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SABININ'S METHOD FOR CLASSIFICATION OF LOCAL BOL LOOPS

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ABSTRACT. Our aim is to explain how the theory of local smooth Bol loops developed by L.V. Sabinin [Sa], [M&S2] and the methods introduced by P.T. Nagy and K. Strambach [N&S1] can be used for classification of analytic local Bol loops. The most of the paper is of expository character but, in the end, we apply the general theory to classification of local hyperbolic loops (on a hyperbolic space) having $SL_2(\mathbb{C})$ as the group topologically generated by left translations.

1. Loops

Definition 1.1. *A loop* $(L, \cdot)$ is an algebraic system with one binary operation such that the equations $a \cdot y = c$ and $x \cdot b = c$ are uniquely solvable for all a, b, $c \in L$, and there exists a unit element $e \in L$ such that $e \cdot x = x \cdot e = x$ for $x \in L$. Besides a *principal* operation $\cdot$, a couple of *additional* operations $\backslash, / : L \times L \to L$ is introduced by $x = c/b$, $y = a\backslash c$. A *left* (or *right*) *Bol loop* is a loop satisfying the identity

$$(1) \qquad (x \cdot (y \cdot (x \cdot z))) = (x \cdot (y \cdot x)) \cdot z, \text{ or } (((z \cdot x) \cdot y) \cdot x) = z \cdot ((x \cdot y) \cdot x)$$

for all x, y, $z \in L$.

Let $\mathcal{C}$ be respectively the category of topological spaces and continuous mappings, or C^r-manifolds and morphisms where $r \in \mathbb{N} \cup \{\infty, \omega\}$ (differentiable, smooth, or real analytic manifolds and morphisms, respectively).

Definition 1.2. *A $\mathcal{C}$-loop* is a loop $(L, \cdot, \backslash, /)$ and at the same time a $\mathcal{C}$-manifold such that all three binary loop operations are $\mathcal{C}$-morphisms.

A *continuous multiplication (product) with unit* on a pointed topological space (X, e), $e \in X$ is a continuous mapping $\mu : X \times X \to X$ such that $\mu(x, e) = \mu(e, x) = x$, that is, both left and right translation by e is equal to identity, $\varrho_e = \lambda_e = 1_X$. An *H-space* is a topological space which admits a continuous multiplication with unit. Any $\mathcal{C}$-loop is an H-space.

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Definition 1.3. A *local* topological ($\mathcal{C}$-) loop is a topological space ($\mathcal{C}$-manifold) L together with continuous ($\mathcal{C}$-) mappings $\cdot$, $\backslash$, $/$ from open domains of $L \times L$ to L and with an element $e \in L$ such that for $x, y \in L$

$$(x/y) \cdot y = x, \quad y \cdot (y\backslash x) = x, \quad (x \cdot y)/y = x, \quad y\backslash(y \cdot x) = x, \\ x \cdot e = e \cdot x = x/e = e\backslash x = x \tag{2}$$

whenever the left hand side of the identity is defined.

Definition 1.4. *An isotopy* of a ($\mathcal{C}$-) loop $(L, \cdot)$ upon a ($\mathcal{C}$-) loop $(M, \circ)$ is a triple (f, g, h) of bijections (resp. category isomorphisms) f, g, $h: L \to M$ provided

$$f(x \cdot y) = g(x) \circ h(x), \qquad x, y \in L. \tag{3}$$

We also say that $(L, \cdot)$ is an isotope of $(M, \circ)$ via an isotopy $(f, g, h): L \to M$ (isotopy is an equivalence relation). If $M = L$ and $f = \mathrm{id}_L$ we speak about a *principal* isotopy. Let $(\mathrm{id}_L, \alpha^{-1}, \beta^{-1})$ be a principal isotopy of $(L, \cdot)$ to $(L, \circ)$ with unit ϵ. Then $x \cdot y = \alpha^{-1}(x) \circ \beta^{-1}(y)$, that is $x \circ y = \alpha(x) \cdot \beta(y)$. Let us denote $a = \alpha(\epsilon)$, $b = \beta(\epsilon)$. Then $\alpha = {\varrho_b}^{-1}$, $\beta = {\lambda_a}^{-1}$, and the multiplication $\circ$ with unit $a \cdot b = \epsilon$ can be written in the form $x \circ y = x \cdot_{(b,a)} y$ where

$$x \cdot_{(b,a)} y = {\varrho_b}^{-1}(x) \cdot {\lambda_a}^{-1}(y) = (x/b) \cdot (a\backslash y). \tag{4}$$

Theorem 1.1. *Any ($\mathcal{C}$-) loop isotopic to a given ($\mathcal{C}$-) loop L is ($\mathcal{C}$-) isomorphic to a ($\mathcal{C}$-) loop $\tilde{\tilde{L}} = (L, \cdot_{(b,a)})$ (principal isotope of L) for some a, $b \in L$,* [B].

In fact, if $(f, g, h): (L, \cdot) \to (M, \circ)$ is an isotopy then $(\mathrm{id}_L, g^{-1}f, h^{-1}f): (L, \cdot) \to (L, *)$ is a principal isotopy where the star operation is introduced by $x * y = g^{-1}f(x) \cdot h^{-1}f(y)$. Now $f: (L, *) \to (M, \circ)$ is an isomorphism, $f(x*y) = f(g^{-1}f(x) \cdot h^{-1}f(y)) = f(x) \circ f(y)$.

Especially, given a loop $(L, \cdot)$ with unit e and $a \in L$ we can introduce a new loop $(L, \cdot^{(a)}) = (L, \cdot_{(e,a)})$ with a new unit a the multiplication of which is given by the formula $x \cdot^{(a)} y = x \cdot_{(e,a)} y = x \cdot (a\backslash y) = x \cdot {\lambda_a}^{-1}(y)$. A mapping $(\mathrm{id}, \mathrm{id}, {\lambda_a}^{-1}): (L, \cdot^{(a)}) \to (L, \cdot)$ is an isotopy.

Similarly for any $b \in L$, a multiplication $x \cdot_{(b)} y = x \cdot_{(b,e)} y = (x/b) \cdot y = {\varrho_b}^{-1}(x) \cdot y$ gives rise to a loop $(L, \cdot_{(b)}) = (L, \cdot_{(b,e)})$ with a new unit element b, isotopic with the original loop via an isotopy $(\mathrm{id}, {\varrho_b}^{-1}, \mathrm{id}): (L, \cdot_{(b)}) \to (L, \cdot)$ (a *left principal isotope*). Now $(L, \cdot_{(b,a)}) = ((L, {\cdot_{(b)}}^{(a)})$ is an isotope of $(L, \cdot)$ via an isotopy $(\mathrm{id}, {\varrho_b}^{-1}, {\lambda_a}^{-1})$ which arises as a composition of the above isotopies.

Definition 1.5. We say that a loop $(M, \circ)$ is *left isotopic* to $(L, \cdot)$ if it is isomorphic to $(L, \cdot_{(b)})$ for some $b \in L$.

2. Groups generated by translations of a loop

2.1 The complete left translation group. Let $(L, \cdot)$ be a topological loop with a unit e on a differentiable manifold L. In comparison to groups where left translations $\lambda_x : y \mapsto x \cdot y$ (right translations ϱ_x) always form a group isomorphic to the original one, the set of left translations $\Lambda = \{\lambda_x; x \in L\}$ of a proper (non-associative) loop L forms only a loop (isomorphic to the initial one) since in general it may happen $\lambda_x \lambda_y \neq \lambda_{xy}$. Under additional assumptions (e.g. for locally compact C-loops) the original loop can be reconstructed from the set Λ.

Consider the left translation group $\mathcal{L}(L) = \langle \lambda_x; x \in X \rangle$ generated by the set Λ of all left translations in the group Homeo(X) of all autohomeomorphisms of X [Ki, H&S]. Similarly denote by Tran$(L) = \langle \lambda_x, \rho_x; x \in X \rangle$ the group generated by all left and right translations. These groups yield a useful tool if they are equipped with a natural topology making them topological transformation groups on L. For locally compact loops (which cover a wide area of important examples), a sufficiently developed theory is available [H&S].

In the following, let X be locally compact. Then on Homeo(X), the so called Arens topology can be introduced as a (unique) coarsest topology $\mathcal{T}$ which contains the compact open topology and for which the inverse image U^{-1} under the inversion map of homeomorphisms, $f \mapsto f^{-1}$, is open for any subset $U \subset$ Homeo(X) open in compact-open topology on Homeo(X), [H&S, p. 217].

Let us introduce the so called *complete translation group* as a closure of Tran(L) in Homeo(X), $\mathcal{M}(L) = \overline{\mathrm{Tran}\,(L)}^{\mathrm{Homeo}\,(X)}$, and a *complete left translation group* $G = G(L) = \overline{\mathcal{L}(L)}^{\mathcal{M}(L)}$. The group $G(L)$ is mostly of high dimension, often infinite-dimensional.

Let $(g, x) \mapsto g(x)$ denote the action of $G(L)$ on L. Denote by $G_e = \{g \in G(L); g(e) = e\}$ the stabilizer (=isotropy subgroup) in $G = G(L)$ of the unit $e \in L$. Consider the factor space G/G_e of left cosets with a natural projection $q: G \to G/G_e$, with the left action $(g, hG_e) \mapsto g \cdot hG_e$ of G on $G \to G/G_e$. A mapping $\beta: G/G_e \to L$ given by $\beta(gG_e) = g(e)$ is a continuous bijection, and is equivariant under the action of G, $\beta(g \cdot hG_e) = g(\beta(hG_e))$. The mappings $\sigma: L \to G$, $x \mapsto \lambda_x$ and $x \mapsto \lambda_x{}^{-1}$ are continuous relative to the compact open topology on $C(L, L) = \{f : L \to L; f \text{ continuous}\}$. For any locally compact connected topological loop, $\sigma = x \mapsto \lambda_x$ is a topological embedding, and L is a retract of G in the category of topological spaces. In fact, the composition $\beta q \sigma = \varrho_e = \mathrm{id}_L$, hence the function $\pi_e: G \to L$, $g \mapsto g(e)$ is a retraction, σ is a global section of π_e, and $q\sigma = \beta^{-1}$. That is, β is a homeomorphism, and $\sigma' = \sigma\beta$, $\xi = gG_e \mapsto \sigma(g(e)) = \lambda_{g(e)}$ is a section of the natural projection q corresponding to the natural section σ. So L is embedded to G, and G is homeomorphic to the topological product of L and the stabilizer of unit, $G \approx L \times G_e$.

Moreover, the image $\Lambda = \sigma(L) \subset G$ is a *simply transitive* family of homeomorphisms in G, $g = \lambda_{b/a}$ is a unique element in G with $g(a) = b$ for $a, b \in L$.

Now let us introduce a loop structure on $G(L)/G_e$ by $\xi * \eta = q(\sigma'(\xi)\sigma'(\eta))$. It can be verified that $\beta: (G(L)/G_e, *) \to (L, \cdot)$ is a loop isomorphism, $\beta(\xi * \eta) = \beta(\xi) \cdot \beta(\eta)$ for $\xi, \eta \in G/G_e$. Moreover, G *acts effectively* on G/G_e by left multiplication, $(g, \eta) \mapsto g\eta$, and $\xi * \eta = \sigma'(\xi)\eta$. We usually identify $x \in L$ with the coset $\lambda_x G_e$ determined by the corresponding left translation. In the image of the cross-section $\Lambda = \sigma'(G/H) = \sigma(L)$ a loop multiplication arises via $\lambda_x \circ \lambda_y = \sigma'(q(\lambda_x \lambda_y))$.

2.2 Complete triples. There is a bijection between locally compact topological loops and so called *complete* triples (G, H, σ) which can be turned out to a category equivalence [H&S].

Definition 2.1. *A complete triple* (G, H, σ) consists of a topological group G, its closed subgroup H and a cross-section $\sigma: G/H \to G$ of the natural projection $q: G \to G/H$ such that $\sigma(H) = 1 \in G$, the subgroup generated by $\sigma(G/H)$ in G is dense in G, the factor space G/H is locally compact, and the set of all functions $\xi \mapsto g \cdot \xi$ in Homeo(G/H) satisfying $g \cdot kH = gk\,H$, g, $k \in G$ is closed in the Arens topology on Homeo(G/H).

If the underlying topological space of a loop L is locally compact then there always exists a corresponding continuous section $\sigma': G \to G/G_e$ of q such that $\sigma'(G/G_e) = \lambda_e = \mathrm{id}_L$ and the loop operations $*$, $\backslash$, $/$ given by σ' on G/G_e are continuous, $(G(L), G/G_e, \sigma')$ is a complete triple. On the other hand given a topological group G, its closed subgroup H and a continuous section $\sigma: G/H \to G$ of the natural projection $g \mapsto gH$ with $\sigma(H) = 1 \in G$ such that for any ξ, $\eta \in G/H$, the elements $\omega = \eta/\xi$, $\tilde{\omega} = \xi\backslash\eta = \sigma(\xi)^{-1}\eta$ satisfying $\sigma(\omega)\,\xi = \eta$, $\sigma(\xi)\,\tilde{\omega} = \eta$ are determined uniquely then $(\xi, \eta) \to \xi * \eta = \sigma(\xi)\eta$ is a continuous mapping, and $(G/H, *)$ is a loop. If $/$ is also continuous then $(G/H, *)$ is a topological loop with additional operations $\backslash$, $/$.

If the starting triple (G, H, σ) is complete the above construction yields a topological loop. Moreover G acts on G/H as a topological transformation group. With respect to the operation $*$, the mapping $\eta \mapsto \sigma(\xi)\eta$, ξ, $\eta \in G/H$ is precisely the left translation λ_ξ on the loop $L = (G/H, *)$, $\sigma(\xi)\eta = \lambda_\xi(\eta)$. That is, left translations λ_ξ may be identified with the corresponding elements $\sigma(\xi)$ of the group G, $\sigma(G/H) = \Lambda$.

A topological loop multiplication obtained by $\xi * \eta = \sigma(\xi)\eta$ on the manifold $L = G/H$ can be transported to the image of the section $\Lambda = \sigma(G/H)$ by introducing a binary operation $\circ: \Lambda \times \Lambda \to \Lambda$ uniquely determined by the formula $(a \circ b)H = ab\,H$ for a, $b \in G/H$, $a \circ b = \sigma(q(ab))$ (ab is a product in G).

2.3 Applications - the idea of constructing examples. The above considerations support us by a method how to construct examples. If we wish to find a topological (differentiable) loop on a manifold L (which is an H-space), at first we choose a suitable group G admitting a *simply transitive action* on L and such that $L \approx G/H$ where H is a subgroup of G containing no normal subgroups of G (since the complete left translation group acts effectively on the loop); $G \approx L \times H$. Then we can try to determine all sections $\sigma: G/H \to G$ such that $\sigma(H) = 1$, the action of $\sigma(G/H)$ on L is simply transitive, and the corresponding loop multiplication $gH *_\sigma kH = \sigma(gH)\,kH$ is (at least) continuous. To simplify the situation we can suppose that $G = G(L)$ is a *Lie group* of low-dimension, "as small as possible". The restriction is not too strong since $G(L)$ is a Lie group for differentiable loops satisfying some weak associativity (e.g. L is a left A-loop or a left (right) Bol loop, [M&S2], Prop. XII.2.14.).

If L is a connected differentiable loop and its complete left translation group G is a *Lie group* then the mapping $\sigma: L \to G$, $\sigma(x) = \lambda_x$, $x \in L$ is a differentiable embedding, that is, the image of the section $\Lambda = \{\lambda_x : x \in L\}$ is an embedded differentiable submanifold in G [N&S3]. Loops on manifolds of dimensions 1 and 2 respectively are investigated by Nagy and Strambach in [N&S1]. In [N&S1], all loops of the class C^1 on a real line $\mathbb{R}$ and on a 1-sphere $\mathbb{S}^1$, with $G(L) = PSL_2(\mathbb{R})$ as the complete left tanslation group were found, and proper loops were distinguished. Moreover, it was proved that any differentiable connected Bol loop on a 1-dimensional manifold is isomorphic either to the real line $(\mathbb{R}, +)$ or to the rotation group $SO(2)$.

The following statements enables us to classify loop multiplications on a manifold L up to isotopy. Let $\sigma: L \to G$, $\sigma(x) = \lambda_x$ denote a section corresponding to the original multiplication on L, let $\sigma': G/H \to G$ be an associated section, and let H denote a stabilizer in G of the unit e. As mentioned above any loop isotopic to a given C-loop $(L, \cdot)$ with unit e is isomorphic with a loop $\tilde{\tilde{L}} = (L, \cdot_{(b,a)})$ which itself is a principal isotope of L. The section $\sigma_{(b,a)} = \sigma_{(b)}\sigma(a)^{-1}$ of the projection $\pi_{a \cdot b}: g \to g(a \cdot b)$ corresponding to the multiplicaiton $\cdot_{(b,a)}$ satisfies $\sigma_{(b,a)} \times 1_L = \sigma\, \varrho_b{}^{-1} \times \lambda_a{}^{-1}$, $\sigma_{(b,a)}(x) = \sigma(x/b)\sigma(a)^{-1}$. The stabilizer in G of the new unit $a \cdot b$ is of the form gHg^{-1} where $g = \lambda_a\lambda_b \in G$, $g(e) = a \cdot b$. The images of the sections coincide, $\sigma_{(b,a)}(L) = \sigma(L)$. Especially if $a = e$, the loop multiplication $x \cdot_{(b)} y = x/b \cdot y$ with a unit b corresponds to a section $\sigma_{(b)} = \sigma\, \varrho_b{}^{-1}: L \to G$, $\sigma_{(b)}(x) = \sigma(x/b)$ of the projection $\pi_b: G \to L$, $\pi_b(g) = g(b)$ where $b \in L$, $G = G(L)$. Moreover, a stabilizer of the new unit b in the isotopic loop $(L, \cdot_{(b)})$ is of the form gHg^{-1} where $g \in G$ is a (unique) element with $g(e) = b$, namely $g = \lambda_b$.

Theorem 2.1. ([N&S1].) *Let L be a C-loop with the complete left translation group G and the stabilizer H, let $\sigma: x \mapsto \lambda_x$ denote the corresponding section. Then $\sigma_{(b)}$ is a section corresponding to the loop $\tilde{L} = (L, \cdot_{(b)})$ isotopic with L, the stabilizer $\tilde{H}$ of the unit element of the loop $\tilde{L}$ is conjugated to H, and images of the sections coincide, $\sigma(L) = \sigma_{(b)}(L)$.*

Similarly in the case $b = e$.

Theorem 2.2. ([N&S1]). *Let L be a loop with the complete left translation group G, the stabilizer H, the section $\sigma: L \to G$, the associated section $\sigma': G/H \to G$ and the image of the section $\Lambda = \sigma(L) = \sigma'(G/H) = \{\lambda_x, x \in L\}$.*

(i) *Let $\alpha \in \mathrm{Aut}(G)$ be an automorphism of G. Then on $G/\alpha(H)$, the section $\hat{\sigma} = \alpha \circ \sigma' \circ \alpha^{-1}$ determines a loop multiplication with the identity $\alpha(H)$, and $\alpha(\Lambda)$ as the image of the section. The loops $(G/\alpha(H), *_{\hat{\sigma}})$ and $(G/H, *_{\sigma'})$ are isomorphic.*

(ii) *If the complete left translation groups of isotopic loops coincide, $G = G(\tilde{L}) = G(L)$, then the stabilizer $\tilde{H}$ of the unit $\tilde{e} \in \tilde{L}$ is of the form $\tilde{H} = \alpha(H)$ where $\alpha \in \mathrm{Aut}(G)$ is an automorphism of G.*

For any $g \in G$ denote by $\mathrm{ad}(g) \in \mathrm{Aut}\, G$ the inner automorphism of G, $\mathrm{ad}(g)(x) = gxg^{-1}$. If $g \in G$ then the section $\tilde{\sigma} = \sigma_{(g(e))}$, $\tilde{\sigma}(x) = \sigma(x/g(e))$ for $x \in L$ together with the conjugated subgroup $gHg^{-1} = \mathrm{ad}(g)H$ as a stabilizer of unit $g(e)$ determines a left isotopic loop $(L, \tilde{\cdot})$ with the associated section $\tilde{\sigma}': G/\mathrm{ad}(g)H \to G$ and the same image of the section,

$$\Lambda = \sigma(L) = \sigma'(G/H) = \tilde{\sigma}'(G/\mathrm{ad}(g)H).$$

Theorem 3.4. *Let G be a Lie group and H its Lie subgroup containing no normal subgroups. Let L be a Bol loop on G/H with unit H, $G(L) = G$, given by a section $\sigma': G/H \to G$. Let $\Lambda = \sigma'(G/H)$. Then all loops which have H as a stabilizer of unit and belong to the isotopy class determined by L are realized precisely on the sets $g\Lambda g^{-1}$ where $g \in G$.*

Analytic local Bol loops can be classified via classification of their tangent objects, so called Bol algebras [H&S], [Bo&M], [Sa].

3.2 Tangent Bol algebra of a left Bol loop.

A *Bol algebra* $\mathfrak{B} = (\mathfrak{m}, \cdot, (\, , \, , \,)))$ is a finite-dimensional real vector space $\mathfrak{m}$ endowed with a bilinear operation $u \cdot v$ and a trilinear operation (u, v, w) such that for all $u, v, w, z, y \in \mathfrak{m}$

$$(u,u,v) = 0,$$
$$(u,v,w) + (v,w,u) + (w,u,v) = 0,$$
$$(u,v,w) \cdot z + w \cdot (u,v,z) - (u,v,w \cdot z) + (w,z,u \cdot v) + uv \cdot wz = 0,$$
$$(u,v,(w,z,y)) - (w,z,(u,v,y)) = ((u,v,w),z,y)) + (w,(u,v,z),y)).$$

For any local analytic (left, right) Bol loop L, a structure of a Bol algebra can be introduced on the tangent space T_eL at unit in a canonical way and is called *the tangent Bol algebra* $\mathfrak{B}$ *of* L, [Bo&M], [Sa]. The structure of a local analytic Bol loop is determined by its tangent Bol algebra uniquely up to isomorphism [M&S1], [M&S2]:

Theorem 3.5. ([M&S1], p. 425, [Sa].) *Any Bol algebra is isomorphic to a tangent Bol algebra of some local analytic Bol loop.*

Theorem 3.6. ([M&S1], p. 424.) *Two local analytic (left, right) Bol loops are isomorphic if and only if the corresponding Bol algebras are isomorphic.*

Theorem 3.7. ([M&S1], p. 425.) *If A is a subloop of a local analytic Bol loop L then T_eA is a subalgebra in $\mathfrak{B} = T_eL$. Conversely, if $\mathfrak{A}$ is a subalgebra in $\mathfrak{B}$ then there exists a (unique up to equivalence, isomorphism) subloop in L such that $T_eA = \mathfrak{A}$.*

For any Bol algebra $\mathfrak{B} = (\mathfrak{m}, \cdot, (\, , \, , \,))$ there exists a (unique up to isomorphism) *canonical enveloping pair* $(\mathfrak{G}_{\mathfrak{B}}, \mathfrak{h})$, i.e. a real finite-dimensional Lie algebra $\mathfrak{G}_{\mathfrak{B}} = (\mathfrak{G}, [\cdot, \cdot])$ containing $\mathfrak{m}$ and its subalgebra $\mathfrak{h}$ containing no non-trivial ideals of $\mathfrak{G}$ so that $\mathfrak{G} = \mathfrak{m} \dot{+} \mathfrak{h}$ (direct sum of vectos spaces), a vector subspace $\mathfrak{m}$ spans $\mathfrak{G}$, $[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] \subset \mathfrak{m}$, $u \cdot v = [u, v]_{\mathfrak{m}}$ (a projection of the Lie product $[u, v]$ in $\mathfrak{B}$ onto $\mathfrak{m}$ parallel to $\mathfrak{h}$) and $(u, v, w) = [[u, v], w]$ for u, v, $w \in \mathfrak{m}$. Remark that the canonical enveloping algebra of an associative Bol algebra is the algebra itself.

Definition 3.2. Bol algebras $\mathfrak{B}$ and $\tilde{\mathfrak{B}}$ are called *isotopic* if there exists a Lie algebra isomorphism of their canonical enveloping algebras $\varphi: \mathfrak{G}_{\mathfrak{B}} \to \mathfrak{G}_{\tilde{\mathfrak{B}}}$, $\mathfrak{G}_{\tilde{\mathfrak{B}}} = \varphi(\mathfrak{G}_{\mathfrak{B}})$ such that $\varphi(\mathfrak{h}) = \mathrm{ad}\,(\xi)\tilde{h}$ for some $\xi \in \tilde{\mathfrak{B}}$.

The definition is justified by the following.

Now an application of the inner automorphism $\mathrm{ad}(g)^{-1}$ yields a loop $(G/H, *_{\tilde{\tilde{\sigma}}'})$ on G/H with the unit (and the stabilizer) H and the section $\tilde{\tilde{\sigma}}' = \mathrm{ad}(g)^{-1}\tilde{\sigma}'\mathrm{ad}(g)$ which is isomorphic with the loop $(G/\mathrm{ad}(g)H, *_{\tilde{\sigma}'})$, and consequently left isotopic with L. The image of the section is of the form

$$\tilde{\tilde{\Lambda}} = \tilde{\tilde{\sigma}}'(G/H) = \mathrm{ad}(g)^{-1}(\Lambda) = g^{-1}\Lambda g.$$

Conversely, each left isotope $(L, \cdot_{(b)})$ gives a left isotopic loop on G/H with $g^{-1}\Lambda g$ as an image of the section where $g(e) = b$. We conclude:

Theorem 2.3. ([N&S1]). (i) *In the above notation, let $\sigma': G/H \to G$ be the section defined by $\sigma: L \to G$, and let $g \in G$. Then the section $\tilde{\tilde{\sigma}}' = \mathrm{ad}(g)^{-1}\tilde{\sigma}'\mathrm{ad}(g)$ together with H determines a loop which is left isotopic to the original one. The images of the sections are conjugated, $\Lambda = \sigma'(G/H) = \tilde{\tilde{\sigma}}'(G/H) = \mathrm{ad}(g)(\tilde{\tilde{\Lambda}})$.*

(ii) *If L and $\tilde{L}$ are two left isotopic loops on G/H with the same unit H then the corresponding images of the sections are conjugated by an element $g \in G$.*

Theorem 2.4. ([N&S1]). *Let $(L, \cdot)$ be a C-loop with unit $e \in L$, the left translation group $G = G(L)$ and the stabilizer $H = G_e$. Let $\tilde{L} = (M, \circ)$ be a C-loop isotopic to L via (f, g, h). Then there exists $a, b \in L$ such that $\tilde{L}$ is isomorphic to the principal isotope $\tilde{\tilde{L}} = (L, \cdot_{(b,a)})$, $f(x \circ y) = f(x) \cdot_{(b,a)} f(y)$, and the stabilizer $\tilde{H}$ in G of the unit $\tilde{e} \in M$ is conjugated to H, $\tilde{H} = gHg^{-1}$ where $g(e) = a \cdot b \in L$.*

3. Bol algebras and Bol loops

3.1 Left Bol loops. Here we will restrict ourselves to the subclass of (analytic) left Bol loops. The one-to-one correspondence between the classes of right and left Bol loops is well known.

Definition 3.1. A (local) C-loop $(L, \cdot, \backslash, /, e)$ is called a *left Bol loop* if for any x, y from L (from a sufficiently small neighborhood of the unit e) the following condition is satisfied:

$$\lambda_x \lambda_y \lambda_x = \lambda_{x \cdot (y \cdot x)}. \tag{5}$$

The following statements are well known:

Theorem 3.1. ([Pf], p. 119). *Any loop isotopic to a left (right) Bol loop is itself a left (right) Bol loop.*

Theorem 3.2. ([M&S2]). *The complete left translation group of a differentiable Bol loop is a Lie group.*

Theorem 3.3. ([N&S1]). *Any loop $\tilde{L}$ isotopic to a differentiable Bol loop L is left isotopic to L.*

As a corollary we obtain the following.

Theorem 3.8. *For each couple $L(\cdot)$, $\tilde{L}(\tilde{\cdot})$ of global analytic Bol loops with isotopic tangent Bol algebras, there exists an analytic Bol loop $\tilde{\tilde{B}}$ such that $\tilde{B}$ is locally isomorphic to $\tilde{\tilde{B}}$, and $\tilde{\tilde{B}}$ is analytically isotopic to B.*

Let $(G, \circ, 1)$ be a (local) Lie group with the Lie algebra $\mathfrak{g}$, and let H be a connected Lie subgroup with a Lie algebra $\mathfrak{h}$, $\mathfrak{h} \subset \mathfrak{g}$. Denote by $q: G \to G/H$ a natural projection onto left cosets. Let us consider a vector subspace $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. Let U be a (sufficiently small) neighborhood of 0 in $\mathfrak{m}$. On the set $B = \exp U$ (regarded as an image of a local cross-section of q) a structure of a (left, right monoalternative) local analytic loop can be introduced ([M&S1], p. 427). If $a \circ b \circ a \in L = \exp U$ for $a, b \in L$ sufficiently close to the unit $e \in L$ then the local analytic loop L is a (left, right) Bol loop ([M&S1], XII.8.21 Proposition, p. 427). If we use the terminology introduced in the theory of left translation groups, $(\mathfrak{g}, \mathfrak{h})$ is an enveloping pair for the Bol algabra $\mathfrak{B}$ of L.

The following yields a useful tool for constructing examples.

Theorem 3.9. ("Sabinin's Theorem", [M&S1], p. 427, [N&S1], Lemma 1.9.) *Let L be a connected differentiable loop such that the set Λ of left translations topologically generates a connected Lie group G with the Lie algebra $\mathfrak{g}$. Denote by $\mathfrak{m} = T_1\Lambda \subset \mathfrak{g}$ the tangent space at $1 \in G$, $\Lambda = \exp \mathfrak{m}$. The loop L is a Bol loop if and only if*

$$[\mathfrak{m}, [\mathfrak{m}, \mathfrak{m}]] \subset \mathfrak{m}. \tag{6}$$

For any Lie group G and any inner automorphism $\mathrm{ad}(g)$, $g \in G$, a tangent mapping at unit $T_1\mathrm{ad}(g) = \mathrm{Ad}(g)$ is an automorphism of the tangent Lie algebra $\mathfrak{g} = T_1G$; $g \mapsto \mathrm{Ad}(g)$ is a linear representation, so called *adjoint representation* of G in $\mathfrak{g}$. We will also make use of the fact that for a *linear* Lie group, the adjoint representation is given by the formula $\mathrm{Ad}(g)(\mathfrak{a}) = g\mathfrak{a}g^{-1}$ for $\mathfrak{a} \in \mathfrak{g}$, $g \in G$, [F&V].

As a consequence of Theorems 2.3 and 3.3 we obtain:

Theorem 3.10. *Two loops $L = \exp \mathfrak{m}$, $\tilde{L} = \exp \tilde{\mathfrak{m}}$ with the same $G(L)$ are isotopic if and only if there exists an element $g \in G$ such that* $\mathrm{Ad}(g)(\mathfrak{m}) = \tilde{\mathfrak{m}}$.

Theorem 3.11. *Two Bol loops L_1 and L_2 with the same linear Lie group G as the complete left translation group and the same stabilizer are isotopic if and only if there exists an element $g \in G$ such that $g^{-1}\mathfrak{m}_1 g = \mathfrak{m}_2$.*

4. Hyperbolic Bol loops

Now let us illustrate the general theory. We will use the methods explained above for classificaiton of all 3-dimensional local hyperbolic Bol loops L (in the hyperbolic space $H^3 = \mathbb{R}^{3+}$) having the complete left translation group isomorphic to $PSL_2(\mathbb{C})$ and the stabilizer $H = G_e$ of unit element $e \in L$ isomorphic with the Lie subgroup SU_2.

4.1 The Iwasawa decomposition. It is well known that the (non-compact) connected Lie group $PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\pm E$ operates simply transitively on the 3-dimensional hyperbolic space H^3 where $SL_2(\mathbb{C}) = \{A \in GL_2(\mathbb{C}); \det A = 1\}$. The group of lower triangle matrices

$$\mathcal{K} = \left\{ \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}; a > 0,\ b \in \mathbb{C} \right\}, \quad \mathcal{K} \approx \mathbb{R}^{3^+} = \{(x, y, z) \in \mathbb{R}^3; z > 0\}, \mathcal{K} \simeq H^3$$

can be regarded as an underlying space of the 3-dimensional hyperbolic space. The well-known action of $SL_2(\mathbb{C})$ on $\mathbb{R}^3$ is given by linear rational functions

$$\gamma(w) = \frac{aw + b}{cw + d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ ad - bc = 1,\ w = x + jy,\ x \in \mathbb{C},\ y \in \mathbb{R}.$$

The restriction of this action onto an invariant subspace $\mathbb{R}^{3^+}$ defines the action of $SL_2(\mathbb{C})$ on H^3, and $(w, \pm\gamma) \mapsto \gamma(w)$ is the action of $PSL_2(\mathbb{C})$. $PSL_2(\mathbb{C})$ and $SL_2(\mathbb{C})$ are locally isomorphic, and have the same Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

For any connected semisimple complex Lie group G there exists a unique (Iwasawa) decomposition $G = T \cdot \mathfrak{A}$ where $\mathfrak{A}$ is a maximal compact subgroup of G and T is a maximal simply connected solvable subgroup in G [Iw, p. 525], [Na], [Zh], moreover all maximal compact subgroup in G are conjugated. Each $g \in G$ determines a unique couple $(t, a) \in T \times \mathfrak{A}$ such that $g = ta$, and the mapping $T \times \mathfrak{A} \to G$ given by $(t, a) \mapsto ta$ is a diffeomorphism. Consequently, any connected semisimple Lie group G is diffeomorphic to a direct product of an arbitrary maximal compact subgroup and a locally Euclidean space.

In our case, SU_2 is a maximal compact subgroup in $SL_2(\mathbb{C})$ [Zh&St p. 211], the Iwasawa decomposition is of the form

$$SL_2(\mathbb{C}) = (SU_2) \cdot \mathcal{K}, \tag{8}$$

and $SL_2(\mathbb{C}) \simeq (SU_2) \times \mathcal{K}$. Let us choose $H = SU_2$ as the stabilizer of unit and $\mathcal{K} \simeq SL_2/SU_2$ as the underlying space of a hyperbolic loop. The group $SL_2(\mathbb{C})$ acts simply transitively on the family of left cosets $SL_2(\mathbb{C})/SU_2 \simeq \mathcal{K}$.

Since we are interested in classification of Bol loops having $G(L)$ isomorphic to $G = PSL_2(\mathbb{C})$ by means of their tangent Bol algebras, we must give a survey on all 3-dimenisonal proper Bol algebras $\mathfrak{B}$ in a 6-dimensional Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Remark that the enveloping algebra of a 3-dimenisonal Bol algebra is at most 6-dimensional; remark that 3-dimensional Bol algebras of a special type with at most 5-dimensional canonical enveloping algebra were classified in [Bo].

Let $\mathfrak{B}$ be a 3-dimensional *proper* Bol algebra (i.e. no subalgebra) with the underlying vector subspace $\mathfrak{m}$ and the enveloping algebra $\mathfrak{g}$. With each of its elements, $u \in \mathfrak{m}$, the subspace $\mathfrak{m}$ contains the corresponding 1-dimensional subalgebra $\langle u \rangle$ and may contain 2-dimensional subalgebras, abelian or non-abelian.

4.2 The Lie group $SL_2(\mathbb{C})$ and the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Now let us characterize all 1- and 2-dimensional subalgebras in the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of trace-less matrices. In $\mathfrak{sl}_2(\mathbb{C})$ regarded as a 6-dimenisonal real vector space let us consider a "classical" basis $\{H, iH, T, iT, U, iU\}$ formed by the following Pauli spin matrices:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$iH = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad iT = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad iU = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The antisymmetric Lie algebra multiplication is given by the multiplication table

$$[H,T]=2U,\quad [H,U]=2T,\quad [U,T]=2H.$$

All 1-dimensional Lie subgroups in $SL_2(\mathbb{C})$ form three conjugate classes determined by the following Lie groups (involved in a complement to the sabilizer H):

$$\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}; a\in\mathbb{R}-\{0\}\right\},\ \left\{\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}; b\in\mathbb{R}\right\},\ \left\{\begin{pmatrix} e^{ia} & 0 \\ 0 & e^{-ia}\end{pmatrix}; a\in\mathbb{R}\right\}\approx\mathbb{S}^1.$$

The corresponding Lie algebras determine three conjugacy classes of 1-dimensional subalgebras in $\mathfrak{sl}_2(\mathbb{C})$:

$$\langle H\rangle=\left\{\begin{pmatrix} \sigma & 0 \\ 0 & -\sigma\end{pmatrix}\sigma\in\mathbb{R}\right\},\ \langle T+U\rangle=\left\{\begin{pmatrix} 0 & \tau \\ 0 & 0\end{pmatrix}\right\},\ \langle iH\rangle=\left\{\begin{pmatrix} i\sigma & 0 \\ 0 & -i\sigma\end{pmatrix}\right\}$$

where σ, $\tau\in\mathbb{R}$. All 2-dimensional non-abelian Lie subalgebras in $\mathfrak{sl}_2(\mathbb{C})$ form a conjugacy class given by the Lie algebra $\mathfrak{n}$ of the Lie group $\mathcal{N}$ determining the unique conjugacy class of 2-dimensional non-abelian Lie subgroups in $SL_2(\mathbb{C})$:

$$\mathfrak{n}=\langle H,T+U\rangle=\left\{\begin{pmatrix} \sigma & \mu \\ 0 & -\sigma\end{pmatrix}; \sigma,\mu\in\mathbb{R}\right\},\quad \mathcal{N}=\left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}; a>0, b\in\mathbb{R}\right\}.$$

Further, in $SL_2(\mathbb{C})$ there exist two conjugacy classes of 2-dimensional abelian Lie subgroups represented by

$$\left\{\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix}; z\in\mathbb{C}\right\}\approx\mathbb{C},\qquad \left\{\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix}; z\in\mathbb{C}\right\}$$

with the corresponding Lie algebras

$$\langle H,iH\rangle=\left\{\begin{pmatrix} \alpha+i\beta & 0 \\ 0 & -\alpha-i\beta\end{pmatrix}\right\},\ \langle T+U,i(T+U)\rangle=\left\{\begin{pmatrix} 0 & \alpha+i\beta \\ 0 & 0\end{pmatrix}\right\},$$

α, $\beta\in\mathbb{R}$. The Lie algebra of the stabilizer is $\mathfrak{h}=\mathfrak{su}_2=\langle U,iT,iH\rangle$, and one particular complement of $\mathfrak{h}$ is the Lie algebra $\mathfrak{k}=\langle T,iU,H\rangle$ of $\mathcal{K}$. So we can write an arbitrary complement in a general form

$$\begin{aligned}\mathfrak{m}=&\langle T+a_1U+b_1iT+c_1iH, iU+a_2U+b_2iT+c_2iH, H+a_3U+b_3iT+c_3iH\rangle\\ =&\{\mu(T+a_1U+b_1iT+c_1iH), \nu(iU+a_2U+b_2iT+c_2iH),\\ &\sigma(H+a_3U+b_3iT+c_3iH);\ \mu,\nu,\sigma\in\mathbb{R}\}.\end{aligned}$$

4.3 The classification of hyperbolic Bol loops with $SL_2(\mathbb{C})$ as the complete left translation group. As mentioned above, the classification of hyperbolic Bol loops with $SL_2(\mathbb{C})$ as the group topologically generated by left translations can be done through classification of 3-dimensional Bol algebras contained in $\mathfrak{sl}_2(\mathbb{C})$ and complementary to $\mathfrak{h}$. It is sufficient to verify which of the subalgebras listed above are involved in some of our complements, and for the particular complement, to find necessary and sufficient conditions under which the Bol property (7) holds. If a Lie algebra, let us say $\mathfrak{U}$, coincides with some of the subalgebras $\langle H\rangle$, $\langle T+U\rangle$, $\langle iH\rangle$, $\langle H, iH\rangle$, or $\langle T+U, i(T+U)\rangle$, and is involved in some of our complements $\mathfrak{m}$ satisfying the Bol condition then any conjugated algebra $\mathrm{Ad}(g)\mathfrak{U} = g\mathfrak{U}g^{-1}$ is contained in a conjugated complement $\mathrm{ad}(g)\mathfrak{m} = g\mathfrak{m}g^{-1}$ determining an isotopic Bol loop.

Case (I1): If the algebra $\langle H\rangle$ is contained in a cmplement $\mathfrak{m}$ to the Lie algebra $\mathfrak{h}$ of the stabilizer then $a_3 = b_3 = c_3 = 0$ (and the parameters are $\mu = \nu = 0$, $\sigma = \tau$). The complement

$$\mathfrak{m} = \langle T + a_1U + b_1iT + c_1iH, iU + a_2U + b_2iT + c_2iH, H\rangle$$

satisfies the Bol condition if and only if $a_1 = b_2$, $a_2 = b_1 = c_1 = c_2 = 0$. The choice $a_1 = 1$ yields a subalgebra $\langle U+T, i(U+T), H\rangle \supset \langle U+T, i(U+T)\rangle$. For $a = a_1 \neq 1$ the complements are no subalgebras since the Lie products $[T+aU, H] = -2(U+aT)$, $[iU + aiT, H] = -2(iT + aiU)$, $[T + aU, iU + aiT] = (2a^2 - 2)iH$ are not involved in $\mathfrak{m}$, and each subspace

$$\text{(9)} \qquad \mathfrak{m}_a = \langle T + aU, iU + aiT, H\rangle, \qquad a = a_1 \in \mathbb{R} - \{1\}$$

spans the whole algebra $\mathfrak{sl}_2(\mathbb{C})$.

Case (I2): The algebra $\langle U+T\rangle$ is involved (for parameters $\mu = \frac{1}{2}\tau$, $\nu = \sigma = 0$) in complements with $a_1 = 1$, $b_1 = c_1 = 0$,

$$\mathfrak{m} = \langle T + U, iU + a_2U + b_2iT + c_2iH, H + a_3U + b_3iT + c_3iH\rangle.$$

The Bol condition is satisfied if and only if either

Case (I2a): $b_2 = 1$, $a_2 = c_2 = a_3 = b_3 = 0$, that is $\tilde{\mathfrak{m}}_c = \langle U+T, i(T+U), H+ciH\rangle = i\mathfrak{m}_c$, $c = c_3 \in \mathbb{R}$ is a subalgebra containing $\langle U+T, i(U+T)\rangle$ (and gives *no* proper Bol loop), or

Case (I2b): $b_2 = 1$, $a_2 = b_3 = c_3 = 0$, $d = a_3 = c_2$. In this case we obtain either the subalgebra $\langle U+T, i(U+T), H\rangle$ if $d = 0$, or a family of *proper* Bol algebras

$$\text{(10)} \qquad \mathfrak{m}_d = \langle U + T,\ diH + i2(U+T),\ H + dU\rangle \qquad \text{for } d \in \mathbb{R} - \{0\}$$

with Lie products $[H + dU, U + T] = dH + 2(U+T)$, $[diH + 2i(U+T)] = 2d(U+T)$, $[H + dU, diH + 2i(U+T)] = 4i(U+T) - 2d^2iT + 2diH$.

Case (I3): No complement contains $\langle iH\rangle$.

Case (II1): No of our complements contains a 2-dimensional algebra $\langle H, T+U\rangle$.

Case (II2): $\langle H, iH\rangle$ is involved in no complements.

Case (II3): The subalgebra $\langle T+U, i(T+U)\rangle$ is (for parameters $\mu = \frac{1}{2}\tau_1$, $\nu = \frac{1}{2}\tau_2$, $\sigma = 0$) involved in complements of the form

$$\mathfrak{m} = \langle T + U, iT + iU, H + a_3U + b_3iT + c_3iH\rangle.$$

The Bol condition is satisfied if and only if moreover the conditions $a_3 = b_3 = 0$ hold. This conditions determine subalgebras $\tilde{\mathfrak{m}}_c = i\mathfrak{m}_c = \langle T+U, i(T+U), H+ciH\rangle$, $c = c_3 \in \mathbb{R}$.

Example 4.1. The subspaces $\mathfrak{m}_a$, $a \neq 1$ and $\mathfrak{m}_d$, $d \neq 0$ are examples of 3-dimensional Bol algebras with a 6-dimensional enveloping Lie algebra and an enveloping pair $(\mathfrak{sl}_2(\mathbb{C}), \mathfrak{su}_2)$.

Theorem 4.1. *In $\mathfrak{sl}_2(\mathbb{C})$ there exists two conjugacy classes of proper Bol algebras complementary to $\mathfrak{h}$ given by 1-parameter families of Lie algebras*

$$\mathfrak{m}_a, \quad a \neq 1, \quad \text{respectively } \mathfrak{m}_d, \quad d \neq 0.$$

Theorem 4.2. *On the 3-dimensional hyperbolic space H^3 there exist two isotopy classes of (local) proper Bol loops given by local sections $SL_2(\mathbb{C})/SU_2 \to SL_2(\mathbb{C})$ They correspond to 1-parameter families of Lie algeras*

$$\mathfrak{m}_a, \quad a \neq 1, \quad \text{respectively } \mathfrak{m}_d, \quad d \neq 0. \tag{11}$$

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