q M$, $u \in \mathcal{X}^p_q N$ and for any injective immersion $\varphi : M \longrightarrow N$ the commutativity of the diagram*

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle u} \\ T^p_q M & \xrightarrow[T^p_q\varphi]{} & T^p_q N \end{array}$$

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implies the commutativity of the diagram

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ {\scriptstyle T_M(t)}\downarrow & & \downarrow{\scriptstyle T_N(u)} \\ T^r_s M & \xrightarrow[T^r_s\varphi]{} & T^r_s N \end{array}$$

If t is an affinor on M then $a_1(t), ..., a_n(t) : M \longrightarrow \mathbf{R}$ denote the coefficients of the characteristic polynomial $\det(\lambda \cdot id_{TM} - t) = \lambda^n + \sum_{i=1}^n a_i(t) \cdot \lambda^{n-i}$ and $t^k = t \circ ... \circ t$ (k times) where t is interpreted as a linear endomorphism of $\mathcal{X}^1_0 M$.

If t is tensor of type (2,2) then we define the following four operations by the formulas:

$$(t^S_S)^{ij}_{kl} = \frac{1}{4}(t^{ij}_{kl} + t^{ji}_{kl} + t^{ij}_{lk} + t^{ji}_{lk})$$
$$(t^S_A)^{ij}_{kl} = \frac{1}{4}(t^{ij}_{kl} + t^{ji}_{kl} - t^{ij}_{lk} - t^{ji}_{lk})$$
$$(t^A_S)^{ij}_{kl} = \frac{1}{4}(t^{ij}_{kl} - t^{ji}_{kl} + t^{ij}_{lk} - t^{ji}_{lk})$$
$$(t^A_A)^{ij}_{kl} = \frac{1}{4}(t^{ij}_{kl} - t^{ji}_{kl} - t^{ij}_{lk} + t^{ji}_{lk})$$

Theorem 1.2. *There is a one-to-one correspondence between natural transformations of affinors into tensor fields of type (2,2) and all system of $2n^2 - n$ smooth functions*

$$(1) \qquad \begin{aligned} &\alpha_{ij} : \mathbf{R}^n \longrightarrow \mathbf{R} \ \mathit{for}\ 0 \le i \le j \le n-1 \\ &\beta_{ij} : \mathbf{R}^n \longrightarrow \mathbf{R} \ \mathit{for}\ 0 \le i < j \le n-1 \\ &\gamma_{ij} : \mathbf{R}^n \longrightarrow \mathbf{R} \ \mathit{for}\ 0 \le i < j \le n-1 \\ &\delta_{ij} : \mathbf{R}^n \longrightarrow \mathbf{R} \ \mathit{for}\ 0 \le i \le j \le n-2 \end{aligned}$$

The natural transformation T corresponding to the system of functions (1) is defined by

$$(2) \qquad \begin{aligned} T_M(t) = &\sum_{0 \le i \le j \le n-1} \alpha_{ij}(a_1(t), ..., a_n(t)) \cdot (t^i \otimes t^j)^S_S \\ &+ \sum_{0 \le i < j \le n-1} \beta_{ij}(a_1(t), ..., a_n(t)) \cdot (t^i \otimes t^j)^S_A \\ &+ \sum_{0 \le i < j \le n-1} \gamma_{ij}(a_1(t), ..., a_n(t)) \cdot (t^i \otimes t^j)^A_S \\ &+ \sum_{0 \le i \le j \le n-2} \delta_{ij}(a_1(t), ..., a_n(t)) \cdot (t^i \otimes t^j)^A_A \end{aligned}$$

for any n-dimensional manifold M, $t \in \mathcal{X}^1_1 M$.

The group $GL(n,\mathbf{R})$ acts on $\bigotimes^p \mathbf{R}^n \otimes \bigotimes^q \mathbf{R}^{n*}$ on the right in the standard way.

In the paper [1] it is shown that the above theorem is equivalent to the following:

Lemma 1.3. *There is a one-to-one correspondence between all systems of $2n^2 - n$ smooth functions (1) and maps $E : \mathbf{R}^n \otimes \mathbf{R}^{n*} \longrightarrow \mathbf{R}^n \otimes \mathbf{R}^n \otimes \mathbf{R}^{n*} \otimes \mathbf{R}^{n*}$ such that*

(3) $E(t \cdot A) = E(t) \cdot A$ for $t \in \mathbf{R}^n \otimes \mathbf{R}^{n*}$, $A \in GL(n, \mathbf{R})$,

(4) for every smooth map $f : \mathbf{R}^n \longrightarrow \mathbf{R}^n \otimes \mathbf{R}^{n*}$ the composition $E \circ f$ is smooth.

If the map E corresponds to the system of functions (1) then $E(t)$ is equal to the right hand side of the equality (2) for every $t \in \mathbf{R}^n \otimes \mathbf{R}^{n*}$.

Proof: It is clear that for every system of smooth functions (1) the right hand side of (2) defines a map E such that the conditions (3) and (4) hold. At first we prove that for every map E there exists at most one system of functions (1) such that $E(t)$ is equal to the right hand side of (2).

Let $E(t)$ is equal to the right hand side of (2). We denote

$$R : \mathbf{R}^n \ni x \longrightarrow \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -x_n & \dots & -x_2 & -x_1 \end{bmatrix} \in \mathbf{R}^n \otimes \mathbf{R}^{n*}$$

If we compute $(E(R(x))^S_S)^{11}_{pq}$ for p, q such that $1 \le p \le q \le n$ then we obtain

$$\alpha_{rs}(x) = \begin{cases} (E(R(x))^S_S)^{11}_{r+1\, s+1} & \text{if } r = s \\ 2(E(R(x))^S_S)^{11}_{r+1\, s+1} & \text{if } r < s \end{cases} \tag{5}$$

for r, s such that $0 \le r \le s \le n-1$.

If we compute $(E(R(x))^S_A)^{11}_{pq}$ for p, q such that $1 \le p < q \le n$ then we obtain

$$\beta_{rs}(x) = 2(E(R(x))^S_A)^{11}_{r+1\, s+1} \tag{6}$$

for r, s such that $0 \le r < s \le n-1$. Analogously, for

$$S : \mathbf{R}^n \ni x \longrightarrow \begin{bmatrix} 0 & & & -x_n \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & -x_2 \\ & & 1 & -x_1 \end{bmatrix} \in \mathbf{R}^n \otimes \mathbf{R}^{n*}$$

we obtain

$$\gamma_{rs}(x) = 2(E(S(x))^A_S)^{r+1\, s+1}_{11} \tag{7}$$

for r, s such that $0 \leq r < s \leq n-1$. If we compute $(E(R(x))_A^A)^{12}_{pq}$ for p, q such that $1 \leq p < q \leq n$ then we obtain

$$\delta_{0s}(x) = \begin{cases} 2(E(R(x))_A^A)^{12}_{1\,s+2} & \text{if } s = 0 \\ 4(E(R(x))_A^A)^{12}_{1\,s+2} & \text{if } s \neq 0 \end{cases} \tag{8}$$

for $s = 1, ..., n-2$,

$$\delta_{r\,n-2}(x) = \begin{cases} 2(E(R(x))_A^A)^{12}_{r+1\,n} & \text{if } r = n-2 \\ 4(E(R(x))_A^A)^{12}_{r+1\,n} & \text{if } r \neq n-2 \end{cases} \tag{9}$$

for $r = 1, ..., n-2$,

$$\delta_{rs}(x) = \begin{cases} 2(E(R(x))_A^A)^{12}_{r+1\,s+2} + \frac{1}{2}\delta_{r-1\,s+1}(x) & \text{if } r = s \\ 4(E(R(x))_A^A)^{12}_{r+1\,s+2} + \delta_{r-1\,s+1}(x) & \text{if } r \neq s \end{cases} \tag{10}$$

for r, s such that $1 \leq r \leq s \leq n-3$. We can compute the functions δ_{rs} for r, s such that $0 \leq r \leq s \leq n-2$ from the formulas (8)—(10) by induction on $\min\{r,\ n-s-2\}$. From the formulas (5)—(10) we conclude that for every map E there exists at most one system of functions (1) such that $E(t)$ is equal to the right hand side of (2).

Let now E be a map satisfying (3) and (4). We define the system of functions (1) by the formulas (5)—(10). These functions are smooth because if R and S are smooth then from (4) the compositions $E \circ R$ and $E \circ S$ are smooth. It is sufficient to show that for these functions the right hand side of equality (2) is equal to $E(t)$.

At first we consider the case of a matrix $t \in \mathbf{R}^n \otimes \mathbf{R}^{n*}$ with n different complex eingenvalues. In this case Jordan's theorem ensures that there exists $A \in GL(n, \mathbf{R})$ such that $t = J \cdot A$, where

$$J = \begin{bmatrix} \pi_1 & & & & & \\ & \ddots & & & & \\ & & \pi_p & & & \\ & & & \begin{bmatrix} \varrho_1 & -\sigma_1 \\ \sigma_1 & \varrho_1 \end{bmatrix} & & \\ & & & & \ddots & \\ & & & & & \begin{bmatrix} \varrho_q & -\sigma_q \\ \sigma_q & \varrho_q \end{bmatrix} \end{bmatrix}$$

Let $Y \subset \mathbf{R}^n \otimes \mathbf{R}^n \otimes \mathbf{R}^{n*} \otimes \mathbf{R}^{n*}$ be the vector space consisting of all tensors $F(J)$ for $F : \mathbf{R}^n \otimes \mathbf{R}^{n*} \longrightarrow \mathbf{R}^n \otimes \mathbf{R}^n \otimes \mathbf{R}^{n*} \otimes \mathbf{R}^{n*}$ which satisfy the conditions (3) and (4).

We define the map $\varepsilon : \{1, ..., n\} \longrightarrow \{1, ..., p+q\}$ by the formula

$$\varepsilon(i) = \begin{cases} i & \text{if } i \leq p \\ [\frac{i+p+1}{2}] & \text{if } i > p \end{cases}$$

Let $a_r = p + 2r - 1$, $b_r = p + 2r$. We define two maps $\zeta_r : \{1, ...n\} \longrightarrow \{1, ..., n\}$ by the formula

$$\zeta_r(i) = \begin{cases} b_r & \text{if } i = a_r \\ a_r & \text{if } i = b_r \\ i & \text{if } i \neq a_r,\ i \neq b_r \end{cases}$$

and $\eta_r : \{1, ..., n\}^4 \longrightarrow \{0, ..., 4\}$ such that $\eta_r(i, j, k, l)$ is the number of terms of the sequence (i, j, k, l) which are equal to a_r.

A standard computation shows that $Y \subset Z$ where $Z \subset \mathbf{R}^n \otimes \mathbf{R}^n \otimes \mathbf{R}^{n*} \otimes \mathbf{R}^{n*}$ is a vector space consisting of all tensors u such that for $i, j, k, l = 1, ..., n$, $r = 1, ..., q$

$$u^{ij}_{kl} = 0 \text{ if } \{\varepsilon(i), \varepsilon(j)\} \neq \{\varepsilon(k), \varepsilon(l)\}$$
$$u^{\zeta_r(i)\zeta_r(j)}_{\zeta_r(k)\zeta_r(l)} = (-1)^{\eta_r(i,j,k,l)} \cdot u^{ij}_{kl}$$
$$(u^S_S)^{a_r b_r}_{a_r b_r} = \frac{(u^S_S)^{a_r a_r}_{a_r a_r} - (u^S_S)^{a_r a_r}_{b_r b_r}}{2}$$
$$(u^S_S)^{a_r a_r}_{a_r b_r} = -(u^S_S)^{a_r b_r}_{a_r a_r}$$

It is easy to compute that $\dim Z = 2n^2 - n$.

Repeated the argument demonstrating that the functions (1) are unique gives that the tensors

$$(J^i \otimes J^j)^S_S \text{ for } 0 \leq i \leq j \leq n-1$$
$$(J^i \otimes J^j)^S_A \text{ for } 0 \leq i < j \leq n-1$$
$$(J^i \otimes J^j)^A_S \text{ for } 0 \leq i < j \leq n-1 \tag{11}$$
$$(J^i \otimes J^j)^A_A \text{ for } 0 \leq i \leq j \leq n-2$$

are linearly independent. Hence the tensors (11) form a basis of the vector space Z and $E(J)$ is a linear combination of the tensors (11). From this we conclude that $E(t)$ is equal to the right hand side of (2).

We now turn to the general case. Let u be an arbitrary matrix and let v be a matrix which has n different complex eingenvalues. Let P be an n-dimensional affine subspace in the $\mathbf{R}^n \otimes \mathbf{R}^{n*}$ such that $u, v \in P$. Suppose that $D(w)$ denotes the discriminant of characteristic polynomial of matrix $w \in \mathbf{R}^n \otimes \mathbf{R}^{n*}$. Then D is a polynomial and $D(w) \neq 0$ if and only if w has n different complex eingenvalues. We have $D|P \neq 0$ because $D(v) \neq 0$. Hence $Q = \{w \in P | D(w) \neq 0\}$ is a dence subset of P. We know that $E|Q = F|Q$ where $F(t)$ is equal to the right hand side of equality (2). Suppose G denotes an affine parametrization of P. From (4) $E \circ G$ is smooth and $E|P = E \circ G \circ G^{-1}$ is smooth too. If two smooth maps are equal on a dense set then these maps are equal. In particular $E(u) = F(u)$. This ends the proof.

2. Natural transformations of affinors into linear forms.

Definition 2.1. *A natural transformation T of tensor fields of type (p,q) into tensor fields of type (r,s) satisfies the regularity condition if for a manifold M, an n-dimensional manifold N and smooth map $M \times N \ni (\alpha, x) \longrightarrow t_\alpha(x) \in T^p_q N$ such that $t_\alpha \in \mathcal{X}^p_q N$ for every $\alpha \in M$, the map $M \times N \ni (\alpha, x) \longrightarrow T_N(t_\alpha)(x) \in T^r_s N$ is smooth.*

We can now formulate our main result.

Theorem 2.2. *There is a one-to-one correspondence between natural transformations of affinors into linear forms satisfying the regularity condition and all systems of n^2 smooth functions $f_{ij} : \mathbf{R}^n \longrightarrow \mathbf{R}$ for $i,j = 1, ..., n$.*

The natural transformations T corresponding to the functions f_{ij} is defined by

$$T_M(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(a_1(t), ..., a_n(t)) \cdot d(a_i(t)) \circ t^{n-j}$$

for any n-dimensional manifold M, $t \in \mathcal{X}^1_1 M$.

Let L^∞_n be the group of infinite jets of local diffeomorphism of $\mathbf{R}^n$ with source and target $0 \in \mathbf{R}^n$. The group L^∞_n acts on $J^\infty_0(\mathbf{R}^n, \bigotimes^p \mathbf{R}^n \otimes \bigotimes^q \mathbf{R}^{n*})$ on the right in the following way: if $J_x\varphi$ denotes the Jacobi matrix of φ at x then $j^\infty_0 t \cdot j^\infty_0 \varphi$ is equal to the infinite jet at 0 of $\mathbf{R}^n \ni x \longrightarrow t(x) \cdot J_x \varphi \in \bigotimes^p \mathbf{R}^n \otimes \bigotimes^q \mathbf{R}^{n*}$.

From Krupka's theorem (see [3]) we conclude that Theorem 2.2. is equivalent to the following:

Lemma 2.3. *There is a one-to-one correspondence between all systems of n^2 smooth functions $f_{ij} : \mathbf{R}^n \longrightarrow \mathbf{R}$ for $i,j = 1, ..., n$ and maps $E : J^\infty_0(\mathbf{R}^n, \mathbf{R}^n \otimes \mathbf{R}^{n*}) \longrightarrow \mathbf{R}^{n*}$ such that*

(12) $E(j^\infty_0 t \cdot j^\infty_0 \varphi) = E(j^\infty_0 t) \cdot J_0 \varphi$ for $j^\infty_0 t \in J^\infty_0(\mathbf{R}^n, \mathbf{R}^n \otimes \mathbf{R}^{n*})$, $j^\infty_0 \varphi \in L^\infty_n$,

(13) for smooth map $\mathbf{R}^k \times \mathbf{R}^n \ni (\alpha, x) \longrightarrow t_\alpha(x) \in \mathbf{R}^n \otimes \mathbf{R}^{n*}$ the map $\mathbf{R}^k \ni \alpha \longrightarrow E(j^\infty_0 t_\alpha) \in \mathbf{R}^{n*}$ is smooth.

The map corresponding to the system of functions f_{ij} is defined by

$$E(j^\infty_0 t) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(a_1(t), ..., a_n(t)) \cdot d_0(a_i(t)) \circ t_0^{n-j} \tag{14}$$

Proof: It is clear that for every system of smooth functions f_{ij} the formula (14) defines a map E such that the conditions (12) and (13) hold.

At first we prove that for every map E there exists at most one system of functions

f_{ij} such that the equality (14) holds. We denote for $i = 1, ..., n$

$$R_i : \mathbf{R}^n \times \mathbf{R}^n \ni (\alpha, x) \longrightarrow \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ \beta_n & \dots & \beta_2 & \beta_1 \end{bmatrix} \in \mathbf{R}^n \otimes \mathbf{R}^{n*}$$

where

$$\beta_j = \begin{cases} -\alpha_j - x_1 & \text{if } j = i \\ -\alpha_j & \text{if } j \neq i \end{cases}$$

If we compute $E(j_0^\infty R_{p\alpha})_q$ for $p, q = 1, ..., n$ then we obtain

$$f_{rs}(\alpha) = E(j_0^\infty R_{r\alpha})_{n-s+1} \tag{15}$$

for $r, s = 1, ..., n$. From the formula (15) we conclude that for every map E there exists at most one system of functions f_{ij} such that the equality (14) is true.

Let now E be a map satisfying (12) and (13). We define the functions f_{ij} by the formula (15). From (13) it follows that these functions are smooth. It is sufficient to show that for these functions the equality (14) holds.

For $c \in \mathbf{R} \setminus \{0\}$ we define the homothety $\kappa_c : \mathbf{R}^n \ni x \longrightarrow \frac{1}{c}x \in \mathbf{R}^n$. A trivial verification shows that for a tensor field t of type (1,1) on $\mathbf{R}^n$

$$\frac{\partial^k (\kappa_{c*} t)^i_j}{\partial x^{k_1} ... \partial x^{k_n}}(0) = c^k \cdot \frac{\partial^k t^i_j}{\partial x^{k_1} ... \partial x^{k_n}}(0)$$

where $k = k_1 + ... + k_n$ and for a tensor t of type (0,1) on $\mathbf{R}^n$ $t \cdot J_0 \kappa_c = c \cdot t$. From (12) we have $E(j_0^\infty(\kappa_{c*} t)) = c \cdot E(j_0^\infty t)$ and we know that the condition (13) is satisfied. Hence the theorem about homogeneous functions ensures that there exist smooth functions $G^{ij}_{kl} : \mathbf{R}^n \otimes \mathbf{R}^{n*} \longrightarrow \mathbf{R}$ for $i, j, k, l = 1, ..., n$ such that

$$E(j_0^\infty t)_k = G^{\alpha\beta}_{k\gamma}(t(0)) \frac{\partial t^\gamma_\alpha}{\partial x^\beta}(0) \tag{16}$$

An easy computation shows that

$$G^{ij}_{kl}(t(0) \cdot J_0 \varphi) = \frac{\partial \varphi^i}{\partial x^\alpha}(0) \frac{\partial \varphi^j}{\partial x^\beta}(0) G^{\alpha\beta}_{\gamma\delta}(t(0)) \frac{\partial \varphi^{-1\gamma}}{\partial x^k}(0) \frac{\partial \varphi^{-1\delta}}{\partial x^l}(0) \tag{17}$$

$$-G^{\alpha j}_{kl}(t) t^i_\alpha - G^{\alpha i}_{kl}(t) t^j_\alpha + G^{ij}_{k\alpha}(t) t^\alpha_l + G^{ji}_{k\alpha}(t) t^\alpha_l = 0 \tag{18}$$

Since the equality (17) is true, Lemma 1.3. now shows that there exist a system of $2n^2 - n$ smooth functions (1) such that $G(t)$ is equal to the right hand side of (2) for $t \in \mathbf{R}^n \otimes \mathbf{R}^{n*}$.

It is easy to show that if we put either $G(t) = (t^p \otimes t^q)^S_S$ or $G(t) = (t^p \otimes t^q)^A_A$ then the equality (18) has a form $((t^p \otimes t^{q+1})^S_A - (t^{p+1} \otimes t^q)^S_A)^{ij}_{kl} = 0$. If we put either $G(t) = (t^p \otimes t^q)^S_A$ or $G(t) = (t^p \otimes t^q)^A_S$ then the equality (18) has a form $((t^p \otimes t^{q+1})^S_S - (t^{p+1} \otimes t^q)^S_S)^{ij}_{kl} = 0$. Using these formulas and the equalities $t^n = -\sum_{i=1}^n a_i(t) \cdot t^{n-i}$, $(t^q \otimes t^p)^S_S = (t^p \otimes t^q)^S_S$, $(t^q \otimes t^p)^S_A = -(t^p \otimes t^q)^S_A$ we can represent the left hand side of the equality (18) for G such that $G(t)$ is equal to the right hand side of (2) as a linear combination of

$$(t^i \otimes t^j)^S_S \text{ for } 0 \le i \le j \le n-1$$
$$(t^i \otimes t^j)^S_A \text{ for } 0 \le i < j \le n-1$$
$$(t^i \otimes t^j)^A_S \text{ for } 0 \le i < j \le n-1$$
$$(t^i \otimes t^j)^A_A \text{ for } 0 \le i \le j \le n-2$$

We know that the coefficients of this linear combination are unique. Therefore these coefficients are equal to zero. If we compute these coefficients and denote

$$\bar{\alpha}_{ij}(t) = \alpha_{ij}(a_1(t), ..., a_n(t))$$
$$\bar{\beta}_{ij}(t) = \beta_{ij}(a_1(t), ..., a_n(t))$$
$$\bar{\gamma}_{ij}(t) = \gamma_{ij}(a_1(t), ..., a_n(t))$$
$$\bar{\delta}_{ij}(t) = \delta_{ij}(a_1(t), ..., a_n(t))$$

then we obtain the following system of linear equations:

for $n = 2$

$$\begin{aligned} &2\bar{\alpha}_{00} - a_1\bar{\alpha}_{01} + 2a_2\bar{\alpha}_{11} + 2\bar{\delta}_{00} = 0 \\ &-a_2(\bar{\beta}_{01} + \bar{\gamma}_{01}) = 0 \\ &-a_1(\bar{\beta}_{01} + \bar{\gamma}_{01}) = 0 \\ &-\bar{\beta}_{01} - \bar{\gamma}_{01} = 0, \end{aligned} \tag{19}$$

for $n = 3$

$$\begin{aligned} &2\bar{\alpha}_{00} - a_2\bar{\alpha}_{01} + a_3\bar{\alpha}_{12} + 2\bar{\delta}_{00} = 0 \\ &\bar{\alpha}_{01} - a_1\bar{\alpha}_{02} + \bar{\delta}_{01} = 0 \\ &-\bar{\alpha}_{02} + 2\bar{\alpha}_{11} - a_1\bar{\alpha}_{12} + 2a_2\bar{\alpha}_{22} + 2\bar{\delta}_{11} = 0 \\ &-a_3(\bar{\beta}_{02} + \bar{\gamma}_{02}) = 0 \\ &-a_2\bar{\beta}_{02} - a_3\bar{\beta}_{12} - a_2\bar{\gamma}_{02} - a_3\bar{\gamma}_{12} = 0 \\ &\bar{\beta}_{01} - a_1\bar{\beta}_{02} + \bar{\gamma}_{01} - a_1\bar{\gamma}_{02} = 0 \\ &-\bar{\beta}_{12} - \bar{\gamma}_{12} = 0 \\ &-\bar{\beta}_{01} - a_2\bar{\beta}_{12} - \bar{\gamma}_{01} - a_2\bar{\gamma}_{12} = 0 \\ &-\bar{\beta}_{02} - a_1\bar{\beta}_{12} - \bar{\gamma}_{02} - a_1\bar{\gamma}_{12} = 0, \end{aligned} \tag{20}$$

for $n = 4$

$$
\begin{aligned}
&2\bar{\alpha}_{00} - a_3\bar{\alpha}_{02} + a_4\bar{\alpha}_{13} + 2\bar{\delta}_{00} = 0\\
&\bar{\alpha}_{01} - a_2\bar{\alpha}_{03} + a_4\bar{\alpha}_{23} + \bar{\delta}_{01} = 0\\
&\bar{\alpha}_{02} - a_1\bar{\alpha}_{03} + 2a_4\bar{\alpha}_{33} + \bar{\delta}_{02} = 0\\
&-\bar{\alpha}_{13} + 2\bar{\alpha}_{22} - a_1\bar{\alpha}_{23} + 2a_2\bar{\alpha}_{33} + 2\bar{\delta}_{22} = 0\\
&-\bar{\alpha}_{02} + 2\bar{\alpha}_{11} - a_2\bar{\alpha}_{13} + a_3\bar{\alpha}_{23} - \bar{\delta}_{02} + 2\bar{\delta}_{11} = 0\\
&\bar{\alpha}_{12} - \bar{\alpha}_{03} - a_1\bar{\alpha}_{13} + 2a_3\bar{\alpha}_{33} + \bar{\delta}_{12} = 0\\
&-a_4(\bar{\beta}_{03} - \bar{\gamma}_{03}) = 0\\
&-a_3\bar{\beta}_{03} - a_4\bar{\beta}_{13} - a_3\bar{\gamma}_{03} - a_4\bar{\gamma}_{13} = 0\\
&\bar{\beta}_{02} - a_1\bar{\beta}_{03} + \bar{\gamma}_{02} - a_1\bar{\gamma}_{03} = 0\\
&\bar{\beta}_{01} - a_2\bar{\beta}_{03} - a_4\bar{\beta}_{23} + \bar{\gamma}_{01} - a_2\bar{\gamma}_{03} - a_4\bar{\gamma}_{23} = 0\\
&-\bar{\beta}_{23} - \bar{\gamma}_{23} = 0\\
&-\bar{\beta}_{01} - a_3\bar{\beta}_{13} - \bar{\gamma}_{01} - a_3\bar{\gamma}_{13} = 0\\
&-\bar{\beta}_{12} - a_2\bar{\beta}_{23} - \bar{\gamma}_{12} - a_2\bar{\gamma}_{23} = 0\\
&-\bar{\beta}_{13} - a_1\bar{\beta}_{23} - \bar{\gamma}_{13} - a_1\bar{\gamma}_{23} = 0\\
&-\bar{\beta}_{02} - a_2\bar{\beta}_{13} - a_3\bar{\beta}_{23} - \bar{\gamma}_{02} - a_2\bar{\gamma}_{13} - a_3\bar{\gamma}_{23} = 0\\
&\bar{\beta}_{12} - \bar{\beta}_{03} - a_1\bar{\beta}_{13} + \bar{\gamma}_{12} - \bar{\gamma}_{03} - a_1\bar{\gamma}_{13} = 0,
\end{aligned}
\tag{21}
$$

for $5 \le n$

$$2\bar{\alpha}_{00} - a_{n-1}\bar{\alpha}_{0\,n-2} + a_n\bar{\alpha}_{1\,n-1} + 2\bar{\delta}_{00} = 0 \tag{22}$$

$$\bar{\alpha}_{0\,n-2} - a_1\bar{\alpha}_{0\,n-1} + 2a_n\bar{\alpha}_{n-1\,n-1} + \bar{\delta}_{0\,n-2} = 0 \tag{23}$$

$$\bar{\alpha}_{0\,p-1} - a_{n-p}\bar{\alpha}_{0\,n-1} + a_n\bar{\alpha}_{p\,n-1} + \bar{\delta}_{0\,p-1} = 0 \text{ for } 2 \le p \le n-2 \tag{24}$$

$$
\begin{aligned}
&-\bar{\alpha}_{n-3\,n-1} + 2\bar{\alpha}_{n-2\,n-2} - a_1\bar{\alpha}_{n-2\,n-1} + 2a_2\bar{\alpha}_{n-1\,n-1}\\
&+ 2\bar{\delta}_{n-2\,n-2} = 0
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
&-\bar{\alpha}_{p-1\,p+1} + 2\bar{\alpha}_{pp} - a_{n-p-1}\bar{\alpha}_{p\,n-1} + a_{n-p}\bar{\alpha}_{p+1\,n-1}\\
&-\bar{\delta}_{p-1\,p+1} + 2\bar{\delta}_{pp} = 0 \text{ for } 1 \le p \le n-3
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&\bar{\alpha}_{p\,n-2} - \bar{\alpha}_{p-1\,n-1} - a_1\bar{\alpha}_{p\,n-1} + 2a_{n-p}\bar{\alpha}_{n-1\,n-1}\\
&+ \bar{\delta}_{p\,n-2} = 0 \text{ for } 1 \le p \le n-3
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
&-\bar{\alpha}_{p-1\,q} + \bar{\alpha}_{p\,q-1} - a_{n-q}\bar{\alpha}_{p\,n-1} + a_{n-p}\bar{\alpha}_{q\,n-1}\\
&-\bar{\delta}_{p-1\,q} + \bar{\delta}_{p\,q-1} = 0 \text{ for } 1 \le p,\ p+2 \le q \le n-2
\end{aligned}
\tag{28}
$$

$$-a_n(\bar{\beta}_{0\,n-1} + \bar{\gamma}_{0\,n-1}) = 0 \tag{29}$$

$$-a_{n-1}\bar{\beta}_{0\,n-1} - a_n\bar{\beta}_{1\,n-1} - a_{n-1}\bar{\gamma}_{0\,n-1} - a_n\bar{\gamma}_{1\,n-1} = 0 \tag{30}$$

$$\bar{\beta}_{0\,n-2} - a_1\bar{\beta}_{0\,n-1} + \bar{\gamma}_{0\,n-2} - a_1\bar{\gamma}_{0\,n-1} = 0 \tag{31}$$

$$\begin{aligned}&\bar{\beta}_{0\,p-1} - a_{n-p}\bar{\beta}_{0\,n-1} - a_n\bar{\beta}_{p\,n-1}\\ &+ \bar{\gamma}_{0\,p-1} - a_{n-p}\bar{\gamma}_{0\,n-1} - a_n\bar{\gamma}_{p\,n-1} = 0 \text{ for } 2 \le p \le n-2\end{aligned} \tag{32}$$

$$-\bar{\beta}_{n-2\,n-1} - \bar{\gamma}_{n-2\,n-1} = 0 \tag{33}$$

$$\bar{\beta}_{p-1\,p} - a_{n-p}\bar{\beta}_{p\,n-1} - \bar{\gamma}_{p-1\,p} - a_{n-p}\bar{\gamma}_{p\,n-1} = 0 \text{ for } 1 \le p \le n-2 \tag{34}$$

$$-\bar{\beta}_{n-3\,n-1} - a_1\bar{\beta}_{n-2\,n-1} - \bar{\gamma}_{n-3\,n-1} - a_1\bar{\gamma}_{n-2\,n-1} = 0 \tag{35}$$

$$\begin{aligned}&-\bar{\beta}_{p-1\,p+1} - a_{n-p-1}\bar{\beta}_{p\,n-1} - a_{n-p}\bar{\beta}_{p+1\,n-1}\\ &-\bar{\gamma}_{p-1\,p+1} - a_{n-p-1}\bar{\gamma}_{p\,n-1} - a_{n-p}\bar{\gamma}_{p+1\,n-1} = 0 \text{ for } 1 \le p \le n-3\end{aligned} \tag{36}$$

$$\begin{aligned}&\bar{\beta}_{p\,n-2} - \bar{\beta}_{p-1\,n-1} - a_1\bar{\beta}_{p\,n-1}\\ &+ \bar{\gamma}_{p\,n-2} - \bar{\gamma}_{p-1\,n-1} - a_1\bar{\gamma}_{p\,n-1} = 0 \text{ for } 1 \le p \le n-3\end{aligned} \tag{37}$$

$$\begin{aligned}&-\bar{\beta}_{p-1\,q} + \bar{\beta}_{p\,q-1} - a_{n-q}\bar{\beta}_{p\,n-1} - a_{n-q}\bar{\beta}_{q\,n-1} - \bar{\gamma}_{p-1\,q} + \bar{\gamma}_{p\,q-1}\\ &- a_{n-q}\bar{\gamma}_{p\,n-1} - a_{n-p}\bar{\gamma}_{q\,n-1} = 0 \text{ for } 1 \le p,\ p+2 \le q \le n-2\end{aligned} \tag{38}$$

An trivial verification shows that there are $n^2 - n$ linearly independent equations in each of the systems (19), (20), (21). We now prove that there are $n^2 - n$ linearly independent equations in the system of linear equations (22)-(38).

We can compute $\bar{\delta}_{00}$ from (22), $\bar{\delta}_{0\,n-2}$ from (23), $\bar{\delta}_{0s}$ for $1 \le s \le n-3$ from (24) for $p = s+1$, $\bar{\delta}_{n-2\,n-2}$ from (25), $\bar{\delta}_{r\,n-2}$ for $1 \le r \le n-3$ from (27) for $p = r$. We next can compute $\bar{\delta}_{rs}$ for $1 \le r \le s \le n-3$ by induction on r. Namely if $r = s$ then we compute $\bar{\delta}_{rs}$ from (26) for $p = r$, if $r+1 \le s$ then we compute $\bar{\delta}_{rs}$ from (28) for $p = r$, $q = s+1$.

We can compute $\bar{\beta}_{n-2\,n-1}$ from (33), $\bar{\beta}_{n-3\,n-2}$ from (34) for $p = n-2$, $\bar{\beta}_{n-3\,n-1}$ from (35), $\bar{\beta}_{n-4\,n-3}$ from (34) for $p = n-3$, $\bar{\beta}_{n-4\,n-2}$ from (36) for $p = n-3$, $\bar{\beta}_{n-4\,n-1}$ from (37) for $p = n-3$. We next can compute $\bar{\beta}_{rs}$ for $r \le n-5$ and $0 \le r < s \le n-1$ by induction on $n-r-4$. Namely if $r+1 = s$ then we compute $\bar{\beta}_{rs}$ from (34) for $p = r+1$, if $r+2 = s$ then we compute $\bar{\beta}_{rs}$ from (36) for $p = r+1$, if $s = n-1$ then we compute $\bar{\beta}_{rs}$ from (37) for $p = r+1$, if $r+3 \le s$ and $s \le n-2$ then we compute $\bar{\beta}_{rs}$ from (38) for $p = r+1$, $q = s$.

Therefore the equations (22)—(28) and (33)—(38) are linearly independent.

Let M be a set consisting of all smooth maps $H : \mathbf{R}^n \otimes \mathbf{R}^{n*} \longrightarrow \mathbf{R}^n \otimes \mathbf{R}^n \otimes \mathbf{R}^{n*} \otimes \mathbf{R}^{n*}$ such that $H(t \cdot A) = H(t) \cdot A$ for $t \in \mathbf{R}^n \otimes \mathbf{R}^{n*}$, $A \in GL(n, \mathbf{R})$. It is obvious that M is an R-module where R is a ring consisting of all smooth functions $F : \mathbf{R}^n \otimes \mathbf{R}^{n*} \longrightarrow \mathbf{R}$ such that $F(t \cdot A) = F(t)$. Lemma 1.3. ensures that $\dim M = 2n^2 - n$. Let N be a submodule of M consisting of all maps of the form

$$H(t) = \sum_{0\leq i\leq j\leq n-1} \bar{\alpha}_{ij}(t)\cdot(t^i\otimes t^j)_S^S + \sum_{0\leq i<j\leq n-1} \bar{\beta}_{ij}(t)\cdot(t^i\otimes t^j)_A^S + \sum_{0\leq i<j\leq n-1} \bar{\gamma}_{ij}(t)\cdot(t^i\otimes t^j)_S^A + \sum_{0\leq i\leq j\leq n-2} \bar{\delta}_{ij}(t)\cdot(t^i\otimes t^j)_A^A$$

where $\bar{\alpha}_{ij}$, $\bar{\beta}_{ij}$, $\bar{\gamma}_{ij}$, $\bar{\delta}_{ij}$ denote arbitrary elements of R such that the system of linear equations (19) if $n = 2$, (20) if $n = 3$, (21) if $n = 4$, (22)-(38) if $5 \leq n$ holds. We see that $\dim N = (2n^2 - n) - (n^2 - n) = n^2$. Let $K_{ij} : \mathbf{R}^n \otimes \mathbf{R}^n \longrightarrow \mathbf{R}^n \otimes \mathbf{R}^n \otimes \mathbf{R}^{n*} \otimes \mathbf{R}^{n*}$ be maps such that

$$(d_0(a_i \circ t) \circ t_0^{n-j})_k = K_{ij}(t(0))_{k\gamma}^{\alpha\beta} \frac{\partial t_\alpha^\gamma}{\partial x^\beta}(0)$$

for $j_0^\infty t \in J_0^\infty(\mathbf{R}^n, \mathbf{R}^n \otimes \mathbf{R}^{n*})$. It is evident that the maps K_{ij} are elements of N. Repeating the argument proving that the functions f_{ij} are unique give that the maps K_{ij}, for $i, j = 1, ..., n$, are linearly independent. Hence these maps form a basis of the R-module N. Consequently $G \in N$ is a linear combination of K_{ij} and it follows immediately that the equality (14) is true. This finished the proof.

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