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GENERAL NIJENHUIS TENSOR AN EXAMPLE OF A SECONDARY INVARIANT

VÁCLAV STUDENÝ

ABSTRACT. All bilinear natural operators with Nijenhuis tensor domain an codomain which are secondary invariants are found.

1.1 This short note aims to demonstrate the power of the secondary invariants theory on the Nijenhuis tensor case. Why, whereas other problems of the invariants theory are an aim of extraordinary interest, secondary invariants, which are without question worthy to note, have not been systematically studied yet. The theme was opened by Jiří Vanžura and this paper is, as far as I know, first publication containing solution of this theory. What do we call a secondary invariant? Let us reason about a common domain: smooth left action of a Lie group on a smooth manifold[1], where a number of invariant mappings can be defined on. *A secondary invariant (with respect to invariant mapping f and invariant set A)* is a mapping which is invariant on the inverse image of A with respect to f and defined on some neighbourhood of this set.

If $Codom(f) \subset A$ then every secondary invariant with respect to f and A is an invariant in the conventional meaning. If f is a polynomial and A is a set of single point of some Euclidean space, then the secondary invariant with respect to f and A is an invariant whose area of invariance is *algebraic manifold* but a general case can be very unusual and not very easy to study.

1.2 Let us recall well known correspondence between natural operators and invariants of type fibers [2,3] and let us introduce some useful notations: X is a smooth manifold and T is the tangent functor. $F^r_Q X$ indicates the fibre bundle associated with the manifold of frames of order r over X whose type fibre is Q, as well as $T^r_n X$ indicates the manifold of r-jets with the source at origin of $\mathbb{R}^n$ and the target in X and $\mathcal{X}(X)$ denoted the set of all sections of $TX \longrightarrow X$. The set of all smooth sections of $F_Q(X)$ will be indicated by $C^\infty F_Q(X)$.

G^r_n is the r-th order differential group in the dimension n with canonical jets coordinates $(a^i_{j_1 \ldots j_p})_{u<v \implies j_u \le j_v}$ and with mappings $(b^i_{j_1 \ldots j_p})_{u<v \implies j_u \le j_v}$ which are the composition of coordinates with the group inversion.

Let us denote by $\otimes$ the usual tensor product and by $\odot$ the symmetric one.

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[0]This paper is in final form and no version of it will be submitted for publication elsewhere
[1]All manifolds here are supposed to have constant dimension indicated n if there is nothing different said explicitly and all mappings here are supposed to be smooth.

$A = \mathbb{R}^n \otimes \mathbb{R}^{n*}$ as well as $B = \mathbb{R}^n \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^{n*}$ are manifolds with the standard tensor action of G^1_n. The canonical coordinates on $T^r_n A$ will be denoted by $(\alpha^i_j, \alpha^i_{jk_1}, \alpha^i_{jk_1k_2}, \dots)_{p<q \Longrightarrow k_p \le k_q}$ in the obvious sense. The type fibre of linear symmetric connections will be denoted $Q = \mathbb{R}^n \otimes \mathbb{R}^{n*} \odot \mathbb{R}^{n*}$ with the standard action. The coordinates on it are denoted by (Γ^i_{jk}).

$[-,-] : \mathcal{X}(X) \times \mathcal{X}(X) \longrightarrow \mathcal{X}(X)$ is the Lie bracket,

$$[-,-] : \begin{cases} C^\infty F_A X \times C^\infty F_A X & \longrightarrow & C^\infty F_A X \\ (\alpha,\beta) & \longmapsto & [\alpha,\beta] \end{cases} .$$

$$\text{where } [\alpha,\beta](\xi) = \alpha(\beta(\xi)) - \beta(\alpha(\xi))$$

is the commutator and the mapping $[-,-] : A \times A \longrightarrow A$ is induced by the preceding mapping in the case where X degenerates to a point and is called a commutator, too.

1.3 Let α and β be two tensor fields of type $(1,1)$ on manifold X.

$$\{\alpha,\beta\} : \begin{cases} \mathcal{X}(X) \times \mathcal{X}(X) & \longrightarrow & \mathcal{X}(X) \\ (\xi,\zeta) & \longmapsto & [\alpha(\xi),\beta(\zeta)] + \alpha\circ\beta([\xi,\zeta]) - \\ & & \alpha([\xi,\beta(\zeta)]) - \beta([\alpha(\xi),\zeta]) + \\ & & [\beta(\xi),\alpha(\zeta)] + \beta\circ\alpha([\xi,\zeta]) - \\ & & \beta([\xi,\alpha(\zeta)]) - \alpha([\beta(\xi),\zeta]) \end{cases}$$

is a tensor field of type (2,1) whose construction was discovered by A. Nijenhuis[2].

The mapping

$$< \alpha,\beta >: \begin{cases} \mathcal{X}(X) \times \mathcal{X}(X) & \longrightarrow & \mathcal{X}(X) \\ (\xi,\zeta) & \longmapsto & [\alpha(\xi),\beta(\zeta)] + \alpha\circ\beta([\xi,\zeta]) \\ & & - \alpha([\xi,\beta(\zeta)]) - \beta([\alpha(\xi),\zeta]) \end{cases}$$

is additive in each component but not homogeneous — in general. Nevertheless, if $\alpha\circ\beta = \beta\circ\alpha$ we have (using formula $[f\cdot\xi, g\cdot\zeta] = f\cdot g\cdot[\xi,\zeta] + f\cdot\xi(g)\cdot\zeta - g\cdot\zeta(f)\cdot\xi$, where $\xi(g)$ is derivative of g along ξ)

$$\begin{aligned} \langle\alpha,\beta\rangle(f\cdot\xi, g\cdot\zeta) = {} & [f\cdot\alpha(\xi), g\cdot\beta(\zeta)] + \\ & \alpha\circ\beta\big([f\cdot\xi, g\cdot\zeta]\big) - \alpha\big([f\cdot\xi, g\cdot\beta(\zeta)]\big) - \beta\big([f\cdot\alpha(\xi), g\cdot\zeta]\big) = \\ & f\cdot g\Big([\alpha(\xi),\beta(\zeta)] + \alpha\circ\beta([\xi,\zeta]) - \alpha\big([\xi,\beta(\zeta)]\big) - \beta\big([\alpha(\xi),\zeta]\big)\Big) + \\ & f\cdot\big(\alpha(\xi)(g)\cdot\beta(\zeta) + \xi(g)\cdot\alpha\circ\beta(\zeta) - \xi(g)\cdot\alpha\big(\beta(\zeta)\big) - \alpha(\xi)(g)\beta(\cdot\zeta)\big) - \\ & - g\cdot\big(\beta(\zeta)(f)\cdot\alpha(\xi) + \zeta(f)\cdot\alpha\circ\beta(\xi) - \beta(\zeta)(f)\alpha(\cdot\xi)\zeta(f)\cdot\beta\big(\alpha(\xi)\big) - \big) \end{aligned}$$

[2] Albert Nijenhuis X_{n-1}-forming sets of eigenvectors — Indag. Math., 1951, **13**, p. 200 - 212.

but $\zeta(f)\cdot\alpha\circ\beta(\xi)=\zeta(f)\cdot\beta(\alpha(\xi))-$ consequently, the last line vanishes as well as the penultimate line does. That is why the mapping

$$\langle -,-\rangle : \begin{cases} C^\infty F_A X \times C^\infty F_A X & \longrightarrow \quad C^\infty F_B X \\ \quad (\alpha,\beta) & \longmapsto \quad \langle\alpha,\beta\rangle \end{cases},$$
$$\text{where } \langle\alpha,\beta\rangle(\xi,\zeta)=[\alpha(\xi),\beta(\zeta)]+\alpha\circ\beta([\xi,\zeta])- \\ \alpha([\xi,\beta(\zeta)])-\beta([\alpha(\xi),\zeta])$$

is a tensor field of type (1,2) (known as Nijenhuis tensor) only if $(\alpha,\beta)\in ker([-,-])$, just as the mapping $T^r_n A\times T^r_n A\longrightarrow B$ with a coordinate expression $\alpha^p_j\beta^i_{kp}-\alpha^i_p\beta^p_{kj}-\alpha^i_{jp}\beta^p_k+\alpha^p_{jk}\beta^i_p$ is a secondary invariant with respect to the commutator and $\{0\in\mathbb{R}^n\}\subset Codom[-,-]$, but it is not an invariant. Moreover, there is a circumstance which is worthy to note: $[-,-]^{-1}(0)$ is not a submanifold of $T^r_n A\times T^r_n A$, because it is not a manifold at all!

2.0 But even if $[-,-]^{-1}(0)$ is not a manifold, this set contains a manifold which is dense in it.

Let us fix some base of $A\times A$. Every element of $A\times A$ can be represented as a pair of matrixes.

2.1 Lemma. *Let $\mathcal{A}=\{(X,Y)\in[-,-]^{-1}(0);\ det(X-\lambda E)$ has n different complex roots, $E=(\delta^i_j)\}$ be a set of pairs of matrixes. Then $\mathcal{A}$ is a manifold of dimension $n(n-1)$ and $\mathcal{A}$ is dense in $[-,-]^{-1}(0)$.*

Proof. Let $Mat(n)$ denote the set of all square matrices of order n, and let GCD denote *the greatest common divisor* of polynomials. The mapping $X=(x^i_j)\longmapsto det(X-\lambda E)$ is a polynomial in x^i_j as well as the mapping which maps coefficients of polynomial of fixed degree to coefficients of its derivative is. Two polynomials have a common divisor if and only if their resultant, which is a determinant and consequently a polynomial in its coefficients, vanishes and the polynomial has a multiple root if and only if it has a common divisor with its derivative; consequently the set $\mathcal{B}=\{X; det(X-\lambda E)$ has n different complex roots $\}$ is a dense open submanifold of $Mat(n)$, because it is a set of points at which some polynomials have nonzero value.

Let us reason about the set $\mathcal{A}=\{(X,Z);\ X\in\mathcal{B}, Z$ is a value of some polynomial on $X\}$ and about the well known mappings

$$\phi_0=\widehat{1}:x\longmapsto 1$$
$$\phi_i:\begin{cases} Mat(n) & \longrightarrow \quad Polynomials \\ \quad X & \longmapsto \quad GCD\{det(Z-\lambda X); \\ & \qquad Z\in Mat(i) \text{ is a minor of } X\}\end{cases}$$
$$\psi_{n-i+1}=\frac{\phi_i}{\phi_{i-1}}$$

degree$(\psi_1|_{\mathcal{B}})$ is a constant mapping of value n, but this implies (see for instance: Felix Rubinovitsch Gantmacher: Teorija matric, Moskva, 1988) that any matrix in $\mathcal{B}$ commutes only with the values of polynomials in itself and that the number of linearly independent matrixes the matrix in question commutes with is n.

That is why the mapping

$$F : \begin{cases} \mathcal{A} & \longrightarrow & \mathbb{R}^{n(n+1)} \\ (X = (x^i_j)^n_{i,j=1}, Y & \longmapsto & (x^i_j, y_0, y_1, \dots, y_{n-1}) \\ & & \text{where } Y = y_0 X^0 + y_1 X^1 + \dots + y_{n-1} X^{n-1} \end{cases}$$

provides global coordinates on $\mathcal{A}$ □

Now our problem can be reformulated in this way: we want to find all *bilinear* natural operators which transform pairs of tensor fields of typ $(1,1)$ corresponding mapping between type fiber (see 1.2) of which satisfy the assumption of lemma 2.1, and which can be extended to a smooth mapping defined on some neighbourhood of $[-,-]^{-1}(0)$, to a tensor field of type $(1,2)$ on the same manifold. (The notion of naturality is used here in more general but obvious meaning.)

3.1 So, our problem is transformed to a standard one, and we can use the standard method to solve it. We first compute the order of our operator. The nonlinear Petree theorem (see [2] page 179) shows us that it is finite, say of order r. The proof is quite technical and so we omit it here.

The finiteness of order and the knowledge of the general theory (see [2] or [3]) and its generalization for our case permits to reformulate the task to the following one: describe all secondary invariant mappings $T^r_n A \times T^r_n A \longrightarrow B$ with respect to the commutator and $\{0 \in \mathbb{R}^n\}$.

Let us denote one of these mappings by $f = (f^i_{jk})$ and let ι denote a multiindex $0 \leq |\iota| \leq r$. To be invariant with respect to the action of homotheties means to fulfill the equation

$$f^i_{jk}(K^{|\iota|}\alpha^i_{j\iota}, K^{|\iota|}\beta^i_{j\iota})_{0 \leq |\iota| \leq r} = K f^i_{jk}(\alpha^i_{j\iota}, \beta^i_{j\iota}).$$

Differentiating this equation with respect to k we obtain

$$f^i_{jk}(\alpha^l_{m\iota}, \beta^l_{m\iota}) = \sum_\iota \frac{\partial f^i_{jk}}{\partial \alpha^p_{q\iota}} |\iota| K^{|\iota|-1} \alpha^p_{q\iota} + \frac{\partial f^i_{jk}}{\partial \beta^p_{q\iota}} |\iota| K^{|\iota|-1} \beta^p_{q\iota}$$

(for $\iota = 0$ the expression is zero). Computing the limit $k \longrightarrow 0$ of the last equation we can see:

(1) f is of order 1.
(2) f is linear in (α^i_{jk}) and (β^i_{jk})

Now, we are going to describe all the bilinear secondary invariant mappings $C^\infty F_A X \times C^\infty F_A X \longrightarrow C^\infty F_B X$ with respect to the commutator and $\{0 \in \mathbb{R}^n\}$ (which do not depend on the choice of X).

If the dimension of X is 1, there are of course only linear combinations of $\alpha^1_{11}\beta^1_1$ and $\alpha^1_1\beta^1_{11}$. We will compute all bilinear invariants in higher dimensions.

3.2 Our way will be demonstrated by the following diagram, in which ∇ is the mapping replacing any derivative by a covariant one and pr_i are cartesian projections.

$$
\begin{tikzcd}
Q \times \left(A \oplus (A \otimes \mathbb{R}^{n*})\right)^2 \arrow[rr, "\phi"] & & B \arrow[dd, equal] \\
& Q \times (T^1_n A \times T^1_n A) \arrow[ul, "id_Q \times \nabla"', "\cong"] \arrow[dr, "\varphi"] \arrow[d, "pr_2"] & \\
& T^1_n A \times T^1_n A \arrow[r, "f"] & B \\
& T^1_n A \times T^1_n A \arrow[u, "id"] \arrow[ur, "f"'] \arrow[uu, bend left=60, "s"] &
\end{tikzcd}
$$

$A \oplus (A \otimes \mathbb{R}^{n*})$ with canonical coordinates $(\alpha^i_j, \alpha^i_{j;k}, \dots, \beta^i_j, \dots)_{j \le k, \dots}$ is endowed with the tensor action of G^r_n. The second power is the cartesian one.

We will describe all secondary invariants ϕ. Composing them with the isomorphism $id_Q \times \nabla$ — which is invariant, too — we find all secondary invariants φ. After that we find all mappings f for which there exists φ such that the diagram commutes for any section s. In this way we obtain every secondary invariant we are looking for; since pr_2 is invariant, for any invariant f the mapping $\varphi = f \circ pr_2$ is invariant and $f \circ pr_2 \circ (id_Q \times \nabla)^{-1}$ is invariant and constant on every fibre of $(id_Q \times \nabla)^{-1} \circ pr_2 : Q \times (A \oplus (A \otimes \mathbb{R}^{n*})) \longrightarrow (T^1_n A \times T^1_n A)$. Consequently there exist invariant ϕ with the same expression.

3.3 After differentiating the action we can see that ϕ does not depend on Γ^i_{jk}. The requirement of bilinearity simplifies our problem. It admits only mappings ϕ of the form

$$
\begin{aligned}
\phi^a_{bc} = A^{aij}_{bcpq} \alpha^i_p \beta^q_j + B^{aijkl}_{bcpq} \alpha^p_{i;k} \beta^q_{j;l} + {} \\
C^{aijk}_{pcpq} \alpha^p_i \beta^q_{j;k} + D^{aijk}_{pcpq} \alpha^p_{i;k} \beta^q_j + \dots \\
\text{where } A^{aij}_{bcpq} = A^{aji}_{bcqp}, \ \dots
\end{aligned}
$$

A direct computation shows us (one can set $\alpha^i_j = 1$ if $(i,j) = (p,q)$ else $\alpha^i_j = 0$, $\beta^i_j = 0$ and $\beta^i_{jk} = 1$ if $(i,j,k) = (t,u,v)$ else $\beta^i_{jk} = 0, \dots$) that the coefficients are components of an absolutely invariant tensor. But these are described completely (see [3] page 68 Theorem 4.1).

Consequently, we find that all invariant mappings ϕ have the following coordinate form:

$$
\begin{gathered}
c_1\alpha^a_b\beta^p_{c;p} + c_2\alpha^a_c\beta^p_{b;p} + c_3\alpha^p_b\beta^a_{c;p} + c_4\alpha^p_c\beta^a_{b;p} + c_5\alpha^a_b\beta^p_{p;c} \\
+c_6\alpha^a_c\beta^p_{p;b} + c_7\alpha^p_b\beta^a_{p;c} + c_8\alpha^p_c\beta^a_{p;b} + c_9\alpha^a_p\beta^p_{b;c} + c_{10}\alpha^a_p\beta^p_{c;b} \\
+c_{11}\alpha^p_p\beta^a_{b;c} + c_{12}\alpha^p_p\beta^a_{c;b} + c_{13}\delta^a_b\alpha^p_c\beta^q_{p;q} + c_{14}\delta^a_b\alpha^p_c\beta^q_{q;p} \\
+c_{15}\delta^a_b\alpha^p_p\beta^q_{c;q} + c_{16}\delta^a_b\alpha^p_p\beta^q_{q;c} + c_{17}\delta^a_b\alpha^p_q\beta^q_{c;p} + c_{18}\delta^a_b\alpha^p_q\beta^q_{p;c} \\
+c_{19}\delta^a_c\alpha^p_b\beta^q_{p;q} + c_{20}\delta^a_c\alpha^p_b\beta^q_{q;p} + c_{21}\delta^a_c\alpha^p_p\beta^q_{b;q} + c_{22}\delta^a_c\alpha^p_p\beta^q_{q;b} \\
+c_{23}\delta^a_c\alpha^p_q\beta^q_{b;p} + c_{24}\delta^a_c\alpha^p_q\beta^q_{p;b} + d_1\alpha^p_{c;p}\beta^a_b + d_2\alpha^p_{b;p}\beta^a_c \\
+d_3\alpha^a_{c;p}\beta^p_b + d_4\alpha^a_{b;p}\beta^p_c + d_5\alpha^p_{p;c}\beta^a_b + d_6\alpha^p_{p;b}\beta^a_c + d_7\alpha^a_{p;c}\beta^p_b \\
+d_8\alpha^a_{p;b}\beta^p_c + d_9\alpha^p_{b;c}\beta^a_p + d_{10}\alpha^p_{c;b}\beta^a_p + d_{11}\alpha^a_{b;c}\beta^p_p + d_{12}\alpha^a_{c;b}\beta^p_p \\
+d_{13}\delta^a_b\alpha^q_{p;q}\beta^p_c + d_{14}\delta^a_b\alpha^q_{q;p}\beta^p_c + d_{15}\delta^a_b\alpha^q_{c;q}\beta^p_p + d_{16}\delta^a_b\alpha^q_{q;c}\beta^p_p \\
+d_{17}\delta^a_b\alpha^q_{c;p}\beta^p_q + d_{18}\delta^a_b\alpha^q_{p;c}\beta^p_q + d_{19}\delta^a_c\alpha^q_{p;q}\beta^p_b + d_{20}\delta^a_c\alpha^q_{q;p}\beta^p_b \\
+d_{21}\delta^a_c\alpha^q_{b;q}\beta^p_p + d_{22}\delta^a_c\alpha^q_{q;b}\beta^p_p + d_{23}\delta^a_c\alpha^q_{b;p}\beta^p_q + d_{24}\delta^a_c\alpha^q_{p;b}\beta^p_q
\end{gathered}
$$

To obtain every secondary invariant φ is as easy as to compose ϕ with the mentioned isomorphism: in coordinates it means, of course: $\alpha^i_{j;k} = \alpha^i_{jk} + \Gamma^i_{sk}\alpha^s_j - \Gamma^s_{kj}\alpha^i_s$.

The coordinate expression for φ^a_{bc} can be written in the form $\varphi^a_{bc} = P^a_{bc} + \Gamma^i_{jk}K^{ajk}_{bci}$ where P^a_{bc} and K^{ajk}_{bci} are linear polynomials in the variables $c_1, \dots, c_{24}, d_1, \dots, d_{24}$, which do not depend on Γ^i_{jk}.

To find all secondary invariants f means to solve the system of linear equations $K^{ajk}_{bci} - K^{akj}_{bci} = 0$ in the case when equations $\alpha^i_j\beta^j_k = \beta^i_j\alpha^j_k$, $\alpha^i_{jl}\beta^j_k + \alpha^i_j\beta^j_{kl} = \beta^i_{jl}\alpha^j_k + \beta^i_j\alpha^j_{kl}$ hold and when the resultant of $det((\alpha^i_j - \lambda\delta^i_j)i,j)$ and $\frac{\partial}{\partial\lambda}\big(det((\alpha^i_j - \lambda\delta^i_j)i,j)\big)$ is not zero. This equations look as follows:

$$
\begin{gathered}
c_1\alpha^a_b\beta^w_c\delta^p_u\delta^v_p - c_1\alpha^a_b\beta^p_u\delta^v_c\delta^w_p + c_1\alpha^a_b\beta^v_c\delta^p_u\delta^w_p - c_1\alpha^a_b\beta^p_u\delta^w_c\delta^v_p \\
+ c_2\alpha^a_c\beta^w_b\delta^p_u\delta^v_p - c_2\alpha^a_c\beta^p_u\delta^v_b\delta^w_p + c_2\alpha^a_c\beta^v_b\delta^p_u\delta^w_p - c_2\alpha^a_c\beta^p_u\delta^w_b\delta^v_p \\
+ c_3\alpha^p_b\beta^w_c\delta^a_u\delta^v_p - c_3\alpha^p_b\beta^a_u\delta^v_c\delta^w_p + c_3\alpha^p_b\beta^v_c\delta^a_u\delta^w_p - c_3\alpha^p_b\beta^a_u\delta^w_c\delta^v_p
\end{gathered}
$$

$+\dots$

$$
\begin{gathered}
+ \delta^a_c(c_{23}\alpha^p_q\beta^w_b\delta^q_u\delta^v_p - c_{23}\alpha^p_q\beta^q_u\delta^v_b\delta^w_p + c_{23}\alpha^p_q\beta^v_b\delta^q_u\delta^w_p - c_{23}\alpha^p_q\beta^q_u\delta^w_b\delta^v_p) \\
+ \delta^a_c(c_{24}\alpha^p_q\beta^w_p\delta^q_u\delta^v_b - c_{24}\alpha^p_q\beta^q_u\delta^v_p\delta^w_b + c_{24}\alpha^p_q\beta^v_p\delta^q_u\delta^w_b - c_{24}\alpha^p_q\beta^q_u\delta^w_p\delta^v_b) \\
+ d_1\alpha^w_c\beta^a_b\delta^p_u\delta^v_p - d_1\alpha^p_u\beta^a_b\delta^v_c\delta^w_p + d_1\alpha^v_c\beta^a_b\delta^p_u\delta^w_p - d_1\alpha^p_u\beta^a_b\delta^w_c\delta^v_p \\
+ d_2\alpha^w_b\beta^a_c\delta^p_u\delta^v_p - d_2\alpha^p_u\beta^a_c\delta^v_b\delta^w_p + d_2\alpha^v_b\beta^a_c\delta^p_u\delta^w_p - d_2\alpha^p_u\beta^a_c\delta^w_b\delta^v_p
\end{gathered}
$$

$+\dots$

$$
+ \delta^a_c(d_{24}\alpha^w_p\beta^p_q\delta^q_u\delta^v_b - d_{24}\alpha^q_u\beta^p_q\delta^v_p\delta^w_b + d_{24}\alpha^v_p\beta^p_q\delta^q_u\delta^w_b - d_{24}\alpha^q_u\beta^p_q\delta^w_p\delta^v_b)
$$

$$= 0$$

The general solution is:

$$
\begin{aligned}
c_1 = c_2 = c_7 = c_8 = c_{11} = c_{12} = \\
= c_{13} = c_{15} = c_{17} = c_{19} = c_{21} = c_{23} = \\
= d_1 = d_2 = d_7 = d_8 = d_{11} = d_{12} = \\
= d_{13} = d_{15} = d_{17} = d_{19} = d_{21} = d_{23} = 0 \\
c_3 = d_9 = -d_4 = -c_{10} \\
d_3 = c_9 = -c_4 = -d_{10}
\end{aligned}
$$

$c_5,\ c_6,\ c_{14},\ c_{16},\ c_{18},\ c_{20},\ c_{22},\ c_{24},$

$d_5,\ d_6,\ d_{14},\ d_{16},\ d_{18},\ d_{20},\ d_{22},\ d_{24},$ are arbitrary.

We can summarize:

3.4 Theorem. *Any bilinear secondary invariant mapping*

$$T^1_n A \times T^1_n A \longrightarrow B$$

with respect to the commutator and $\{0 \in \mathbb{R}^n\}$ *can be obtained as a linear combination of the following secondary invariants.*

(1) *Nijenhuis tensors:* $\eta^p_j \theta^a_{kp} - \eta^a_p \theta^p_{kj} - \eta^a_{jp} \theta^p_k + \eta^p_{jk} \theta^a_p$, $\{j,k\} = \{b,c\}$, $\{\eta,\theta\} = \{\alpha,\beta\}$ *which are not invariants;*

(2) $\delta^a_j \eta^p_q \theta^q_{pk}$, $\{j,k\} = \{b,c\}$, $\{\eta,\theta\} = \{\alpha,\beta\}$ *which are not invariants, too;*

(3) $\delta^i_j \eta^l_k \theta^q_{qr}$, $\{i,l\} = \{a,p\}$, $\{j,k,r\} = \{b,c,p\}$, $\{\eta,\theta\} = \{\alpha,\beta\}$.

Consequently, they create a vector space, the base of which is

$$
\begin{gathered}
\alpha^p_b \beta^a_{cp} - \alpha^a_p \beta^p_{cb} - \alpha^a_{bp} \beta^p_c + \alpha^p_{bc} \beta^a_p, \\
\alpha^a_{cp} \beta^p_b - \alpha^p_{cb} \beta^a_p - \alpha^p_c \beta^a_{bp} + \alpha^a_p \beta^p_{bc}, \\
\delta^a_b \alpha^p_q \beta^q_{pc}, \quad \delta^a_b \alpha^q_{pc} \beta^p_q, \quad \delta^a_c \alpha^p_q \beta^q_{pb}, \quad \delta^a_c \alpha^q_{pb} \beta^p_q, \\
\alpha^a_b \beta^p_{pc}, \quad \alpha^a_c \beta^p_{pb}, \quad \alpha^p_{pc} \beta^a_b, \quad \alpha^p_{pb} \beta^a_c, \\
\delta^a_b \alpha^p_c \beta^q_{qp}, \quad \delta^a_b \alpha^p_p \beta^q_{qc}, \quad \delta^a_c \alpha^p_b \beta^q_{qp}, \quad \delta^a_c \alpha^p_p \beta^q_{qb}, \\
\delta^a_b \alpha^q_{qp} \beta^p_c, \quad \delta^a_b \alpha^q_{qc} \beta^p_p, \quad \delta^a_c \alpha^q_{qp} \beta^p_b, \quad \delta^a_c \alpha^q_{qb} \beta^p_p,
\end{gathered}
$$

3.5 To formulate our theorem in terms of operators we introduce some useful notations.

Let $N = (N^i_{jk}) : C^\infty F^2_A X \longrightarrow C^\infty F_B X$ denote the value of the Nijenhuis operator (Nijenhuis tensor) on the manifold X, D the differential, and C the contraction $\big(C(N) = (N^i_{ij}),\ C((\alpha^j_i)) = (\alpha^i_i)\big)$. pr_i are the cartesian projections and $\widetilde{id}$ is the section with value id in every point. ex is the exchange on a cartesian product.

The value of operators on fields $\alpha = (\alpha^i_j, \dots)$ and $\beta = (\beta^i_j, \dots)$ which in coordinates look like 3.4.1 can be described as

$$N(\alpha,\beta) \text{ or } N(\beta,\alpha)$$

The value of operators on fields $\alpha = (\alpha^i_j, \dots)$ and $\beta = (\beta^i_j, \dots)$ which in coordinates look like 3.4.3 can be described as

$$\widetilde{id} \otimes D \circ C(\alpha), \quad D \circ C(\alpha) \otimes \widetilde{id}, \quad \widetilde{id} \otimes D \circ C(\beta), \quad D \circ C(\beta) \otimes \widetilde{id}$$

multiplying by $C(\alpha)$

$$\beta \otimes (D \circ C(\alpha)), \quad (D \circ C(\alpha)) \otimes \beta, \quad \alpha \otimes (D \circ C(\beta)), \quad (D \circ C(\beta)) \otimes \alpha$$
$$\widetilde{id} \otimes \beta(D \circ C(\alpha)), \quad \beta(D \circ C(\alpha)) \otimes \widetilde{id}, \quad \widetilde{id} \otimes \alpha(D \circ C(\beta)), \quad \alpha(D \circ C(\beta)) \otimes \widetilde{id}$$

As regards 3.4.2 we note that $C \circ N(\alpha, \beta) + \beta(D \circ C(\alpha)) = (\beta^j_k \alpha^k_{ji}) \dots$ consequently the value of operators on fields $\alpha = (\alpha^i_j, \dots)$ and $\beta = (\beta^i_j, \dots)$ which in coordinates look like 3.4.2 can be replaced by

$$\widetilde{id} \otimes C \circ N(\alpha, \beta), \quad C \circ N(\alpha, \beta) \otimes \widetilde{id}, \quad \widetilde{id} \otimes C \circ N(\beta, \alpha), \quad C \circ N(\beta, \alpha) \otimes \widetilde{id}.$$

We can finish with a theorem:

3.6 Corollary. *The only bilinear natural operators transforming commuting pairs of tensor fields of type* $(1,1)$ *to fields of type* $(2,1)$ *on the same manifold are*

$$\begin{aligned} F_1 &= N \\ F_2 &= \widetilde{id} \otimes D \circ C(pr_1(-)) \cdot C(pr_2(-)) \\ F_3 &= pr_2(-) \otimes (D \circ C(pr_1(-))) \\ F_4 &= \widetilde{id} \otimes pr_2(-) \circ (D \circ C(pr_1(-))) \\ F_5 &= \widetilde{id} \otimes C \circ N(pr_1(-), pr_2(-)) \end{aligned}$$

the operators

$$F_i \circ ex$$
$$EX(F_i)$$
$$EX(F_i \circ ex)$$

where $EX(F)(x) = F(x) \circ ex$*, and* ex *is transposition on cartesian product.*

(but let us note that for i=1 we obtain only two linearly independent mappings, because $N(\alpha, \beta) \circ ex = -N(\beta, \alpha)$*), and the linear combinations of the previous 18 linearly independent operators.*

Let us note that the only secondary invariants which are not invariant are values of natural operations on Nienhuis tensor.

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