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EIGENVALUES OF CONFORMALLY INVARIANT OPERATORS ON SPHERES*

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1 Introduction

Let M be a smooth oriented compact n-dimensional manifold, $n \geq 3$, endowed with a Riemannian metric and a spin structure. A huge amount of information has been collected concerning spectral properties of the basic invariant differential operators on M. Spectra of the Laplace and the Dirac operators has been computed explicitely on many examples of homogeneous spaces ([B1, B2, BGM, CaH, CFG, Mi, MS, Sa]). Estimates concerning the first eigenvalues and relation to geometry of M has been studied in many papers ([Fri1, Hi, Kir, KSW, Li2, Lo, Su]) In a general case, exact formulae for eigenvalues are not available but their asymptotic behaviour is a classical subject studied for a long time already ([DF, Ga]).

Recently, a growing interest is paid to properties of more complicated invariant first order differential operators on M. A prototype of them is the Rarita-Schwinger operator (see [Fra1, Fra2, FraS, MP, N, NGRN, Pe1, Pe2, Pe3, Pe4, Pe5, RaS, US, Wa]). It acts on sections of the bundle associated to a more complicated representation of the group Spin(n). In the paper, we are going to study spectral properties of a certain class of differential operators on M which has been intensively used in Clifford analysis in connection with monogenic differential forms (see [DSS, Ry1, So1, So2, SoS, So3]). The aim of the paper is to compute explicitely spectra of this class of conformally invariant operators on the flat model, i.e. on spheres.

As for the Dirac and the Laplace operators, methods of representation theory can be used in homogeneous case. The main tool used in the paper are general results of Branson, Ólafsson and Ørsted (see [BOO]) describing a construction of intertwinning operators between principal series representations of semisimple Lie groups. They are able to compute spectra of a wide class of invariant operators up to a normalisation, i.e. they are giving explicit formulae for ratios of eigenvalues. These formulae can be used directly in odd dimensions. In even dimensions, differential operators studied here are not covered by the results in [BOO], nevertheless the methods used there can

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be adapted for our purpose (see Sect.3).

The symbol of the Dirac operator is given by the Clifford multiplication. Hence the question of normalisation is answered here by a choice of the Clifford action. For higher spin representations and the associated invariant operators, the question of normalisation of the studied operators is first to be settled (see Sect.2). To compute exact formulae for spectra, it is then sufficient to find explicitely one eigenvalue. It is done in Sect.4 using methods developed in [VSe].

After submitting the paper to press, we have learnt that a general procedure how to compute spectra of invariant first order operators was described in [Br]. The normalization of operators involved is different from ours and the scale of spectra is computed as a result of a general scheme worked for all operators. For cases, discussed in our paper, we use simpler tools for computing the scale. The strong irreducibility result proved in [Br] is not needed in our cases.

2 First order conformally invariant operators

A classification and a description of first order conformally invariant differential operators was first described by Fegan in ([F]). There is a standard definition of an invariant (homogeneous) operator on homogeneous spaces but there are several different definitions of conformally invariant operators in a curved case (for details see a [BE, S, CSS1, CSS2]). A construction of curved analogues of invariant operators is a difficult task which is not yet completely understood (see [BE, GJM]). For first order operators, however, there are no additional complications in the curved case with respect to the homogenous model. A general scheme for a construction of such invariant operators is as follows (see [F]).

Let M be a compact oriented manifold with a conformal structure. Let us choose a Riemannian metric in the given conformal class and suppose that a spin structure is given on M, i.e. that we have principal fibre bundles

$$\tilde{\mathcal{P}} \equiv \tilde{\mathcal{P}}_{Spin} \to \mathcal{P}_{SO} \to M.$$

on the manifold M.

Finite-dimensional irreducible representations $\mathbf{V}_\lambda$ of the group $H = \mathrm{Spin}(n)$ are classified by their highest weights $\lambda \in \Lambda^+$, where for $n = 2k$ even, we have

$$\Lambda^+ = \{\lambda = (\lambda_1, ..., \lambda_k); \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{k-1} \geq |\lambda_k|\}, \lambda_i \in \mathbf{Z} \cup \frac{1}{2}\mathbf{Z}$$

and for $n = 2k + 1$ odd, we have

$$\Lambda^+ = \{\lambda = (\lambda_1, ..., \lambda_k); \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{k-1} \geq \lambda_k \geq 0\}, \lambda_i \in \mathbf{Z} \cup \frac{1}{2}\mathbf{Z}.$$

Invariant operators are acting among spaces of sections of the corresponding associated bundles

$$V_\lambda = \tilde{\mathcal{P}} \times_H \mathbf{V}_\lambda$$

over M. Let us consider the Levi-Civita connection ω of the chosen Riemannian metric on $\mathcal{P}$ and let $\tilde{\omega}$ be its (unique) lift to $\tilde{\mathcal{P}}$. For any choice of $\lambda \in \Lambda^+$, we have the associated covariant derivative

$$\nabla_\lambda : \Gamma(V_\lambda) \to \Gamma(V_\lambda \otimes T^*(M)).$$

There are standard algorithms (see [Sal]) for a decomposition of the tensor product $\mathbf{V}_\lambda \otimes \mathbf{C}_n$ into irreducible components

$$\mathbf{V}_\lambda \otimes \mathbf{C}_n = \oplus_{\lambda' \in A} \mathbf{V}_{\lambda'},$$

where A is the set of highest weights of all irreducible components (multiplicities included). There are simple rules how to describe $A = A(\lambda)$ explicitely for any λ (see [F, S]). Let $\pi_{\lambda'}$ be the projection from $\mathbf{V}_\lambda \otimes \mathbf{C}_n$ to $\mathbf{V}_{\lambda'}$. Then operators

$$D_{\lambda,\lambda'} : \Gamma(V_\lambda) \to \Gamma(V_{\lambda'}),\ D_{\lambda,\lambda'} := \pi_{\lambda'} \circ \nabla^\lambda$$

are first order conformally invariant differential operators and all such operators can be constructed in this way.

Any conformally invariant first order differential operator is uniquely determined (up to a constant multiple) by a choice of allowed λ and λ' but there is no natural normalization in general. To study spectral properties, it is necessary to remove this ambiguity and to fix a scale of the operator, to choose appropriate normalization. For the Dirac operator, the choice of normalization is given by the Clifford action. By using twisted Dirac operators, we shall extend this normalization to a wide class of first order operators.

Definition 1 *Let* $\mathbf{S}$ *(for* $n = 2k+1$*), resp.* $\mathbf{S} = \mathbf{S}^+ \oplus \mathbf{S}^-$ *(for* $n = 2k$*), denote the basic spinor representations with highest weights* $\sigma = (\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2})$*, resp.* $\sigma^\pm = (\frac{1}{2}, \ldots, \frac{1}{2}, \pm\frac{1}{2})$*.*

Let $\lambda \in \Lambda^+$*, (for* $n = 2k+1$*), resp:* $\lambda^\pm \in \Lambda^+$ *(for* $n = 2k$*) be dominant weights with* $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, \frac{1}{2})$*, resp.* $\lambda^\pm = (\lambda_1, \ldots, \lambda_{k-1}, \pm\frac{1}{2})$*. Denote further* $\lambda' = \lambda - \sigma \in \Lambda^+$*, resp.* $\lambda' = \lambda^+ - \sigma^+ \in \Lambda^+$*. In even dimensions, we shall use the notation*

$$\mathbf{V}_\lambda = \mathbf{V}_{\lambda^+} \oplus \mathbf{V}_{\lambda^-}.$$

The representation $\mathbf{V}_\lambda$ *appears with multiplicity one in the decomposition of the tensor product* $\mathbf{S} \otimes \mathbf{V}_{\lambda'}$ *(it is the Cartan product of both representations). Hence we can write the product as*

$$\mathbf{S} \otimes \mathbf{V}_{\lambda'} = \mathbf{V}_\lambda \oplus \mathbf{W},$$

where $\mathbf{W}$ *is the sum of all other irreducible components in the decomposition.*

Let $D^T_{\lambda'}$ *be the twisted Dirac operator on* $S \otimes V_{\lambda'}$*. If we write the operator* $D^T_{\lambda'}$ *in the block form as*

$$\begin{array}{ccc}
\Gamma(S\otimes V_{\lambda'}) & \xrightarrow{D^T_{\lambda'}} & \Gamma(S\otimes V_{\lambda'}) \\
\| & & \| \\
\Gamma(V_\lambda) & \xrightarrow{D_\lambda} & \Gamma(V_\lambda) \\
\oplus & \times & \oplus \\
\Gamma(W) & \longrightarrow & \Gamma(W)
\end{array}$$

we have defined 4 new invariant operators, one of them being the operator

$$D_\lambda : \Gamma(V_\lambda) \to \Gamma(V_\lambda).$$

Operators D_λ defined in such a way will be called higher spin Dirac operators.

A certain subclass of invariant operators discussed above have appeared often in discussions of higher dimensional generalizations of holomorphic differential forms (see [DSS, So2]). They are arising in the following way. Let us consider spinor valued differential forms, they are coming as elements of the twisted de Rham sequence,

$$\Gamma(S^\pm) \overset{\nabla^S}{\to} \dots \Gamma(\Omega^k_c \otimes S^\pm) \overset{\nabla^S}{\to} \dots \overset{\nabla^S}{\to} \Gamma(\Omega^n_c \otimes S^\pm)$$

where ∇^S denotes the associated covariant derivative on spinor bundles extended to S-valued forms (see [So2, VSe]).

Every representation $\Lambda^k(\mathbf{C}_n) \otimes \mathbf{S}$ can be split into irreducible pieces. There are no multiplicities in the decomposition, so the irreducible pieces are well defined. For k forms ($k \leq [n/2]$), there are k pieces in the decomposition and the decomposition is symmetric with respect to the action of the Hodge star operator. The space of spinor valued k-forms $\Gamma(\Omega^k_c \otimes S^\pm)$ ($k \leq [n/2]$) can be written as the sum $\oplus_{j=1}^k E^{k,j}$ and it can be checked (see [DSS, VSe, So2]) that $E^{k,j}$ is the bundle associated with the representation with the highest weight $\lambda_j = (\frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \pm\frac{1}{2})$, where the number j indicates that the component $\frac{3}{2}$ appears with multiplicity equal to j. Signs $\pm$ at the last components are relevant only in even dimensions (more details can be found in [VSe]). The whole splitting can be described by the following triangle shaped diagram (in odd dimensions, there are two columns of the same length in the middle).

$$\begin{array}{ccccccccccccccccc}
E^{0,0} & \xrightarrow{D_0} & E^{1,0} & \xrightarrow{D_0} & \dots & \xrightarrow{D_0} & E^{k,0} & \xrightarrow{D_0} & \dots & \xrightarrow{D_0} & E^{2k-1,0} & \xrightarrow{D_0} & E^{2k,0} \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
 & & E^{1,1} & \xrightarrow{D_1} & \dots & \xrightarrow{D_1} & E^{k,1} & \xrightarrow{D_1} & \dots & \xrightarrow{D_1} & E^{2k-1,1} & & \\
 & & & & \oplus & & \oplus & & \oplus & & & & \\
 & & & & \dots & \xrightarrow{D_j} & \vdots & \xrightarrow{D_j} & \dots & & & & \\
 & & & & & & \oplus & & & & & & \\
 & & & & & & E^{k,k} & & & & & &
\end{array}$$

The general construction of invariant operators described above can be used in the special case of spinor valued forms. The covariant derivative ∇^S restricted to $E^{k,j}$

and projected to $E^{k+1,j'}$ is an example of this general construction. It can be shown that if $|j-j'|>1$, then the corresponding invariant operator is trivial. We shall be mainly interested in 'horizontal arrows', i.e. in operators D_j given by restriction to $E^{k,j}$ and projection to $E^{k+1,j}$. They are indicated in the above scheme. The simplest cases among them are well known. The operator D_0 is (a multiple of) the Dirac operator. The operator D_1 is (an elliptic version of) the operator called Rarita-Schwinger operator by physicists (see [EP, RaS, Wa]). All of them are elliptic operators (see [So1]). Note that all operators D_j on the same row in the scheme above cannot be identified without further comments. To compare them, it is necessary first to choose an equivariant isomorphism among corresponding bundles. Then they coincide up to a constant multiple.

To compare the operators D_j in the above scheme with the higher spin Dirac operators (see Def.1), we shall choose a certain identification of the corresponding source and target bundles. We shall do it for the first operator D_j in the row.

Let us characterize an algebraic operator $Y:\Gamma(\Omega_c^{k+1}\otimes S)\to\Gamma(\Omega_c^k\otimes S)$ by a local formula

$$Y(\omega\otimes s)=-\sum_i \iota(e_i)\omega\otimes e_i\cdot s,$$

where $\{e_i\}$ is a (local) orthonormal basis of TM and ι denotes the contraction of a differential form by a vector. As shown in [VSe], the map $Y:E^{k+1,j}\to E^{k,j}, j<k<[n/2]$ is an isomorphism.

The twisted Dirac operator D^T maps the space $\Gamma(\Omega_c^k\otimes S)$ to itself. In [VSe], it was proved that we have a relation $\nabla\circ Y+Y\circ\nabla=-D^T$. Let us denote the projection from $\Omega_c^k\otimes S$ onto $E^{k,j}$ by $\pi_{k,j}$. Symbols $\tilde{D}_j$, $0\le j<[n/2]$ will denote operators

$$\tilde{D}_j=Y\circ D_j=\pi_{j,j}\circ Y\circ\nabla^S|_{E^{j,j}},$$

mapping the space of sections of $E^{j,j}$ to itself. Then $Y|_{E^{j,j}}=0$ implies that

$$\tilde{D}_j=\pi_{j,j}\circ Y\circ\nabla^S|_{E^{j,j}}=-\pi_{j,j}\circ D^T|_{E^{j,j}}=-D_{\lambda_j},$$

where D_{λ_j} is the higher spin Dirac operator corresponding to the bundle V_{λ_j}, $\lambda_j=(\frac{3}{2},\ldots,\frac{3}{2},\frac{1}{2},\ldots,\frac{1}{2})$ (component $\frac{3}{2}$ appearing j times). More precisely, there are no signs in odd dimensions, while in even dimension, $V_{\lambda_j}=V_{\lambda_j^+}\oplus V_{\lambda_j^-}$. To compute spectrum of the higher spin Dirac operators, it is hence sufficient to do it for $\tilde{D}_j$.

Now, we shall restrict our study to operators $\tilde{D}_j$ and we shall consider them on spheres. We would like to compute their spectra. The spectrum of the Dirac operator is well-known (see [B2]).

Lemma 1 *The eigenvalues of the Dirac operator on the sphere S_n with standard metric are*

$$\mu_l=\pm\left(\frac{n}{2}+l\right);\ l=0,1,2,\ldots.$$

with multiplicity

$$2^{[\frac{n}{2}]}\binom{l+n-1}{l}.$$

The main result of the paper is given in the following theorem.

Theorem 1 *Let $D_{\lambda_j} = -\tilde{D}_j$, $0 < j < n/2$, be the higher spin Dirac operators defined above, considered on the sphere S_n with the standard metric. Then their eigenvalues are :*

$$\mu_l^1 = \pm\left[\frac{n-2j}{n-2j+2}\left(\frac{n}{2}+l\right)\right]; \; l = 1, 2, \ldots$$

with multiplicity

$$2^{[\frac{n}{2}]}\binom{n+1}{j}\binom{l+n}{l-1}\frac{(n-2j+2)j}{(l+j-1)(l+n-j+1)}$$

and

$$\mu_l^2 = \pm\left(\frac{n}{2}+l\right); \; l = 1, 2, \ldots$$

with multiplicity

$$2^{[\frac{n}{2}]}\binom{n+1}{j+1}\binom{l+n}{l-1}\frac{(n-2j)(j+1)}{(l+j)(l+n-j)}.$$

The rest of the paper will be devoted to the proof of the theorem.

3 Ratios of eigenvalues

A main tool for computation of eigenvalues will be taken from the paper of Branson, Ólafsson and Ørsted (see [BOO]). Their paper is designed to *construct* invariant operators on homogeneous spaces in a diagonal form. They have developed a powerful method of study of invariant operators (not necessarily differential ones!) using representation theoretical methods. They are prescribing the so called spectral function, giving eigenvalues (up to a multiple) of an operator in question on suitably defined finite dimensional spaces of eigenfunctions. It applies to a broad class of homogeneous spaces, which includes the conformally invariant operators considered above in case of odd dimensions. But operators D_{λ_j} in even dimensional case are explicitely excluded from consideration in their paper. Our task here is different so that we can *compute* ratios of eigenvalues using their method also in even dimensions.

The first thing to note is that eigenspaces of our operators can be easily identified and described using representation theory. Let us consider the n-dimensional sphere S^n as a homogeneous space

$$S^n = G/P = K/H$$

where $G = \mathrm{Spin}_o(n+1,1)$, $K = \mathrm{Spin}(n+1)$ is a maximal compact subgroup of G, $H = \mathrm{Spin}(n)$ and P is a (noncompact) maximal parabolic $P = MAN$ with $M \subset K$. The invariant metric g on S^n is constructed by left translation of the Killing form $\tilde{B} = B/2n$, then S^n has constant sectional curvature $K = 1$ with respect to this metric. We shall need here the compact picture $S^n = K/H$ only.

Let $\lambda \in \Lambda^+$ and let V_λ be the corresponding homogeneous bundle on the sphere. The group K acts on the space of sections $\Gamma(V_\lambda)$ by the left regular representation. The group K is compact, hence the space of sections can be decomposed to corresponding isotypic components, which are finite-dimensional. The main case considered in [BOO] is the multiplicity one case, when these isotypic components are irreducible. Then by Schur lemma, any invariant operator (when restricted to these components and acting among identical bundles) is a multiple of identity. Then these components are eigenspaces of the operator. To compute ratios of eigenvalues, the authors use a suitable combination of Casimir operators called the spectrum generating operator.

There are explicit formulas how to find highest weights of isotypic components appearing in the decomposition. They are given by the so called branching rules, which were carefully studied in representation theory. In the conformal case needed below, they are given as follows.

Let us agree first the following notation. Let $\lambda \in \Lambda^+(\mathrm{Spin}(n))$ and $\alpha \in \Lambda^+(\mathrm{Spin}(n+1))$. The symbol $\alpha \downarrow \lambda$ is defined by the following relations:

1) Let $n = 2k$.

$$\alpha \downarrow \lambda \iff \alpha_1 \geq \lambda_1 \geq \alpha_2 \geq \lambda_2 \geq \ldots \geq \alpha_k \geq |\lambda_k|.$$

2) Let $n = 2k+1$.

$$\alpha \downarrow \lambda \iff \alpha_1 \geq \lambda_1 \geq \alpha_2 \geq \lambda_2 \geq \ldots \geq \alpha_k \geq \lambda_k \geq |\alpha_{k+1}|.$$

If we consider now the space of sections $W = \Gamma(V_\lambda)$, $\lambda \in \Lambda^+(H)$ as a K - modul, then all isotypic components $V_\alpha, \alpha \in \Lambda^+(K)$ have multiplcity at most one and are nontrivial iff $\alpha \downarrow \lambda$. The sum $\oplus_{\alpha,\alpha\downarrow\lambda} W_\alpha$ is then dense in $\Gamma(V_\lambda)$.

Methods and results of [BOO] can be used to show

Theorem 2 *Let D_λ, $\lambda \in \Lambda^+, |\lambda_k| = \frac{1}{2}$, be a higher spin Dirac operator (see Def.1) and let μ, μ' be its two different eigenvalues, having both the same sign. Let W, resp. W' be the corresponding spaces of eigenvectors.*

Then there exist isotypic components $W_\alpha, W_{\alpha'}$ with highest weights $\alpha, \alpha' \in \Lambda^+(K)$ such that $W \subset W_\alpha$, $W' \subset W_{\alpha'}$, and

$$\frac{\mu}{\mu'} = \Pi_{a=1}^{[\frac{n+1}{2}]} \frac{\Gamma(\frac{1}{2}(n+3) - a + \alpha_a)\Gamma(\frac{1}{2}(n+1) - a + \alpha'_a)}{\Gamma(\frac{1}{2}(n+3) - a + \alpha'_a)\Gamma(\frac{1}{2}(n+1) - a + \alpha_a)}$$

Proof: Suppose first that n is odd, $n = 2k+1$. Then the space $\mathbf{V}_\lambda$ is irreducible, all isotypic conponents of $\Gamma(V_\lambda)$ have multiplicity one and the formula above for ratios of their eigenvalues was proved in [BOO].

So suppose next that $n = 2k$. In the paper [BOO], they have to exclude this case from their construction. The main reason was that in this case, they had no control over certain compatibility conditions needed for it. Nevertheless, if the aim is not to *construct* intertwining operators but to *compute* their eigenvalues under the assumption that they exist, it is not necessary to verify corresponding compatibility conditions and their methods are applicable.

So only problem to discuss is that isotypic components of the space of sections $\Gamma(V_\lambda)$ have multiplicity two. Indeed, $\mathbf{V}_\lambda = \mathbf{V}_{\lambda^+} \oplus \mathbf{V}_{\lambda^-}$. Isotypic components of $\Gamma(V_{\lambda^+})$, have all multiplicity one, but they are identical with the corresponding isotypic components of $\Gamma(V_{\lambda^-})$.

The operator D_λ intertwines action of K, so it preserves individual isotypic components. If $s = (s^+, s^-)$, $s^\pm \in \Gamma(V_{\lambda^\pm})$ is eigenvector of D_λ with eigenvalue μ, then $(s^+, -s^-)$ is eigenvector with eigenvalue $-\mu$.Hence the restriction of D_λ to any isotypic component W_α has at least two eigenvalues $\pm\mu$. The corresponding eigenspaces are then K-modules and the isotypic component W_α is a direct sum of them. Let us denote by W^+ the closure of the sum of all eigenspaces corresponding to positive eigenvalues. The corresponding isotypic components have multiplicity one, W^+ is an invariant subspace with respect to the action of G and the computation in [BOO] can be repeated to prove the result.

□

4 Normalisation of eigenvalues

To finish the computation of spectra, it is necessary to compute at least one eigenvalue of a given operator. The spectrum of the Dirac operator is known. We shall show how to compute inductively one eigenvalue for operators $\tilde{D}_j = -D_{\lambda_j}$, $0 < j < [n/2]$. It will lead then in next section to a formula for their full spectrum.

A useful relation among spectra operators $\tilde{D}_j$ was shown in [VSe], the following theorem is proved there.

Theorem 3 *Let M be aconformally flat Einstein spin manifold, dim $M = n$.. Let us define a first order differential operator T_j by*

$$T_j = \pi_{j+1,j+1} \circ \nabla^S|_{E^{j,j}},\ 0 \le j \le [n/2] - 1.$$

If s is an eigenvector of the operator $\tilde{D}_j$ corresponding to an eigenvalue μ and if $T_j(s) \neq 0$, then $s' = T_j(s)$ is an eigenvector of the operator $\tilde{D}_{j+1}$ corresponding to the eigenvalue $\mu' = \frac{n-2j-2}{n-2j}\mu$.

As a consequence, if we are able to find eigenvectors of $\tilde{D}_j$ which does not belong to the kernel of T_j, we can compute at least one eigenvalue of $\tilde{D}_{j+1}$. The following theorem shows that it is always possible.

Theorem 4 *The operators T_j, $0 \le j \le \frac{n}{2} - 1$ have nontrivial symbol, hence their kernel is a proper subset of $\Gamma(V_{\lambda_j})$.*

Proof: Let $\varepsilon(v) : \Omega^j_c \to \Omega^{j+1}_c$, $v \in \Omega^1_c$ denote the outer multiplication by the element v. Then symbol σ of the operator $\tilde{D}^j$ is given by

$$\sigma(v)(\omega) = \pi_{j+1,j+1} \circ \varepsilon(v)(\omega),\ v \in \Omega^1_c,\ \omega \in E^{j,j} \subset \Omega^j \otimes S.$$

Let v_j denote a nontrivial weight vector of the fundamental representation $\mathbf{C}^n$ corresponding to a weight $\lambda_j = (0, \ldots, 1, \ldots, 0)$ with 1 on the j-th place, resp. corresponding associated element in Ω^1_c. Then $v_1 \wedge \ldots \wedge v_j$ is a (nontrivial) weight vector of $\Lambda^j \mathbf{C}^n$. Denote further by s_0 a nontrivial weight vector for the highest weight of $\mathbf{S}$. Then

$$w = v_1 \wedge \ldots \wedge v_j \otimes s_0$$

is a (nontrivial) weight vector for the highest weight of $\Lambda^j \mathbf{C}^n \otimes \mathbf{S}$. Hence w belongs to the Cartan product of $\Omega^j \otimes S$, which is just equal to $E^{j,j}$.

Hence

$$\sigma(v_{j+1})(v_1 \wedge \ldots \wedge v_j \otimes s_0) = (-1)^j v_1 \wedge \ldots \wedge v_{j+1} \otimes s_0$$

is a nontrivial vector and the theorem is proved. □

5 The proof of Theorem 1

Now we can finish the proof of the main theorem. It is necessary to distinguish even and odd dimensional cases.

Proof:

1) Let first $n = 2k + 1$.

The highest weight of the space $S = E^{0,0}$ is $\lambda_0 = (\frac{1}{2}, \ldots, \frac{1}{2})$ (k components) and the space of sections of $\Gamma(E^{0,0})$ is a sum of K-types

$$A_{\alpha_0(\pm,l)},\ \alpha_0(\pm, l) = \left(\frac{2l+1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \pm\frac{1}{2}\right),\ l = 0, 1, 2, \ldots,$$

($\alpha_0(\pm, l)$ having $k+1$ components).

The highest weight of the space $E^{j,j}$, $j > 0$ is $\lambda_j = \left(\frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$ (with $\frac{3}{2}$ appearing j times) and the space of sections of $\Gamma(E^{j,j})$ is a sum of K-types

$$A_{\alpha_j(\pm,l)},\ \alpha_j(\pm, l) = \left(\frac{2l+1}{2}, \frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \pm\frac{1}{2}\right),\ l = 1, 2, \ldots,$$

and

$$B_{\beta_j(\pm,l)},\ \beta_j(\pm, l) = \left(\frac{2l+1}{2}, \frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, \pm\frac{1}{2}\right),\ l = 1, 2, \ldots,$$

where the component $\frac{3}{2}$ is appearing $j-1$ times in the weight $\alpha_j(\pm, l)$ and j times in the weight $\beta_j(\pm, l)$.

Using the formula for the ratio of eigenvalues from Th.2, we get first for $\alpha = \alpha_j(\pm, l)$

$$\Pi_{a=1}^{\frac{n+1}{2}} \frac{\Gamma(\frac{1}{2}(n+3) - a + \alpha_a)}{\Gamma(\frac{1}{2}(n+1) - a + \alpha_a)} = \pm(\frac{n}{2} + l) 2^{-\frac{n-1}{2}} (n!!)(\frac{n}{2} - j)^{-1},$$

hence there are constant C_1, C_2 (independent of a K type chosen, but depending on n and j) such that the eigenvalues $\mu^1_{\pm,l}(j)$, resp. $\mu^2_{\pm,l}(j)$, corresponding to the eigenspace $A_{\alpha(\pm,l)}$, resp. $B_{\beta(\pm,l)}$ are equal to

$$\mu^1_{\pm,l} = \pm C_1\,(l + \frac{n}{2})$$

resp.

$$\mu^2_{\pm,l} = \pm C_2\,(l + \frac{n}{2}).$$

Moreover, Th.2 implies that

$$\frac{\mu^1_{\pm,l}(j)}{\mu^2_{\pm,l}(j)} = \frac{n-2j}{n-2j+2}.$$

The unknown constants C_1, C_2 will be computed inductively (with respect to j). For the Dirac operator, the spectrum is known (see e.g. [B2]), the eigenvalue corresponding to K-typ with $\alpha(\pm, l)$ is equal to $(\frac{n}{2} + l)$.

For the Rarita-Schwinger operator ($j = 1$), there are two sequences of K-types, one of them being a subset of that for the Dirac operator (only the first term is missing).

The twistor operator T_0 is invariant (hence should preserve the label of a K-type) and has a finite dimensional kernel (hence is nontrivial for at least one K-type). Th.3 is then saying that

$$\mu^1_{\pm,l}(1) = \pm\left(\frac{n}{2} + l\right)\frac{n-2}{n},$$

hence the theorem is valid for $j = 1$.

Due to the preceding theorem, operators T_j are nontrivial for all j, so the proof can be finished in the same way by induction.

2) Let $n = 2k$. In even dimensions, $E^{j,j}, j > 0$ is a sum of two $\pm$ spaces with highest weights $\lambda^\pm_j = \left(\frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \pm\frac{1}{2}\right)$ (k components, $\frac{3}{2}$ appearing j times). Hence the space of sections will be a sum of K-types (α's having also k components)

$$A_{\alpha(l)}, \alpha(l) = \left(\frac{2l+1}{2}, \frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right), \, l = 1, 2, \dots,$$

and

$$B_{\beta(l)}, \beta(l) = \left(\frac{2l+1}{2}, \frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right), \, l = 1, 2, \dots,$$

where $\frac{3}{2}$ is appearing $j-1$ times in the weight $\alpha(l)$ and j times in the weight $\beta(l)$. This time, however, each type will appear with a multiplicity two.

As in the proof of Th.2, we can split each isotypic component with respect to K as a sum of eigenspaces corresponding to opposite eigenvalues. The sum of spaces corresponding to positive ones will be invariant with respect to G and the same proof as in odd dimensional case will go through.

The formula for the dimension of the space of eigenvectors is the consequence of the Weyl dimensional formula for the representation with the corresponding highest weight. □

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