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ON HOLOMORPHICALLY PROJECTIVE MAPPINGS ONTO KÄHLERIAN SPACES

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ABSTRACT. In this paper we consider holomorphically projective mappings from equiaffine spaces onto (pseudo-) Kählerian spaces. We found the equations of these mappings in the form of a system of linear Cauchy equations. These results generalize the results obtained for holomorphically projective mappings of Kählerian spaces by J. Mikeš and V.V. Domashev. We continue the investigations of the F-planar mappings onto Hermitian and Riemannian spaces.

The concept of degree of mobility of equiaffine space relative to holomorphically projective mappings was defined and some intervals confining its value were found.

1. INTRODUCTION

Diffeomorphisms and automorphisms of geometrically generalized spaces constitute one of the current main directions in differential geometry. A large number of papers are devoted to geodesic, quasigeodesic, almost geodesic, holomorphically projective and other mappings (see [3], [8], [9], [12], [14], [16]). On the other hand, one line of thought is now the most important one, namely, the investigation of special affine-connected, Riemannian, Kählerian and Hermitian spaces.

In this paper, we present some new results obtained for holomorphically projective mappings from equiaffine spaces A_n onto Kählerian spaces $\bar{K}_n$.

By the term *Hermitian spaces* H_n, we denote all (pseudo-) Riemannian spaces, where an affinor structure F_i^h exists, for which the conditions

$$F_\alpha^h F_i^\alpha = -\delta_i^h\,, \quad g_{\alpha(i}F_{j)}^\alpha = 0 \tag{1}$$

hold. Here g_{ij} is the metric tensor on H_n, δ_i^h is the Kronecker symbol and (i,j) denotes a symmetrization without division.

A natural classification containg 16 types of Hermitian spaces has been done by A. Gray and L.M. Hervella [4]. *Kählerian spaces* K_n are special cases of Hermitian spaces, which have a covariantly constant structure F_i^h.

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In many papers holomorphically projective mappings and transformations of Hermitian spaces $H_n \to \bar{H}_n$ are studied (for example see [2], [5], [6], [9], [11], [14], [16]). These are special cases of F_1-planar mappings. In [7], [9], F_1-planar mappings from the space A_n with affine connection onto a Riemannian space $\bar{V}_n$ are defined and studied. These are characterized w.r.t. a common coordinate system x by the following equations

$$\text{(2)} \qquad \text{a)}\ \ \bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_{(i}\psi_{j)} + F^h_{(i}\varphi_{j)}\,, \qquad \text{b)}\ \ \bar{g}_{\alpha(i}F^\alpha_{j)} = 0\,,$$

where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are the objects of affine connection on A_n and $\bar{V}_n$, respectively, $\bar{g}_{ij}$ is the metric tensor of $\bar{V}_n$, $\psi_i(x)$, $\varphi_i(x)$ are covectors, and $F^h_i(x)$ $(\text{Rank}\|F^h_i - \rho\delta^h_i\| > 1)$ is the affinor structure on A_n and $\bar{V}_n$.

Equations (2) are equivalent to the equations

$$\text{(3)} \qquad \text{a)}\ \ \bar{g}_{ij,k} = 2\psi_k\bar{g}_{ij} + \psi_{(i}\bar{g}_{j)k} + \varphi_{(i}\bar{F}_{j)k}\,, \qquad \text{b)}\ \ \bar{g}_{\alpha(i}F^\alpha_{j)} = 0\,,$$

where $\bar{F}_{ij} \overset{\text{def}}{=} \bar{g}_{i\alpha}F^\alpha_j$. Here and in what follows comma denotes covariant derivative on A_n.

In [7] it is proved that a general solution of the system (3) for a given space A_n and a given structure F^h_i depends on finitely many parameters.

An F_1-planar mapping is called *F_2-planar* if the covector ψ_i is a gradient, i.e. $\psi_i = \partial\psi/\partial x^i$, and *$F_3$-planar* if $\psi_i = \varphi_\alpha F^\alpha_i$. If A_n is an equiaffine space, then an F_3-planar mapping is F_2-planar. The following Theorem holds [7]:

Theorem 1. *An equiaffine space A_n admits F_3-planar mappings onto $\bar{V}_n$ if only if a regular symmetric tensor a^{ij} and a vector ξ^i satisfy the following equations:*

$$\text{(4)} \qquad \text{a)}\ \ a^{ij}{}_{,k} = \xi^\alpha F^{(i}_\alpha\delta^{j)}_k + \xi^{(i}F^{j)}_k\,, \qquad \text{b)}\ \ a^{\alpha(i}F^{j)}_\alpha = 0\,.$$

Solutions of (3) and (4) are connected by relations $a^{ij} = \mathrm{e}^{-2\psi}\bar{g}^{ij}$, $\xi^i = -\mathrm{e}^{-2\psi}\bar{g}^{i\alpha}\varphi_\alpha$, where $\|\bar{g}^{ij}\| = \|\bar{g}_{ij}\|^{-1}$.

2. Holomorphically projective mappings

An F_3-planar mapping from a space A_n with affine connection onto a Hermitian space $\bar{H}_n$, for which formulas (2) are satisfied and F^h_i is the almost complex structure of $\bar{H}_n$, is called a *holomorphically projective mapping* (*HPM*).

For this mapping it holds:

$$\text{(5)} \qquad F^h_{i,j} = F^h_{i|j}\,,$$

where "," and "|" are the covariant derivatives in A_n and $\bar{H}_n$ respectively.

In what follows we will study holomorphically projective mappings from an equiaffine space A_n onto a Kählerian space $\bar{K}_n$. In this case (5) implies

$$\text{(6)} \qquad F^h_{i,j} = 0\,.$$

The integrability condition of (6) has the form $F^h_\alpha R^\alpha_{ijk} = F^\alpha_i R^h_{\alpha jk}$, where R^h_{ijk} is the Riemannian tensor on A_n. These conditions are equivalent to

$$\text{(7)} \qquad F^h_\alpha F^\beta_i R^\alpha_{\beta jk} = -R^h_{ijk}\,.$$

We shall investigate the integrability condition of equation (4a). Let us derive them covariantly by x^l and then alternate them w.r. to the indices k and l. With respect to the Ricci identity we find the following:

$$\lambda^{(i}_{,l}\delta^{j)}_{k} - \lambda^{(i}_{,k}\delta^{j)}_{l} + \lambda^{\alpha}_{,l}F^{(i}_{\alpha}F^{j)}_{k} - \lambda^{\alpha}_{,k}F^{(i}_{\alpha}F^{j)}_{l} = -a^{\alpha(i}R^{j)}_{\alpha kl}\,, \tag{8}$$

where $\lambda^i \overset{\text{def}}{=} \xi^\alpha F^i_\alpha$ $(\equiv -e^{-2\psi}\bar{g}^{i\alpha}\psi_\alpha)$.

Contracting (8) by the indices j and k, we obtain

$$(n-1)\lambda^i_{,l} + \lambda^\alpha_{,\beta}F^\beta_l F^i_\alpha = \mu\delta^i_l + \lambda^\alpha_{,\beta}F^\beta_\alpha F^i_l + a^{\alpha i}R_{\alpha l} - a^{\alpha\beta}R^i_{\alpha\beta l}\,, \tag{9}$$

where $\mu \overset{\text{def}}{=} \lambda^\alpha_{,\alpha}$, $R_{ij} \overset{\text{def}}{=} R^\alpha_{ij\alpha}$ is the Ricci tensor, which is symmetric in the equaffine space A_n.

When we contract (9) with F^l_i and then we use properties of Riemannian and Ricci tensors, we can see $\lambda^\alpha_{,\beta}F^\beta_\alpha = 0$. Therefore we can simplify (9)

$$(n-1)\lambda^i_{,l} + \lambda^\alpha_{,\beta}F^\beta_l F^i_\alpha = \mu\delta^i_l + a^{\alpha i}R_{\alpha l} - a^{\alpha\beta}R^i_{\alpha\beta l}\,, \tag{10}$$

After contracting (10) with $F^{i'}_i F^j_{j'}$ we shall omit symbols the primes and add the obtained form to (10). After editing we have

$$\lambda^\alpha_{,\beta}F^i_\alpha F^\beta_l = -\lambda^i_{,l} + \frac{1}{n}(a^{\alpha i}R_{\alpha l} - a^{\alpha\beta}R^i_{\alpha\beta l} + a^{\alpha\gamma}R_{\alpha\delta}F^i_\gamma F^\delta_l - a^{\alpha\beta}R^\gamma_{\alpha\beta\delta}F^i_\gamma F^\delta_l)\,. \tag{11}$$

Substituting (11) to (10), we find

$$n\,\lambda^i_{,l} = \mu\,\delta^i_l + a^{\alpha\beta}\,\overset{1}{T}{}^i_{l\alpha\beta}\,, \tag{12}$$

where

$$\overset{1}{T}{}^i_{l\alpha\beta} \overset{\text{def}}{=} \frac{n-1}{n}(\delta^i_\beta R_{\alpha l} - R^i_{\alpha\beta l}) + \frac{1}{n}(\delta^i_\beta R_{\gamma\delta}F^\gamma_l F^\delta_\alpha + R^\gamma_{\alpha\beta\delta}F^i_\gamma F^\delta_l)\,. \tag{13}$$

Further we covariantly differentiate (12) by x^m, and after alternation of the indices l and m and application of Ricci identities and (4) we obtain:

$$-n\lambda^\alpha R^i_{\alpha lm} = \delta^i_l\mu_{,m} - \delta^i_m\mu_{,l} + a^{\alpha\beta}(\overset{1}{T}{}^i_{l\alpha\beta,m} - \overset{1}{T}{}^i_{m\alpha\beta,l}) + \lambda^\alpha\,\overset{2}{T}{}^i_{\alpha lm}\,, \tag{14}$$

where

$$\begin{aligned}\overset{2}{T}{}^i_{\alpha lm} \overset{\text{def}}{=}\ & \overset{1}{T}{}^i_{l\alpha m} + \overset{1}{T}{}^i_{lm\alpha} + F^\beta_\alpha\,\overset{1}{T}{}^i_{l\beta m} + F^\beta_\alpha\,\overset{1}{T}{}^i_{lm\beta} - \\ & \overset{1}{T}{}^i_{m\alpha l} - \overset{1}{T}{}^i_{ml\alpha} - F^\beta_\alpha\,\overset{1}{T}{}^i_{m\beta l} - F^\beta_\alpha\,\overset{1}{T}{}^i_{ml\beta}\,.\end{aligned} \tag{15}$$

We contract formula (14) w.r. to the indices i and m, and we get

$$(n-1)\,\mu_{,l} = n\lambda^\alpha R_{\alpha l} + a^{\alpha\beta}(\overset{1}{T}{}^\gamma_{l\alpha\beta,\gamma} - \overset{1}{T}{}^\gamma_{\gamma\alpha\beta,l}) + \lambda^\alpha\,\overset{2}{T}{}^\gamma_{\alpha l\gamma}\,.$$

The following theorem is the result of previous computations and Theorem 1.

Theorem 2. *Let A_n be an equiaffine space with affine connection and let be defined a covariantly constant affinor F^h_i such that $F^h_\alpha F^\alpha_i = -\delta^h_i$. Then A_n admits a holomorphically projective mapping onto a Kählerian spaces $\bar{K}_n$ if only if the following system* *of linear differential equations of Cauchy type is solvable with respect to the unknown functions a^{ij}, λ^i and μ:*

$$
\begin{aligned}
& a^{ij}{}_{,k} = \lambda^{(i}\delta^{j)}_k + \lambda^\alpha F^{(i}_\alpha F^{j)}_k\,; \\
(16) \qquad & n\,\lambda^i{}_{,l} = \mu\,\delta^i_l + a^{\alpha\beta}\,\overset{1}{T}{}^i_{l\alpha\beta}\,; \\
& (n-1)\,\mu_{,l} = n\,\lambda^\alpha R_{\alpha l} + a^{\alpha\beta}(\overset{1}{T}{}^\gamma_{l\alpha\beta,\gamma} - \overset{1}{T}{}^\gamma_{\gamma\alpha\beta,l}) + \lambda^\alpha\,\overset{2}{T}{}^\gamma_{\alpha l\gamma}\,;
\end{aligned}
$$

where the matrix (a^{ij}) should further satisfy $\det\|a^{ij}\| \neq 0$ *and the algebraic conditions*

$$(17) \qquad a^{ij} = a^{ji}; \quad a^{ij} = a^{\alpha\beta}F^i_\alpha F^j_\beta\,.$$

Here $\overset{\sigma}{T}$, $\sigma = 1, 2$, are tensors which are explicitly expressed as of the objects defined in A_n: (13) *and* (15), *i.e. affine connection A_n and affinor F^h_i.*

This Theorem is a generalization of results in [2], [9], [14], [15].

The system (16) does not have more than one solution for the initial Cauchy conditions $a^{ij}(x_o) = a^{ij}_o$, $\lambda^i(x_o) = \lambda^i_o$, $\mu(x_o) = \mu_o$ under the conditions (17). Therefore the general solution of (16) does not depend on more than $N_o = 1/4\,(n+1)^2$ parameters. The question of existence of a solution of (16) leads to the studium of integrability conditions, which are linear equations w.r. to the unknowns a^{ij}, λ^i and μ with coefficients from the space A_n.

3. On the degrees of mobility of equiaffine spaces relative to holomorphically projective mappings onto Kählerian spaces

The number of essential parameters $r \leq N_o$ of a solution of (16) is called *degree of mobility of an equiaffine space A_n relative to holomorphically projective mappings onto Kählerian spaces.* For Kählerian spaces K_n of constant holomorphic curvature holds $r = N_o$ [2], [9], [14], [15].

Let in an equiaffine space A_n the condition

$$(18) \qquad R^h_{ijk} = \delta^h_i\,\overset{1}{v}_{jk} + \delta^h_j\,\overset{2}{v}_{ik} - \delta^h_k\,\overset{2}{v}_{ij} + F^h_i\,\overset{3}{v}_{jk} + F^h_j\,\overset{4}{v}_{ik} - F^h_k\,\overset{4}{v}_{ij}$$

hold, where $\overset{\sigma}{v}$ are tensors.

It is easy to prove the following

Lemma 1. *If an equiaffine space A_n with condition* (18) *admits a holomorphically projective mapping onto a Kählerian space $\bar{K}_n$, then $\bar{K}_n$ has constant holomorphic curvature.*

Thus if an equiaffine space A_n admit a holomorphically projective mapping onto Kählerian spaces of constant holomorphic curvature then from the previous Lemma and remark follows, that A_n has degree $r = N_o$.

In the following we will show, that the degree r of A_n, which does not admit holomorphically projective mappings onto Kählerian spaces of constant holomorphic curvature, is essentially reduced. First we will show the next

Lemma 2. *If A_n does not admit holomorphically projective mappings onto Kählerian spaces of constant holomorphic curvature, then all components of the vector λ^i depend on components of the tensor a^{ij} and geometric objects of A_n.*

This Lemma follows from the first continuation of the integrability condition (8).

The symmetric tensor a^{ij} must correspond, beside condition (17), to equations (8). These conditions give some new requirements for a^{ij}. By complicated computations from (8), analogically to [6], [14], we can prove the validity of

Lemma 3. *If A_n does not admit a holomorphically projective mapping onto Kählerian spaces of constant holomorphic curvature, then the system* (8) *contains at least $n-6$ independent equations with components of a symmetric tensor a^{ij}, which is conform to* (17).

The next theorem follows from Lemma 2 and Lemma 3.

Theorem 3. *If A_n does not admit a holomorphically projective mapping onto Kählerian spaces of constant holomorphic curvature, then the degree of A_n of a holomorphically projective mapping onto Kählerian spaces corresponds to the inequality:*

$$r \leq \frac{n^2}{4} - n + 5 .$$

This Theorem is a generalization of results which we have for Kählerian spaces.

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