Andreas Čap

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REPORT ON K–THEORY FOR CONVENIENT ALGEBRAS II

Andreas Cap[1]

0. Introduction

In [Ca] I gave an overview over the first steps of a generalization of K–theory for Banach algebras to a much more general class of algebras, the so called convenient algebras. In this paper I continue this overview with the discussion of the two fundamental long exact sequences in K–theory, the one induced by a smooth map and the one induced by a bounded algebra homomorphism. Throughout this paper we will use the notions, notations and results of [Ca].

This paper splits into two parts: In the first part we develop some basic homotopy theory for smooth spaces, in particular the theory of fibrations and cofibrations. The main result is that there are smooth versions of the Puppe sequences, long exact sequences of certain sets of smooth homotopy classes of smooth mappings.

In the second part we discuss higher K–groups and relative K–groups and interpret these groups in terms of homotopy theory. Then we derive the fundamental long exact sequences from the Puppe sequences.

1. Smooth fibrations and cofibrations

Simple examples show that the obvious analog of the classical definition of a cofibration would not lead to a reasonable theory in the smooth category. (For example the inclusion of the point 0 into the unit interval I is not a cofibration in this sense.) Instead one is lead to the following weakening:

1.1. Definition. (1): Let X and Y be smooth spaces. A smooth map $i : Y \to X$ is called a *smooth cofibration* iff it has the following homotopy extension property: If Z is an arbitrary smooth space and $H : Y \times I \to Z$ and $f : X \to Z$ are smooth maps such that $f \circ i := H|_{Y\times\{0\}}$ then there is a smooth map $\tilde{H} : X \times I \to Z$ such that $\tilde{H} \circ (i \times Id) = H$ and $\tilde{H}|_{X\times\{0\}}$ is smoothly homotopic to f relative to Y, i.e. there is a smooth homotopy $h : X \times I \to Z$ such that $h(x,0) = \tilde{H}(x,0)$, $h(x,1) = f(x)$ and $h(i(y),t) = f(i(y))$ for all $x \in X$, $y \in Y$ and $t \in I$.
(2): For a smooth cofibration $i : Y \to X$ we define the *cofiber* of the cofibration to be the quotient space $X/i(Y)$ with the final smooth structure with respect to the canonical projection $X \to X/i(Y)$.

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1.2. Now several basic results on smooth cofibrations can be proved: Smooth cofibrations are always initial morphisms and for Hausdorff spaces they are injective, compositions and push outs of smooth cofibrations are again smooth cofibrations and so on. To see that there are nontrivial examples note that the inclusion of the smooth manifold S^{n-1} into the closed unit ball E^n is a smooth cofibration.

It is also interesting that similar as in topology, inclusion maps which are smooth cofibrations can be characterized by a property analogous to the notion of NDR–pairs (c.f. [**Steenrod**, 1967] or [**Whitehead**, 1978]).

1.3. Mapping cylinders. Let X and Y be arbitrary smooth spaces, $g : X \to Y$ a smooth map. We define the *mapping cylinder* M_g of g to be the push out:

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & M_g \end{array}$$

The inclusion of $X \times \{1\}$ into $X \times I$ induces a smooth map $i : X \to M_g$.

As a special case we define the *cone* CX over X to be the mapping cylinder of the unique smooth map $X \to pt$, where *pt* denotes the smooth space consisting of a single point.

1.4. Proposition. *Let M_g be the mapping cylinder of a smooth map $g : X \to Y$. Then we have:*
(1): The map $i : X \to M_g$ is a smooth cofibration.
(2): The natural map $Y \to M_g$ is a smooth homotopy equivalence.
(3): If X and Y are base spaces then M_g is a base space.

1.5. Using this result one shows that as in topology any smooth map factors into a composition of a smooth homotopy equivalence and a smooth cofibration.

Next for a smooth map $g : X \to Y$ one defines the *homotopy cofiber* C_g of g to be the cofiber of the smooth cofibration $i : X \to M_g$. It turns out that this space coincides with the mapping cone, i.e. the space obtained by attaching the cone over X to Y along g. Using this one concludes that the natural map $Y \to C_g$ is always a smooth cofibration. Moreover for a smooth cofibration the cofiber and the homotopy cofiber are smoothly homotopy equivalent and for a map between base spaces the homotopy cofiber is again a base space.

Dual to smooth cofibrations we define smooth fibrations as follows:

1.6. Definition. (1): Let X and Y be smooth spaces. A smooth map $p : X \to Y$ is called a *smooth fibration* iff it satisfies the following homotopy lifting property: If Z is an arbitrary smooth space and $H : Z \times I \to Y$ and $f : Z \to X$ are smooth maps such that $p \circ f = H|_{Z\times\{0\}}$, then there is a smooth map $\tilde{H} : Z \times I \to X$ such that $p \circ \tilde{H} = H$ and such that $\tilde{H}|_{Z\times\{0\}}$ is homotopic to f via a fiber preserving homotopy, i.e. there is a smooth map $h : Z \times I \to X$ such that $h|_{Z\times\{0\}} = \tilde{H}|_{Z\times\{0\}}$, $h|_{Z\times\{1\}} = f$ and $p \circ h = p \circ f \circ pr_1$.
(2): If $p : X \to Y$ is a smooth fibration and $y \in Y$ is a point then we define the *fiber* of p over y to be the set $p^{-1}(y)$ with the initial smooth structure with respect to the inclusion into X.

1.7. Again several basic properties of smooth fibrations can be proved: If the target space is smoothly path connected then any smooth fibration is a surjective final morphism, compositions and pullbacks of smooth fibrations are again smooth fibrations and so on.

The analog of the construction of the mapping cylinder now looks as follows:

1.8. Mapping cocylinders. Let X and Y be smooth spaces, $g : X \to Y$ an arbitrary smooth map. We define the *mapping cocylinder* M^g of g to be the pullback:

$$\begin{array}{ccc} M^g & \longrightarrow & C^\infty(I,Y) \\ \big\downarrow & & \big\downarrow{\scriptstyle ev_0} \\ X & \xrightarrow{g} & Y \end{array}$$

The composition of the evaluation at 1 and the canonical map $M^g \to C^\infty(I,Y)$ defines a smooth map $p : M^g \to Y$.

As a special case we define for a pointed smooth space (X, x_0) the *path space* PX over X to be the mapping cocylinder of the inclusion of x_0 into X.

1.9. PROPOSITION. *Let M^g be the mapping cocylinder of a smooth map $g : X \to Y$. Then we have:*
(1): The map $p : M^g \to Y$ is a smooth fibration.
(2): The natural map $q : M^g \to X$ is a smooth homotopy equivalence.

1.10. From this result one easily concludes that any smooth map factors into a composition of a smooth fibration and a smooth homotopy equivalence.

The definition of the *homotopy fiber* is now a little more subtle than the one of the homotopy cofiber. It can be proved that for a smoothly path connected target space the fibers over any two points are smoothly homotopy equivalent. But using this result we would get a definition of the homotopy fiber which is only up to smooth homotopy equivalence. To avoid this we restrict to base point preserving smooth maps $g : X \to Y$, where (X, x_0) and (Y, y_0) are pointed smooth spaces. For such a map we define the homotopy fiber C^g of g to be the fiber over y_0 of the smooth fibration $p : M^g \to Y$. It can be shown that C^g is naturally diffeomorphic to the pullback of the path fibration $PY \to Y$ along g, which implies that the natural map $C^g \to X$ is always a smooth fibration. Finally for a base point preserving smooth fibration between pointed smooth spaces the homotopy fiber is smoothly homotopy equivalent to the fiber over the base point.

1.11. Now we start the discussion of long exact sequences of sets of homotopy classes. Let W be a smoothly path connected smooth space. Then for any smooth space X the set $[X, W]$ of free smooth homotopy classes of smooth maps from X to W has a natural base point, namely the class represented by any map which maps the whole space X to a single point.

It is a quite simple consequence of the homotopy extension property that for a smooth cofibration $i : X \to Y$ with cofiber $p : Y \to Y/i(X)$ and for any smoothly path connected space W the sequence $[Y/i(X), W] \xrightarrow{p^*} [Y, W] \xrightarrow{i^*} [X, W]$ is an exact sequence of pointed sets.

Using the construction of the mapping cylinder (c.f. 1.3) one easily shows that one gets a similar exact sequence for an arbitrary smooth map replacing the cofiber by the homotopy cofiber. Clearly this procedure can be iterated to produce a long exact sequence of pointed sets in which the spaces are iterated homotopy cofibers. The main step to get the Puppe sequences is to give a description of these spaces (and the connecting maps) up to homotopy equivalence. This is quite subtle but can be done similar as in topology:

1.12. For a smooth space X the inclusion $X \to CX$ into the cone is a smooth cofibration (c.f. 1.3 and 1.4) and we define the (unreduced) *suspension* SX of X to be the cofiber of this smooth cofibration. Inductively we define higher suspensions S^nX. Clearly the suspensions have functorial properties, so a smooth map $f : X \to Y$ induces smooth maps $S^n(f) : S^nX \to S^nY$.

Now for a smooth map $f : X \to Y$ with homotopy cofiber C_f there is a natural smooth map $q : C_f \to SX$ induced by contracting the image of Y in C_f to a single point. Then it turns out that the iterated homotopy cofibers in the long exact sequence constructed in 1.11 are homotopy equivalent to iterated suspensions and one gets:

1.13. THEOREM. *Let $f : X \to Y$ be a smooth map between arbitrary smooth spaces with homotopy cofiber $g : Y \to C_f$, and let $q : C_f \to SX$ be the map constructed in 1.12. Then for any smoothly path connected smooth space W the sequence*

$$\dots \to [S^{n+1}X, W] \xrightarrow{S^n(q)^*} [S^nC_f, W] \xrightarrow{S^n(g)^*}$$
$$\to [S^nY, W] \xrightarrow{S^n(f)^*} [S^nX, W] \to \dots$$
$$\dots \to [SX, W] \xrightarrow{q^*} [C_f, W] \xrightarrow{g^*} [Y, W] \xrightarrow{f^*} [X, W]$$

of pointed sets is exact.

1.14. Dually to the considerations above we can construct similar sequences based on fibrations. First let $p : X \to Y$ be a base point preserving smooth fibration between pointed smooth spaces and let $i : F \to X$ be the inclusion of the fiber over the base point. Then from the homotopy lifting property one easily concludes that for any smooth space W the sequence $[W, F] \xrightarrow{i_*} [W, X] \xrightarrow{p_*} [W, Y]$ is an exact sequence of pointed sets. (Clearly these sets have natural base points since all spaces on the right hand side are pointed.)

Passing to arbitrary base point preserving maps and homotopy fibers and then iterating the procedure we get long exact sequences of pointed sets in which the spaces are iterated homotopy fibers.

1.15. For any pointed smooth space (X, x_0) we have the path fibration $PX \to X$ (c.f. 1.8 and 1.9) and we define the *loop space* ΩX of X to be the fiber over x_0 of this smooth fibration. Inductively we define iterated loop spaces Ω^nX. Clearly this construction is functorial, so a smooth map $f : X \to Y$ induces smooth maps $\Omega^n(f) : \Omega^nX \to \Omega^nY$.

Next if $f : X \to Y$ is a base point preserving smooth map with homotopy fiber $g : C^f \to X$ then the inclusion of the fiber over the base point of this smooth fibration induces a smooth map $k : \Omega Y \to C^f$. Now it turns out that the iterated homotopy fibers in the long exact sequence constructed in 1.14 are smoothly homotopy equivalent to certain loop spaces and we get:

1.16. THEOREM. *Let $f : X \to Y$ be a base point preserving smooth map between pointed smooth spaces with homotopy fiber $g : C^f \to X$, and let $k : \Omega Y \to C^f$ be the smooth map constructed in 1.15. Then for any smooth space W the sequence*

$$\begin{aligned} \cdots \to [W, \Omega^{n+1} Y] \xrightarrow{\Omega^n(k)_*} [W, \Omega^n C^f] \xrightarrow{\Omega^n(g)_*} \\ \to [W, \Omega^n X] \xrightarrow{\Omega^n(f)_*} [W, \Omega^n Y] \to \dots \\ \dots \to [W, \Omega Y] \xrightarrow{k_*} [W, C^f] \xrightarrow{g_*} [W, X] \xrightarrow{f_*} [W, Y] \end{aligned}$$

is an exact sequence of pointed sets.

2. THE FUNDAMENTAL LONG EXACT SEQUENCES IN K–THEORY FOR CONVENIENT ALGEBRAS

2.1. Definition. Let X be a base space, A a convenient algebra. Then the unique smooth map $X \to pt$ from X to the single point smooth space induces a group homomorphism $K_0(A) = K_A(pt) \to K_A(X)$, and we define the group $\tilde{K}_A(X)$ to be the cokernel of this homomorphism. If X is smoothly path connected then this group coincides with the group $K'_A(X)$ considered in [Ca].

Now in analogy with classical topological K–theory we define the higher K-groups $K_A^{-n}(X)$ of X for $n > 0$ as $\tilde{K}_A(S^n(X^+))$, where X^+ denotes the disjoint union of X and a single point and S^n denotes the n–fold suspension (c.f. 1.12). From the functorial properties of the groups $K_A(X)$ one immediately concludes that the higher K-groups are bifunctors, contravariant in X and covariant in A.

For well pointed base spaces (X, x_0), i.e. spaces for which the inclusion of the base point is a smooth cofibration, there is a nice interpretation of the group $\tilde{K}_A(X)$ in terms of homotopy theory. Let $[X, K_0(A) \times BGL(n, A)]_0$ be the set of smooth homotopy classes of base point preserving smooth maps from X to the product of the discrete space $K_0(A)$ with the classifying space of the smooth group $GL(n, A)$. Then these sets form an inductive system with respect to the same connecting maps as for the free homotopy classes (c.f. [Ca]) and we denote by $[X, K_0(A) \times BGL(A)]_0$ the direct limit of this system. Then one proves that there is an isomorphism of bifunctors $\tilde{K}_A(X) \cong [X, K_0(A) \times BGL(A)]_0$.

For any base space X the space X^+ is well pointed and suspensions of well pointed spaces are well pointed. Moreover, since suspensions are always smoothly path connected we can leave out the $K_0(A)$ and we get $K_A^{-n}(X) \cong [S^n(X^+), BGL(A)]_0$ (where again this has to be understood as a direct limit).

2.2. For the case of a smooth map there is a simple definition of the relative K–group using K–theory of the homotopy cofiber. But the situation is more complicated in the case of a bounded algebra homomorphism, where there is no obvious definition. Thus we take a more general approach. First observe the following facts:

(1): The categories $\mathcal{P}(A)$ of finitely generated projective right modules over a convenient algebra A and $\mathcal{E}_A(X)$ of A–bundles over a base space X are convenient, i.e. every set of morphisms in any of these categories has a natural structure of a convenient vector space.
(2): The functors $\mathcal{E}_A(f)$ induced by a smooth map f and $\mathcal{P}(\varphi)$ and $\mathcal{E}_\varphi(X)$ induced by a bounded algebra homomorphism φ are all convenient in the sense that they induce bounded linear maps on morphism sets.
Now we can adapt the general definition of the K–group of a Banach functor due to Karoubi (c.f. [**Ka**, II.2.13]) to define the K–group of a convenient functor between convenient categories as follows:

2.3. Definition. Let $\mathcal{C}$ and $\mathcal{C}'$ be additive convenient categories, $\varphi : \mathcal{C} \to \mathcal{C}'$ a convenient additive functor. Let $\Gamma(\varphi)$ denote the set of all triples (E, F, α) where E and F are objects of $\mathcal{C}$ and α is an isomorphism between $\varphi(E)$ and $\varphi(F)$. Two triples (E, F, α) and (E', F', α') are called isomorphic iff there are isomorphisms $f : E \to E'$ and $g : F \to F'$ such that the following diagram commutes.

$$\begin{array}{ccc} \varphi(E) & \xrightarrow{\alpha} & \varphi(F) \\ \downarrow{\scriptstyle \varphi(f)} & & \downarrow{\scriptstyle \varphi(g)} \\ \varphi(E') & \xrightarrow{\alpha'} & \varphi(F') \end{array}$$

A triple (E, F, α) is called *elementary* iff $E = F$ and α is homotopic to $id_{\varphi(E)}$ as an automorphism of $\varphi(E)$. (This makes sense since $\mathcal{C}'(\varphi(E), \varphi(E))$ is a convenient algebra and thus the automorphisms form a smooth group.) Finally we define the sum of two triples by

$$(E, F, \alpha) + (E', F', \alpha') := (E \oplus E', F \oplus F', \alpha \oplus \alpha').$$

Now we define the K–group $K(\varphi)$ of the functor φ to be the quotient of $\Gamma(\varphi)$ with respect to the equivalence relation defined by declaring two elements σ and σ' to be equivalent if and only if there are elementary triples τ and τ' such that $\sigma + \tau$ and $\sigma' + \tau'$ are isomorphic. We write $d(E, F, \alpha)$ for the class of the triple in $K(\varphi)$.

Obviously the addition defined above factors to an addition on $K(\varphi)$ which defines the structure of a commutative monoid on $K(\varphi)$. Moreover one shows (c.f. [**Ka**, II.2.14]) that the elements $d(E, F, \alpha)$ and $d(F, E, \alpha^{-1})$ are inverse in $K(\varphi)$ and thus $K(\varphi)$ is an abelian group.

2.4. Let us first consider the case of a smooth map. We want to give a homotopy interpretation of the group $K(\mathcal{E}_A(f))$ of the functor $\mathcal{E}_A(f) : \mathcal{E}_A(Y) \to \mathcal{E}_A(X)$ induced by a smooth map $f : X \to Y$. This interpretation follows the same lines as for absolute groups but is much more difficult to prove. First we associate to a triple $(E, F, \alpha) \in \Gamma(\mathcal{E}_A(f))$ the locally constant function $Y \to K_0(A)$ given by assigning to any point $y \in Y$ the difference of the classes of the fibers of E and F over y. This construction defines a group homomorphism $K(\mathcal{E}_A(f)) \to H^0(Y, K_0(A))$ and we define $K'(\mathcal{E}_A(f))$ to be the kernel of this homomorphism. One shows that there is a splitting short exact sequence

$$0 \to K'(\mathcal{E}_A(f)) \to K(\mathcal{E}_A(f)) \to \mathrm{Ker}(f^*) \to 0,$$

where $f^* : H^0(Y, K_0(A)) \to H^0(X, K_0(A))$ is the homomorphism induced by f. One immediately verifies that $\mathrm{Ker}(f^*) \cong [C_f, K_0(A)]_0$, where C_f is the homotopy cofiber of f and thus we have an isomorphism $K(\mathcal{E}_A(f)) \cong K'(\mathcal{E}_A(f)) \oplus [C_f, K_0(A)]_0$.

2.5. Next consider the set of all pairs (E, α), where E is an A–bundle over Y with fiber A^n and α is an isomorphism between f^*E and the trivial bundle $X \times A^n$. Two such pairs (E, α) and (E', α') are said to be equivalent if and only if there is an isomorphism $\varphi : E \to E'$ such that $\alpha' \circ f^*\varphi$ is homotopic to α as an isomorphism from f^*E to $X \times A^n$. Let $\Phi_n(\mathcal{E}_A(f))$ be the set of all equivalence classes.

Now $(E, \alpha) \mapsto (E \oplus \theta_1, \alpha \oplus id)$, where θ_1 denotes the trivial 'line' bundle $Y \times A$ over Y, defines a map $\Phi_n(\mathcal{E}_A(f)) \to \Phi_{n+1}(\mathcal{E}_A(f))$ and we define $\Phi(\mathcal{E}_A(f))$ to be the direct limit of the so obtained inductive system. Next $((E, \alpha), (F, \beta)) \mapsto (E \oplus F, \alpha \oplus \beta)$ defines a map $\Phi_n(\mathcal{E}_A(f)) \times \Phi_m(\mathcal{E}_A(f)) \to \Phi_{n+m}(\mathcal{E}_A(f))$ and one shows that this induces the structure of a commutative monoid on $\Phi(\mathcal{E}_A(f))$.

We define a map $\sigma_n : \Phi_n(\mathcal{E}_A(f)) \to K'(\mathcal{E}_A(f))$ by $\sigma_n(E, \alpha) := d(E, Y \times A^n, \alpha)$. This is easily seen to be well defined and clearly it induces a monoid homomorphism $\sigma : \Phi(\mathcal{E}_A(f)) \to K'(\mathcal{E}_A(f))$. Then one proves:

2.6. PROPOSITION. *The homomorphism σ defined above is bijective, so $\Phi(\mathcal{E}_A(f))$ is an abelian group.*

2.7. When working in the topological category one could now easily relate such pairs (E, α) to bundles over the homotopy cofiber C_f via clutching constructions. This would be difficult in the smooth setting since we can perform clutching only over open subsets. Thus we pass to classifying maps as follows: Clearly we may restrict to pairs $(E, \alpha) \in \Phi_n(\mathcal{E}_A(f))$ in which E is the associated bundle to the pullback of the universal $GL(n, A)$ bundle along some smooth map $g : Y \to BGL(n, A)$. Then α is induced by a trivialization of the pullback along f of this principal bundle. Using the canonical section of the trivial $GL(n, A)$ bundle and the inverse of this trivialization we get a smooth map s from X to the total space of the universal $GL(n, A)$ bundle which projects to $g \circ f$. But this total space is contractible, so s must be null homotopic. The projection of such a null homotopy can be viewed as a smooth map $H : CX \to BGL(n, A)$, which together with g induces a base point preserving map $C_f \to BGL(n, A)$, the homotopy class of which we assign to the pair (E, α).

2.8. THEOREM. *For any $n > 0$ the construction described above leads to a well defined map $u_n : \Phi_n(\mathcal{E}_A(f)) \to [C_f, BGL(n, A)]_0$. Together these maps induce a group isomorphism $u : \Phi(\mathcal{E}_A(f)) \to [C_f, BGL(A)]_0$.*

2.9. Putting together this result with the isomorphisms from 2.4 and 2.1 we see that $K(\mathcal{E}_A(f)) \cong \tilde{K}_A(C_f)$. Moreover let $f^+ : X^+ \to Y^+$ be the base point preserving smooth map induced by f. Then one proves that the homotopy cofiber C_{f^+} is smoothly homotopy equivalent to C_f, so we can use this space as well.

Next we define higher relative K–groups by $K^{-n}(\mathcal{E}_A(f)) := \tilde{K}_A(S^n(C_{f^+}))$. Then there are obvious homomorphisms $K_A^{-n-1}(X) \to K^{-n}(\mathcal{E}_A(f)) \to K_A^{-n}(Y)$ induced by the natural smooth maps $S(X^+) \to C_{f^+} \to Y^+$ (c.f. 1.12) by passing to the n–fold suspension. Using the Puppe sequence 1.13 induced by f^+ one proves:

2.10. THEOREM (THE LONG EXACT SEQUENCE OF A SMOOTH MAP). *Let $f : X \to Y$ be a smooth map between base spaces and let A be a convenient algebra. Then there is a long exact sequence of abelian groups and group homomorphisms*

$$\dots \to K_A^{-n-1}(X) \to K^{-n}(\mathcal{E}_A(f)) \to$$
$$\to K_A^{-n}(Y) \xrightarrow{K_A^{-n}(f)} K_A^{-n}(X) \to \dots$$
$$\to K_A^{-1}(X) \to K(\mathcal{E}_A(f)) \to K_A^0(Y) \xrightarrow{K_A^0(f)} K_A^0(X)$$

which is natural in f and A.

2.11. Using the homotopy interpretation of relative K–groups we can now give another homotopy interpretation of higher K–groups: This is based on the fact that one can show that for a base space X the group $\tilde{K}_A(SX)$ is isomorphic to $\Phi(\mathcal{E}_A(i))$, where $i : X \to CX$ is the inclusion into the cone.

For a pointed base space (X, x_0) let $[X, GL(n,A)]_0$ be the set of all pointed smooth homotopy classes of base point preserving smooth maps from X to the smooth group $GL(n,A)$, where we take the identity as the base point of $GL(n,A)$. The natural smooth homomorphisms $GL(n,A) \to GL(n+1,A)$ induce maps $[X, GL(n,A)]_0 \to [X, GL(n+1,A)]_0$ and we denote by $[X, GL(A)]_0$ the direct limit of the so obtained inductive system. Clearly the pointwise multiplication of smooth maps induces a group structure on $[X, GL(A)]_0$. Similarly for an arbitrary base space we define $[X, GL(A)]$, which is also a group. Then one proves:

THEOREM. *For any pointed smooth space (X, x_0) there is a natural isomorphism $\tilde{K}_A(SX) \cong [X, GL(A)]_0$. In particular $[X, GL(A)]_0$ is always an abelian group.*

An easy corollary of this result is that there is a natural isomorphism of bifunctors $K_A^{-1}(X) \cong [X, GL(A)]$. Moreover one can show that the iterated suspensions of the two point smooth space are smoothly homotopy equivalent to the spheres and thus we get: $K_n(A) := K_A^{-n}(pt) \cong [S^{n-1}, GL(A)]_0$.

2.12. So let us now turn to the case of a bounded algebra homomorphism $\varphi : A \to D$ between convenient algebras. For a fixed base space X we consider the functor $\varphi_* = \mathcal{E}_\varphi(X) : \mathcal{E}_A(X) \to \mathcal{E}_D(X)$ induced by φ. We want to give a homotopy interpretation of the K–group of this functor. The first part of this is parallel to the case of a smooth map so we only give a short outline: First we decompose the group into a direct sum as $K(\mathcal{E}_\varphi(X)) \cong K'(\mathcal{E}_\varphi(X)) \oplus \mathrm{Ker}(K_0(\varphi)_*)$, where $K_0(\varphi)_* : H^0(X, K_0(A)) \to H^0(X, K_0(D))$ is the group homomorphism induced by φ. Next for any $n > 0$ we consider the set $\Phi_n(\mathcal{E}_\varphi(X))$ of equivalence classes of pairs (E, α), where E is an A–bundle over X with fiber A^n and $\alpha : \varphi_*(E) \to X \times D^n$ is an isomorphism. Then one constructs connecting maps $\Phi_n(\mathcal{E}_\varphi(X)) \to \Phi_{n+1}(\mathcal{E}_\varphi(X))$ and the structure of a commutative monoid on the direct limit $\Phi(\mathcal{E}_\varphi(X))$. Finally one shows that this monoid is isomorphic to $K'(\mathcal{E}_\varphi(X))$.

2.13. To relate $\Phi(\mathcal{E}_\varphi(X))$ to homotopy theory we proceed as follows: For any $n > 0$ the algebra homomorphism φ induces a smooth homomorphism $\varphi_n : GL(n, A) \to GL(n, D)$ which in turn induces a smooth map $B(\varphi_n) : BGL(n, A) \to BGL(n, D)$. Let $\mathcal{F}_n(\varphi)$ be the homotopy fiber of this smooth map. Recall from 1.10 that $\mathcal{F}_n(\varphi)$ is defined by the pullback

$$\begin{array}{ccc} \mathcal{F}_n(\varphi) & \longrightarrow & P(BGL(n,D)) \\ \downarrow & & \downarrow \\ BGL(n,A) & \xrightarrow{B(\varphi_n)} & BGL(n,D) \end{array} .$$

where P denotes the path fibration. Note that the connecting maps $BGL(n, A) \to BGL(n + 1, A)$ and $BGL(n, D) \to BGL(n + 1, D)$ induce smooth maps $\mathcal{F}_n(\varphi) \to \mathcal{F}_{n+1}(\varphi)$. For a base space X we denote by $[X, \mathcal{F}(\varphi)]$ the direct limit of the inductive system of sets $[X, \mathcal{F}_n(\varphi)]$.

As before we restrict to pairs $(E, \alpha) \in \Phi_n(\mathcal{E}_\varphi(X))$ in which E is the associated bundle to the pullback of the universal $GL(n, A)$ bundle along some smooth map $g : X \to BGL(n, A)$, and α comes from a trivialization of the corresponding $GL(n, D)$ bundle. Similar as in 2.7 we construct from this data a smooth map from X to the total space of the universal $GL(n, D)$ bundle which projects to $B(\varphi_n) \circ g$. Again this map is null homotopic and we consider the map $X \to C^\infty(I, BGL(n, D))$ associated via cartesian closedness to the projection of a null homotopy. Then this map has values in the path space and thus together with g it induces a smooth map $X \to \mathcal{F}_n(\varphi)$, the homotopy class of which we assign to the pair (E, α). Then one proves:

2.14. THEOREM. *For any $n > 0$ the construction of 2.13 gives a well defined bijective map $u_n : \Phi_n(\mathcal{E}_\varphi(X)) \to [X, \mathcal{F}_n(\varphi)]$. Together these maps induce a bijection $u : \Phi(\mathcal{E}_\varphi(X)) \to [X, \mathcal{F}(\varphi)]$.*

2.15. The natural maps $\mathcal{F}_n(\varphi) \to BGL(n, A)$ induce a map $K(\mathcal{E}_\varphi(X)) \to K_A(X)$. On the other hand identifying $[S(X^+), BGL(n, D)]_0$ with $[X^+, \Omega(BGL(n, D))]_0 = [X, \Omega(BGL(n, D))]$ we see that the inclusions of the loop spaces into the path spaces induce a map $K_D^{-1}(X) \to K(\mathcal{E}_\varphi(X))$. It turns out that on the level of bundles these maps can be described as follows: The first one is induced by sending a triple (E, F, α) to the difference of the classes of E and F in $K_A(X)$. On the other hand identifying $K_D^{-1}(X)$ with $[X, GL(D)]$ and viewing a smooth map $f : X \to GL(n, D)$ as an automorphism α_f of the trivial bundle $X \times D^n$, the second map is induced by sending f to the triple $(X \times A^n, X \times A^n, \alpha_f)$. From this description it is obvious that the two maps are in fact group homomorphisms.

2.16. The relative K–group has obvious functorial properties. In particular for any n the unique smooth map $S^n(X^+) \to pt$ induces a group homomorphism $K(\mathcal{E}_\varphi(pt)) \to K(\mathcal{E}_\varphi(S^n(X^+)))$ and we define $K^{-n}(\mathcal{E}_\varphi(X))$ to be the cokernel of this homomorphism. From this definition one immediately concludes that for any n we get a group homomorphism $K^{-n}(\mathcal{E}_\varphi(X)) \to K_A^{-n}(X)$. On the other hand we get a group homomorphism $K_D^{-n-1}(X) \to K^{-n}(\mathcal{E}_\varphi(X))$ using the following result:

LEMMA. *Let X be a base space, D a convenient algebra. Then there is an exact sequence of abelian groups and group homomorphisms*

$$K_1(D) = K_D^{-1}(pt) \to \tilde{K}_D(S(X^+)) \to \tilde{K}_D(S(X)) \to 0$$

which is natural in X.

2.17. Next, using the Puppe sequence 1.16 for the map $B(\varphi_n)$, one shows that for any base space X one gets an exact sequence of abelian groups and group homomorphisms

$$\tilde{K}_A(S(X^+)) \to \tilde{K}_D(S(X^+)) \to K(\mathcal{E}_\varphi(X)) \to K_A(X) \to K_D(X)$$

which is natural in X. This is already the first part of the long exact sequence associated to the algebra homomorphism φ. Putting together this sequence with the one from 2.16 and the exact sequences defining the higher absolute and relative K–groups in a diagram one shows by a diagram chase that for any $n > 0$ there is an exact sequence of abelian groups and group homomorphisms

$$K_A^{-n-1}(X) \to K_D^{-n-1}(X) \to K^{-n}(\mathcal{E}_\varphi(X)) \to K_A^{-n}(X) \to K_D^{-n}(X).$$

Thus we get:

2.18. THEOREM (THE LONG EXACT SEQUENCE OF A BOUNDED ALGEBRA HOMOMORPHISM). *Let $\varphi : A \to D$ be a bounded homomorphism between convenient algebras. Then for any base space X there is a long exact sequence of abelian groups and group homomorphisms*

$$\begin{aligned} \dots \to K_D^{-n-1}(X) \to K^{-n}(\mathcal{E}_\varphi(X)) \to \\ \to K_A^{-n}(X) \xrightarrow{K_\varphi^{-n}(X)} K_D^{-n}(X) \to \dots \\ \to K_D^{-1}(X) \to K(\mathcal{E}_\varphi(X)) \to K_A^0(X) \xrightarrow{K_\varphi^0(X)} K_D^0(X) \end{aligned}$$

which is natural in X.

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Institut für Mathematik der Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria.
email: bitnet: cap@awirap
in: cap@pap.univie.ac.at