Dusa McDuff

Lectures on groups of symplectomorphisms

In: Jan Slovák and Martin Čadek (eds.): Proceedings of the 23rd Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 2004. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 72. pp. [43]--78.

Persistent URL: http://dml.cz/dmlcz/701723

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 72 (2004), pp. 43-78

LECTURES ON GROUPS OF SYMPLECTOMORPHISMS

DUSA MCDUFF

ABSTRACT. These notes combine material from short lecture courses given in Paris, France, in July 2001 and in Srni, the Czech Republic, in January 2003. They discuss groups of symplectomorphisms of closed symplectic manifolds (M,ω) from various points of view. Lectures 1 and 2 provide an overview of our current knowledge of their algebraic, geometric and homotopy theoretic properties. Lecture 3 sketches the arguments used by Gromov, Abreu and Abreu–McDuff to figure out the rational homotopy type of these groups in the cases $M = \mathbb{C}P^2$ and $M = S^2 \times S^2$. We outline the needed J-holomorphic curve techniques. Much of the recent progress in understanding the geometry and topology of these groups has come from studying the properties of fibrations with the manifold M as fiber and structural group equal either to the symplectic group or to its Hamiltonian subgroup $\mathrm{Ham}(M)$. The case when the base is S^2 has proved particularly important. Lecture 4 describes the geometry of Hamiltonian fibrations over S^2, while Lecture 5 discusses their Gromov-Witten invariants via the Seidel representation. It ends by sketching Entov's explanation of the ABW inequalities for eigenvalues of products of special unitary matrices. Finally in Lecture 6 we apply the ideas developed in the previous two lectures to demonstrate the existence of (short) paths in $\mathrm{Ham}(M,\omega)$ that minimize the Hofer norm over all paths with the given endpoints.

CONTENTS

2000 *Mathematics Subject Classification*: 57R17, 53D35.
Key words and phrases: symplectomorphism group, Hamiltonian group, Hofer metric, symplectic fibrations, quantum homology, Seidel representation.
Partially supported by NSF grant DMS 0072512.
The paper is in final form and no version of it will be submitted elsewhere.

Acknowledgements. The author warmly thanks the organisers of the schools in Jussieu and Srni for the wonderful working atmosphere that they created. She also thanks Jarek Kędra for some useful comments.

1. Overview

There are many different aspects to the study of groups of symplectomorphisms. One can consider

- algebraic properties;
- geometric properties;
- their homotopy type;
- their stability under perturbations of the symplectic form.

One could also look at the dynamical properties of individual elements or of one parameter subgroups. But we will not emphasize such questions here, instead concentrating on the above mentioned properties of the whole group. In this article we will first survey some old and new results, mentioning some open problems, and then will sketch some of the relevant proofs. Background information and more references can be found in [45, 46, 47]. Since the survey [44] discusses recent results on the homotopy properties of the action of the Hamiltonian group on the underlying manifold M, this aspect of the theory will be only briefly mentioned here.

1.1. **Basic notions.** Throughout (M,ω) will be a closed (i.e. compact and without boundary), smooth (i.e. C^∞), symplectic manifold of dimension $2n$ unless it is explicitly mentioned otherwise. The symplectomorphism group $\mathrm{Symp}(M,\omega)$ consists of all diffeomorphisms $\phi : M \to M$ such that $\phi^*(\omega) = \omega$, and is equipped with the C^∞-topology. Its identity component is denoted $\mathrm{Symp}_0(M,\omega)$. The latter contains an important subgroup $\mathrm{Ham}(M,\omega)$ whose elements are time 1 maps of Hamiltonian

flows ϕ_t^H. These are the flows $\phi_H^t, t \in [0,1]$, that at each time t are tangent to the symplectic gradient X_t^H of the function $H_t : M \to \mathbb{R}$, i.e.

$$\frac{\partial}{\partial t}\phi_t^H(x) := \dot{\phi}_t^H\big(\phi_t^H(x)\big) = X_t^H\big(\phi_t^H(x)\big), \qquad \omega(X_t^H, \cdot) = -dH_t. \tag{1}$$

Thus we have a sequence of groups and inclusions

$$\mathrm{Ham}(M,\omega) \hookrightarrow \mathrm{Symp}_0(M,\omega) \hookrightarrow \mathrm{Symp}(M,\omega) \hookrightarrow \mathrm{Diff}(M).$$

These groups are all infinite dimensional. As explained in Milnor [50], they can each be given the structure of a Fréchet Lie group. As such they have well defined Lie algebras, with an exponential map. For example, the Lie algebra of $\mathrm{Ham}(M,\omega)$ consists of the space $C_0^\infty(M,\mathbb{R})$ of smooth functions on M with zero mean $\int_M H\omega^n$, and the exponential map takes the (time independent) Hamiltonian H to the time 1-map ϕ_1^H of the corresponding flow. Observe that this map is never surjective.

The most important elementary theorems in symplectic geometry are:

• **Darboux's theorem**: *every symplectic form is locally diffeomorphic to the linear form*

$$\omega_0 := dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_{2n}$$

on Euclidean space; and

• **Moser's theorem**: *any path $\omega_t, t \in [0,1]$, of cohomologous symplectic forms on a closed manifold M is induced by an isotopy $\phi_t : M \to M$ of the underlying manifold, i.e. $\phi_t^*(\omega_t) = \omega_0$, $\phi_0 = \mathrm{id}$. Forms ω_0, ω_1 that are related in this way are called* **isotopic**.

The fact that there are no local invariants of symplectic structures is closely related to the fact that the symplectomorphism group is infinite dimensional. Contrast this with Riemannian geometry in which the curvature is a local invariant and isometry groups are always finite dimensional. Moser's theorem implies that the groups $\mathrm{Symp}(M,\omega)$ and $\mathrm{Ham}(M,\omega)$ depend only on the diffeomorphism class of the form ω. In particular, they do not change their topological or algebraic properties when ω_t varies along a path of cohomologous forms. However, changes in the cohomology class $[\omega]$ can cause significant changes in the homotopy type of these groups: see Proposition 2.3. In turn, this is closely related to the fact that the Moser theorem fails for families of noncohomologous forms. As shown by McDuff [37] (cf. [47, Chapter 9]), there is a family of noncohomologous symplectic forms $\omega_t, 0 \le t \le 1$, on $S^2 \times S^2 \times T^2$ such that ω_0 is cohomologous to ω_1 but not isotopic to it. Here ω_1 is constructed to be diffeomorphic to ω_0, but the construction can be modified to give a family of symplectic forms $\omega_t, 0 \le t \le 1$, on an 8-dimensional manifold such that ω_0 is cohomologous to ω_1 but not diffeomorphic to it.

Another basic point is that each of these groups have the homotopy type of a countable CW complex (cf. [47, Chapter 9.5]).

1.2. **Algebraic aspects.** Let $\widetilde{\mathrm{Symp}_0}(M)$ denote the universal cover of $\mathrm{Symp}_0(M)$.[1] Its elements $\widetilde{\phi}$ are equivalence classes of paths $\{\phi_t\}_{t\in[0,1]}$ starting at the identity, where

[1] We shall often drop ω from the notation when it can be understood from the context.

$\{\phi_t\} \sim \{\phi_t'\}$ iff $\phi_1 = \phi_1'$ and the paths are homotopic with fixed endpoints. We define

$$\mathrm{Flux}(\widetilde{\phi}) = \int_0^1 [\omega(\dot{\phi}_t, \cdot)] \in H^1(M, \mathbb{R}) .$$

One can check that $\mathrm{Flux}(\widetilde{\phi})$ is independent of the choice of representative for $\widetilde{\phi}$, and that the map $\widetilde{\phi} \mapsto \mathrm{Flux}(\widetilde{\phi})$ defines a homomorphism $\widetilde{\mathrm{Symp}_0}(M) \to H^1(M, \mathbb{R})$. This is known as the **Flux homomorphism**. One way to see that $\mathrm{Flux}(\widetilde{\phi})$ is well defined is to use the following alternate description. Since $\mathrm{Flux}(\widetilde{\phi})$ is a cohomology class, it is determined by its values on loops $s \mapsto \gamma(s), s \in S^1$, in M, and one can check that

$$\mathrm{Flux}(\widetilde{\phi})(\gamma) = \int_{\mathrm{tr}_\phi(\gamma)} \omega , \tag{2}$$

where $\mathrm{tr}_\phi(\gamma)$ is the 2-cycle

$$[0,1] \times S^1 \to M : (t, s) \mapsto (\phi_t(\gamma(s)) .$$

(For a proof of this and the other basic results in this section see [46, Chapter 10].)

One of the first results in the theory is that the rows and columns in the following commutative diagram are short exact sequences of groups.

$$\begin{array}{ccccc} \pi_1(\mathrm{Ham}(M)) & \longrightarrow & \pi_1(\mathrm{Symp}_0(M)) & \xrightarrow{\mathrm{Flux}} & \Gamma_\omega \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{\mathrm{Ham}}(M) & \longrightarrow & \widetilde{\mathrm{Symp}_0}(M) & \xrightarrow{\mathrm{Flux}} & H^1(M, \mathbb{R}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ham}(M) & \longrightarrow & \mathrm{Symp}_0(M) & \xrightarrow{\mathrm{Flux}} & H^1(M, \mathbb{R})/\Gamma_\omega . \end{array} \tag{3}$$

Here Γ_ω is the so-called **flux group**. It is the image of $\pi_1(\mathrm{Symp}_0(M))$ under the flux homomorphism, and so far is not completely understood. In particular, it is not yet known whether Γ_ω is always discrete. This question is discussed further in §1.4.

One might wonder what other "natural" homomorphisms there are from $\mathrm{Symp}_0(M)$ to a arbitrary group G. If M is closed, the somewhat surprising answer here is that *every* nontrivial homomorphism must factor through the flux homomorphism. Equivalently, $\mathrm{Ham}(M)$ is *simple*, i.e. it has no proper normal subgroups. The statement that $\mathrm{Ham}(M)$ has no proper closed normal subgroups is relatively easy and was proved by Calabi [9]. The statement that it has no proper normal subgroups at all is much more subtle and was proved by Banyaga [4] following a method introduced by Thurston to deal with the group of volume preserving diffeomorphisms. The proof uses the relatively accessible fact[2] that the commutator subgroup of $\mathrm{Ham}(M)$ is simple and the much deeper result that $\mathrm{Ham}(M)$ is a *perfect* group, i.e. is equal to its commutator subgroup. More recently, Banyaga [5] has shown that the manifold (M, ω) may be recovered from the abstract discrete group $\mathrm{Symp}(M, \omega)$. In other words,

Proposition 1.1. *If $\Phi : \mathrm{Symp}(M, \omega) \to \mathrm{Symp}(M', \omega')$ is a group homomorphism, then there is a diffeomorphism $f : M \to M'$ such that*

$$f^*(\omega') = \pm\omega, \qquad \Phi(\phi) = f \circ \phi \circ f^{-1} \quad \textit{for all} \quad \phi \in \mathrm{Symp}(M, \omega) .$$

[2] A general result due to Epstein [16] states that if a group G of compactly supported homeomorphisms satisfies some natural axioms then its commutator subgroup is simple.

When M is noncompact, the group $\mathrm{Symp}_0(M,\omega)$ has many normal subgroups, for example the subgroup $\mathrm{Symp}_0(M,\omega) \cap \mathrm{Symp}^c(M,\omega)$ of all its compactly supported elements. The identity component $\mathrm{Symp}_0^c(M,\omega)$ of the latter group supports a new homomorphism onto $\mathbb{R}$ called the Calabi homomorphism, and Banyaga showed that the kernel of this homomorphism is again a simple group. However, there is very little understanding of the normal subgroups of the full group $\mathrm{Symp}(M)$. In view of the above discussion of the closed case the most obvious question is the following.

Question 1.2. *Is* $\mathrm{Symp}(\mathbb{R}^{2n},\omega_0)$ *a perfect group?*

It is known that the group of volume preserving diffeomorphisms of $\mathbb{R}^k, k \geq 3$, is perfect and is generated by elements whose support lies in a countable union of disjoint closed balls of radius 1: see McDuff [36] and Mascaro [35]. (The same holds for the group of diffeomorphisms of $\mathbb{R}^n$ in any dimension.) However, the first of these statements is unknown in the symplectic case (even in the case of $\mathbb{R}^2$!) while the second is false: see Barsamian [6].

Some other questions of an algebraic nature are beginning to be tractable. Using ideas of Barge and Ghys, Entov [14] has recently shown that in the closed case the perfect groups $\mathrm{Symp}_0(M,\omega)$, $\mathrm{Ham}(M,\omega)$ as well as their universal covers all have infinite commutator length, while in [62] Polterovich develops methods to estimate the word length of iterates $f^{\circ n}$ in finitely generated subgroups of $\mathrm{Ham}(M,\omega)$. This leds to interesting new restrictions on manifolds that support symplectic actions of nonamenable groups such as $\mathrm{SL}(2,\mathbb{R})$.

Entov and Polterovich [15] have also recently constructed a a nontrivial continuous **quasimorphism** μ on $\mathrm{Symp}(S^2)$. A quasimorphism on a group G is a map $\mu : \mathrm{G} \to \mathbb{R}$ that is a bounded distance away from being a homomorphism, i.e. there is a constant $c = c(\mu) > 0$ such that

$$|\mu(gh) - \mu(g) - \mu(h)| < c, \qquad g,h \in \mathrm{G}.$$

There is no such homomorphism because $\mathrm{Symp}(S^2)$ is simple. Their construction uses the structure of the quantum cohomology ring of S^2, and it is not yet clear whether it can be extended to all symplectic manifolds.

1.3. **Geometric aspects.** The Lie algebra of the group $\mathrm{Symp}(M)$ is the space of all symplectic vector fields X, i.e. the vector fields on M such that the 1-form $\iota_X(\omega)$ is closed. Similarly, the Lie algebra $\mathrm{LieHam}(M)$ is the space of all Hamiltonian vector fields X, i.e. those for which the 1-form $\iota_X(\omega)$ is exact. Since each exact 1-form may be uniquely written as dH where H has zero mean $\int_M H\omega^n$, $\mathrm{LieHam}(M)$ may also be identified with the space $C_0(M)$ of smooth functions on M with zero mean. With this interpretation, one easily sees that $\mathrm{LieHam}(M)$ has a nondegenerate inner product

$$\langle H,K\rangle = \int_M HK\omega^n \tag{4}$$

that is bi-invariant under the adjoint action

$$\mathrm{Ad}_\phi(H) = H \circ \phi$$

of $\phi \in \mathrm{Symp}(M)$. Since a finite dimensional semisimple Lie group is compact if and only if it has such an inner product, this suggests that $\mathrm{Ham}(M)$ is an infinite dimensional analog of a compact group. Of course this analogy is not perfect: as we

shall see in §2.1 below, $\mathrm{Ham}(M)$ does not always have the homotopy type of a compact Lie group since it can have infinite cohomological dimension. Nevertheless, it seems interesting to try to compare its properties with those of a compact Lie group.

One way in which $\mathrm{Ham}(M)$ does resemble a compact Lie group is that it supports a bi-invariant Finsler metric known as the **Hofer metric**: see [20]. To define this, consider a path $\{\phi_t^H\}_{t\in[0,1]}$, in $\mathrm{Ham}(M)$ generated by the function $\{H_t\}_{t\in[0,1]}$. Assuming that each H_t has zero mean $\int_M H_t\omega^n$, we can define the negative and positive parts of its length by setting[3]

$$\mathcal{L}^-(H_t) = \int_0^1 -\min_{x\in M} H_t(x)\,dt\,, \quad \mathcal{L}^+(H_t) = \int_0^1 \max_{x\in M} H_t(x)\,dt\,.$$

Accordingly, we define seminorms $\rho^\pm$ and ρ by taking $\rho^\pm(\phi)$ to be the infimum of $\mathcal{L}^\pm(H_t)$ over all Hamiltonians H_t with time 1 map ϕ, and $\rho(\phi)$ to be the infimum of

$$\mathcal{L}(H_t) = \mathcal{L}^+(H_t) + \mathcal{L}^-(H_t)$$

over all such paths. It is easy to see that

$$\rho^+(\phi) = \rho^-(\phi^{-1})\,, \quad \rho^\pm(\phi\psi) \le \rho^\pm(\phi) + \rho^\pm(\psi)\,, \quad \rho^\pm(\psi^{-1}\phi\psi) = \rho^\pm(\phi)\,.$$

It follows that the metric $d_\rho(\phi,\psi) := \rho(\psi\phi^{-1})$ is bi-invariant and satisfies the triangle inequality. Its nondegeneracy is equivalent to the statement

$$\rho(\phi) = 0 \iff \phi = \mathrm{id}\,.$$

This deep result is the culmination of a series of papers by Hofer [20], Polterovich [56] and Lalonde–McDuff [26]. The key point is the following basic estimate which is known as the **energy–capacity inequality**.

Proposition 1.3. *Let $B = \phi(B^{2n}(r))$ be a symplectically embedded ball in (M,ω) of radius r. If $\phi(B)\cap B = \emptyset$, then $\rho(\phi) \ge \pi r^2/2$.*

There has also been some success in describing the geodesics in $(\mathrm{Ham}(M),\rho)$. This study was first initiated by Bialy–Polterovich, and a good theory has been developed for paths that minimize length in their homotopy class. In §5 we shall sketch the proof that absolutely length minimizing paths exist. Here is a simple form of the result. (It has been recently generalised by Oh [53] to **quasiautonomous** Hamiltonians.)

Proposition 1.4. *The natural 1-parameter subgroups $\{\phi_t^H\}_{t\in\mathbb{R}}$ generated by time independent H minimize length between the identity and ϕ_t^H for all sufficiently small $|t|$. Thus for each H there is $T = T(H) > 0$ such that*

$$\rho(\phi_t^H) = \mathcal{L}(\{H\}_{t'\in[0,|t|]}) = |t|(\max H - \min H)$$

whenever $|t| \le T$.

There are still many interesting open questions about Hofer geometry, some of which are mentioned below. Interested readers should consult Polterovich's book [61] for references and further discussion. There are also beginning to be very interesting dynamical applications of Hofer geometry: see, for example, Biran–Polterovich–Salamon [7].

[3]To simplify the notation we will often write H_t for the path $\{H_t\}_{t\in[0,1]}$ and ϕ_t instead of $\{\phi_t\}$. Since it is seldom that we need to refer to H_t for a fixed t this should cause no confusion.

Question 1.5. *Does* $\mathrm{Ham}(M)$ *always have infinite diameter with respect to the Hofer norm* ρ*?*

This basic question has not yet been answered because of the difficulty of finding lower bounds for ρ. The most substantial result here is due to Polterovich, who gave an affirmative answer in the case $M = S^2$: see [60, 61].

Question 1.6. *Find ways to estimate the maximum value of* T *such that the path* $\{\phi_t^H\}_{t\in[0,T]}$ *minimizes the length between the identity and* ϕ_T^H *among all homotopic paths with fixed endpoints.*

If H is time independent it has been shown by McDuff–Slimowitz [48] and Entov [13] that one can let T be the smallest positive number such that either the flow ϕ_t^H of H or one of the linearized flows at its critical points has a nontrivial periodic orbit of period T. A Hamiltonian that satisfies this condition with $T = 1$ is said to be **slow**. If a compact Lie group G acts effectively on (M,ω) by Hamiltonian symplectomorphisms then its image in $\mathrm{Ham}(M)$ consists of elements that are the time-1 maps of the flows of autonomous Hamiltonians. Hence its image is totally geodesic with respect to the Hofer norm. Moreover the restriction of the Hofer norm to G often has interesting geometric properties. For an example, see the discussion of the ABW inequalities in Section 4.4.

Oh has made considerable progress with the general question (for nonautonomous flows) in his recent paper[53]. He has also defined a refined version of the Hofer norm using spectral invariants, that coincides with it in a neighbourhood of the identity but in general is smaller.

Question 1.7. *What conditions on the Hamiltonian* $H : M \to \mathbb{R}$ *imply that* $\rho(\phi_t^H) \to \infty$ *as* $t \to \infty$*?*

Clearly, we need to assume that H does not generate a circle action, or more generally, that its flow $\phi_t^H, t \geq 0$, is not quasiperiodic in the sense that its elements are not contained in any compact subset of $\mathrm{Ham}(M)$. However, this condition is not sufficient. For example, if H has the form $F - F\circ\tau$ where $F \in \mathrm{LieHam}(M)$ has support in a set U that is disjoined by τ (i.e. $\tau(U)\cap U = \emptyset$), then

$$\phi_t^H \;=\; [\phi_t^F, \tau] \;=\; \phi_t^F \tau(\phi_t^F)^{-1}\tau^{-1}.$$

Hence

$$\rho(\phi_t^H) \;\leq\; \rho(\phi_t^F \tau(\phi_t^F)^{-1}) + \rho(\tau^{-1}) \;\leq\; 2\rho(\tau)\,.$$

Therefore $\rho(\phi_t^H)$ is bounded. But it is easy to construct examples on S^2 for which the sequence $\phi_n^H, n = 1,2,3,\ldots$ has no subsequence that converges in the C^0 topology. Thus the flow is not quasiperiodic.

Exercise 1.8. Suppose that M is a Riemann surface of genus > 0 and that H has a level set that represents a nontrivial element in $\pi_1(M)$. Show that $\rho(\phi_t^H) \to \infty$ as $t \to \infty$ by lifting to the universal cover and using the energy-capacity inequality.

Question 1.9. *Is the sum* $\rho^+ + \rho^-$ *of the one sided seminorms nondegenerate for all* (M,ω)*?*

The seminorms ρ^+ and ρ^- are said to be "one sided" because they do not in general take the same values on an element ϕ and its inverse. (As we show in Proposition 3.7 they also have very natural geometric interpretations in which they measure the size of ϕ from just one side.) Their sum $\rho^+ + \rho^-$ is two sided. Hence its null set

$$\mathrm{null}(\rho^+ + \rho^-) = \{\phi : \rho^+(\phi) + \rho^-(\phi) = 0\}$$

is a normal subgroup of $\mathrm{Ham}(M)$. Therefore it is trivial: in other words, $\rho^+ + \rho^-$ is either identically zero or is nondegenerate. Thus to prove nondegeneracy one just has to find one element on which $\rho^+ + \rho^-$ does not vanish. The paper [43] develops geometric arguments (using Gromov–Witten invariants on suitable Hamiltonian fibrations over S^2) that show that it is nondegenerate in certain cases, for example if (M, ω) is a projective space or is weakly exact, i.e. ω vanishes on $\pi_2(M)$. (For the latter case, see also Schwarz [66].) However the general case is still open. An even harder question is whether the one sided norms $\rho^{\pm}$ are each nondegenerate. Now the null set is only a conjugation invariant semigroup and so could be a proper subset of $\mathrm{Ham}(M)$. Hence it seems that one could only prove nondegeneracy by an argument that would apply to an arbitrary element of $\mathrm{Ham}(M)$.

1.4. **Questions of stability.** One of the first nonelementary results in symplectic topology is due to Eliashberg [11] and Ekeland–Hofer [10]:

- **Symplectic rigidity theorem:** *the group* $\mathrm{Symp}(M, \omega)$ *is* C^0*-closed in* $\mathrm{Diff}(M)$.

This celebrated result is the basis of symplectic topology. The proof shows that there is are invariants $c(U)$ (usually called **symplectic capacities**) of an open subset of a symplectic manifold that are continuous with respect to the Hausdorff metric on sets and that are preserved only by symplectomorphisms. (When n is even, one must slightly modify the previous statement to rule out the case $\phi^*(\omega) = -\omega$.) There are several ways to define suitable c. Perhaps the easiest is to take Gromov's width:

$$c(U) = \sup\{\pi r^2 : B^{2n}(r) \text{ embeds symplectically in } U\}.$$

Here $B^{2n}(r)$ is the standard ball of radius r in Euclidean space.

- *Stability properties of* $\mathrm{Symp}(M)$ *and* $\mathrm{Symp}_0(M)$.

By this we mean that if G denotes either of these groups, there is a C^1-neighbourhood $\mathcal{N}(G)$ of G in $\mathrm{Diff}(M)$ that deformation retracts onto G. This follows from the Moser isotopy argument. In the case $G = \mathrm{Symp}(M)$ take

$$\mathcal{N}(\mathrm{Symp}) = \{\phi \in \mathrm{Diff}(M) : t\phi^*(\omega) + (1-t)\omega \text{ is nondegenerate, for all } t \in [0,1]\}.$$

By Moser, one can define for each such ϕ a unique isotopy ϕ_t (that depends smoothly on $\phi^*(\omega)$) such that $\phi_t^*(t\phi^*(\omega) + (1-t)\omega) = \omega$ for all t. Hence $\phi_1 \in \mathrm{Symp}(M)$. Similarly, when $G = \mathrm{Symp}_0(M)$ one can take $\mathcal{N}(G)$ to be the identity component of $\mathcal{N}(\mathrm{Symp})$.

Question 1.10. *Is there a* C^0*-neighborhood of* $\mathrm{Symp}_0(M)$ *in* $\mathrm{Diff}_0(M)$ *that deformation retracts into* $\mathrm{Symp}_0(M)$*?*

• *The Flux conjecture*

The analogous questions for the Hamiltonian group are much harder, and much less is known. We do not even know if the group $\mathrm{Ham}(M)$ is always closed in the C^1-topology, let alone whether it is C^0-closed. It follows from diagram (3) that $\mathrm{Ham}(M)$ is C^1-closed in $\mathrm{Symp}_0(M)$ if and only if the flux group Γ_ω is closed. Therefore, the most important question here is the following.

Question 1.11. *Is Γ_ω a discrete subgroup of $H^1(M,\mathbb{R})$?*

The hypothesis that it is always discrete is known as the **Flux conjecture.** Γ_ω is known to be discrete for many (M,ω). For example, it follows from (2) that Γ_ω is a subgroup of $H^1(M;\mathcal{P}_\omega)$ where $\mathcal{P}_\omega$ denotes the periods of $[\omega]$, i.e. its values on $H_2(M;\mathbb{Z})$.[4] Thus Γ_ω is discrete if $[\omega] \in H^2(M;\mathbb{Z})$. Another easy case is when

$$\wedge[\omega]^{n-1} : H^1(M,\mathbb{R}) \to H^{2n-1}(M,\mathbb{R})$$

is an isomorphism (eg. if (M,ω) is Kähler.) Nevertheless, this question does not yet have a complete answer: see Lalonde–McDuff–Polterovich [31, 32] and Kędra [23, 24]. It follows that there may be manifolds (M,ω) for which the normal subgroup $\mathrm{Ham}(M)$ of $\mathrm{Symp}(M)$ is not closed with respect to the C^∞ topology. In fact, if Γ_ω is not discrete then one should think of $\mathrm{Ham}(M)$ as a leaf in a foliation of $\mathrm{Symp}_0(M)$ that has codimension equal to the first Betti number $\mathrm{rk}(H^1(M,\mathbb{R})$ of M.

• *Special geometric properties of elements in* $\mathrm{Ham}(M)$.

One indication that Γ_ω may always be discrete is that the elements $\mathrm{Ham}(M)$ have special geometric properties. In particular, according to Arnold's celebrated conjecture (proven by Fukaya–Ono and Liu–Tian based on work by Floer) the number of fixed points of $\phi \in \mathrm{Ham}$ may be estimated as

$$\#\mathrm{Fix}\,\phi \geq \sum_k \mathrm{rank}\, H^k(M,\mathbb{Q})$$

provided that all its fixed points are nondegenerate (i.e. the graph of ϕ is transverse to the diagonal.) The following simple argument shows how this is related to the Flux problem. Denote $r(M) : \sum_k \mathrm{rank}\, H^k(M,\mathbb{Q})$.

Lemma 1.12. *Suppose that $\Lambda \subset H^1(M;\mathbb{R})$ has the property that every element $\alpha \in H^1(M;\mathbb{R}) \setminus \Lambda$ is the flux of some symplectic path $\{\phi_t^\alpha\}_{t\in[0,1]}$ whose time-1 map is nondegenerate and has $< r(M)$ fixed points. Then $\Gamma_\omega \subset \Lambda$.*

Proof. Suppose this is false, and let $\{\psi_t\}$ be a loop in $\mathrm{Ham}(M)$ with flux equal to $\alpha \in H^1(M;\mathbb{R}) \setminus \Lambda$. Then the path $\{\psi_t^{-1}\phi_t^\alpha\}_{t\in[0,1]}$ is Hamiltonian and has time 1-map ϕ_1^α. Therefore ϕ_1^α must have at least $r(M)$ fixed points by Arnold's conjecture, which contradicts the hypothesis. □

For example, one can apply this to the standard torus $(T^{2n},\omega_0) = (\mathbb{R}^{2n}/\mathbb{Z}^{2n},\omega_0)$, taking Λ to be the lattice $H^1(T^{2n};\mathbb{Z})$, and conclude that the flux group must be contained in this lattice. But we knew this anyway. So far, no one has succeeded in getting very far with this kind of geometric argument.

[4] In fact, one can restrict to its set of values on spheres by Proposition 1.14 (i) below.

• *Stability of Hamiltonian loops*

Although what one might call geometric stability (even for fixed ω) has not yet been established for the Hamiltonian group, it does have stability properties on the homotopy level. Here is a typical question. Suppose given continuous map from a finite CW complex X to $\mathrm{Ham}(M,\omega)$. What happens if we perturb ω? If ω' is sufficiently close to ω, then it follows by Moser stability that X is homotopic through maps to $\mathrm{Diff}(M)$ to a map itno $\mathrm{Symp}(M,\omega')$. But can we always deform X into $\mathrm{Ham}(M,\omega')$? Since $\pi_k(\mathrm{Ham}(M)) = \pi_k(\mathrm{Symp}_0(M))$ when $k > 1$ by diagram (3) this is automatic when X is simply connected. However the case $X = S^1$ is not at all obvious, and is proved in [32, 40]. In the version stated below $(M^M)_{id}$ denotes the group of homotopy self-equivalences of M, i.e. the identity component of the space of degree 1 maps $M \to M$.

Proposition 1.13. *Suppose that* $\phi \in \pi_1(\mathrm{Symp}(M,\omega))$ *and* $\phi' \in \pi_1(\mathrm{Symp}(M,\omega'))$ *represent the same element of* $\pi_1((M^M)_{id})$. *Then*

$$\mathrm{Flux}_\omega(\phi) = 0 \iff \mathrm{Flux}_{\omega'}(\phi') = 0\,.$$

This is an easy consequence of a more general vanishing theorem for various actions of the Hamiltonian group. One can generalise the defining equation (2) for the Flux homomorphism to get a general definition of the action $\mathrm{tr}_\phi : H_*(M) \to H_{*+k}(M)$ of an element $\phi \in H_k(M^M)$. Namely if ϕ is represented by the cycle $t \mapsto \phi_t$ for $t \in V^k$ and $c \in H_*(M)$ is represented by $x \mapsto c(x)$ for $x \in C$ then $\mathrm{tr}_\phi(c)$ is represented by the cycle

$$V^k \times Z \to M : (t,x) \mapsto \phi_t(x)\,.$$

To say this action is trivial means that

$$\mathrm{tr}_\phi(c) = 0 \qquad \text{whenever } c \in H_k(M),\ k > 0\,.$$

If $V^k = S^k$ is a sphere then this action is precisely the differential ∂ in the Wang long exact sequence for the associated bundle $M \to P \to S^{k+1}$:

$$\cdots \to H_i(M) \to H_i(P) \to H_{i-k}M \xrightarrow{\partial} H_{i-1}M \to \ldots$$

Therefore the triviality of the action is equivalent to saying that $H_*(P)$ is isomorphic to the tensor product $H_*(M) \otimes H*(S^{k+1})$.

The results to date on these questions are still incomplete. The following proposition states the most important known conclusions. The first result below is a consequence of the proof of the Arnold conjecture. Another more direct proof may be found in Lalonde–McDuff–Polterovich [32]. The second part is proved in Lalonde–McDuff [30] and the third is an easy consequence. Since the main ideas in the proofs are sketched in the survey article McDuff [44] we shall not say more about them here.

Proposition 1.14. (i) *The evaluation map* $\pi_1(\mathrm{Ham}(M)) \to \pi_1(M)$ *is zero.*
(ii) *The natural action of* $H_*(\mathrm{Ham}(M),\mathbb{Q})$ *on* $H_*(M,\mathbb{Q})$ *is trivial.*
(iii) *If* $(M,\omega) \to P \to S^{k+1}$ *is any bundle with structural group* $\mathrm{Ham}(M)$ *then* $H_*(P;\mathbb{Q}) \cong H_*(M;\mathbb{Q}) \otimes H_*(S^{k+1};\mathbb{Q})$.

It would be very interesting to know if there are any similar results in other related categories, such as the category of Poisson manifolds. Even though it does not seem as if there would be a good analogue for the Hamiltonian group as such, there still might

be a notion of something akin to a Hamiltonian bundle and hence some analogue of property (iii) above.

Note that (iii) definitely fails for general symplectic bundles over S^2: if $\{\psi_t\}_{t\in S^1}$ is a symplectic loop with nontrivial flux then the corresponding bundle over S^2 has nontrivial Wang differential. This situation is discussed further in Section 3. But when $k > 1$ there is no difference between Hamltonian and symplectic bundles over S^{k+1}, since, as we already mentioned, $\pi_k(\mathrm{Ham}) = \pi_k(\mathrm{Symp})$ for $k > 1$.

Here is another rather curious consequence of Proposition 1.14. It applies to *any* symplectic fibration $(M,\omega) \to P \to S^2$, not just Hamiltonian fibrations.

Corollary 1.15. *Let $(M,\omega) \to P \to S^2$ be any symplectic fibration and let ∂ be its Wang differential. Then $\partial \circ \partial : H_iM \to H_{i+2}(M)$ is zero.*

It is quite possible that this is always true for any smooth fibration over S^2 (with compact total space), but I do not know a proof.

2. The homotopy type of $\mathrm{Symp}(M)$

2.1. **Ruled 4-manifolds.** Another set of questions concerns the homotopy type of the group $\mathrm{Symp}(M)$. In rare cases this is completely understood. The following results are due to Gromov [19]:

Proposition 2.1. (i) $\mathrm{Symp}^c(\mathbb{R}^4,\omega_0)$ *is contractible;*

(ii) $\mathrm{Symp}(S^2 \times S^2, \sigma + \sigma)$ *is homotopy equivalent to the extension of* $\mathrm{SO}(3) \times \mathrm{SO}(3)$ *by* $\mathbb{Z}/2\mathbb{Z}$ *where this acts by interchanging the factors;*

(iii) $\mathrm{Symp}(\mathbb{C}P^2,\omega)$ *is homotopy equivalent to* $\mathrm{PU}(3)$.

It is no coincidence that these results occur in dimension 4. The proofs use J-holomorphic spheres, and these give much more information in dimension 4 because of positivity of intersections. Abreu [1] and Abreu–McDuff [3] recently extended Gromov's arguments to other ruled surfaces. Here are their main results, stated for convenience for the product manifold $\Sigma \times S^2$ (though there are similar results for the nontrivial S^2 bundle over Σ.) We sketch the easiest proofs in Section 2.4.

Consider the following family of symplectic forms on $M_g = \Sigma_g \times S^2$ (where g is genus(Σ)):

$$\omega_\mu = \mu\sigma_\Sigma + \sigma_{S^2}\,, \qquad \mu > 0\,,$$

where σ_Y denotes (the pullback to the product of) an area form on the Riemann surface Y with total area 1.[5] Denote by $G_\mu = G^g_\mu$ the subgroup

$$G^g_\mu := \mathrm{Symp}(M_g,\omega_\mu) \cap \mathrm{Diff}_0(M_g)$$

of the group of symplectomorphisms of (M_g,ω_μ). When $g > 0$ μ ranges over all positive numbers. However, when $g = 0$ there is an extra symmetry — interchanging the two spheres gives an isomorphism $G^0_\mu \cong G^0_{1/\mu}$ — and so we take $\mu \geq 1$. Although it is not completely obvious, there is a natural homotopy class of maps from G^g_μ to $G^g_{\mu+\varepsilon}$ for all $\varepsilon > 0$. To see this, let

$$G^g_{[a,b]} = \bigcup_{\mu\in[a,b]} \{\mu\} \times G^g_\mu \;\subset\; \mathbb{R} \times \mathrm{Diff}(M_g)\,.$$

[5] Using results of Taubes and Li–Liu, Lalonde–McDuff show in [28] that these are the *only* symplectic forms on $\Sigma \times S^2$ up to diffeomorphism.

It is shown in [3] that the inclusion $G_b^g \to G_{[a,b]}^g$ is a homotopy equivalence. Therefore we can take the map $G_\mu^g \to G_{\mu+\varepsilon}^g$ to be the composite of the inclusion $G_\mu^g \to G_{[\mu,\mu+\varepsilon]}^g$ with a homotopy inverse $G_{[\mu,\mu+\varepsilon]}^g \to G_{\mu+\varepsilon}^g$.

Proposition 2.2. *As $\mu \to \infty$, the groups G_μ^g tend to a limit G_∞^g that has the homotopy type of the identity component $\mathcal{D}_0^g$ of the group of fiberwise diffeomorphisms of $M_g = \Sigma_g \times S^2 \to \Sigma$.*

Proposition 2.3. *When $\ell < \mu \le \ell + 1$ for some integer $\ell \ge 1$,*

$$H^*(G_\mu^0, \mathbb{Q}) = \Lambda(t, x, y) \otimes \mathbb{Q}[w_\ell],$$

where $\Lambda(t, x, y)$ is an exterior algebra over $\mathbb{Q}$ with generators t of degree 1, and x, y of degree 3 and $\mathbb{Q}[w_\ell]$ is the polynomial algebra on a generator w_ℓ of degree 4ℓ.

In the above statement, the generators x, y come from $H^*(G_1^0) = H^*(\mathrm{SO}(3) \times \mathrm{SO}(3))$ and t corresponds to an element in $\pi_1(G_\mu^0), \mu > 1$, found by Gromov in [19]. Thus the subalgebra $\Lambda(t, x, y)$ is the pullback of $H^*(\mathcal{D}_0^0, \mathbb{Q})$ under the map $G_\mu \to \mathcal{D}_0^0$. The other generator w_ℓ is fragile, in the sense that the corresponding element in homology disappears (i.e. becomes null homologous) when μ increases.

There are still many unanswered questions about these groups. Here is a sampling.

Question 2.4. *Is the group G_μ^g connected for all $\mu > 0, g \ge 0$?*

It is shown in McDuff [42] that the answer is "yes" whenever $\mu \ge [g/2]$. (The case $g = 0$ was proved in [3].) This paper also provides an affirmative answer to the next question in the genus zero case.

Question 2.5. *Is the homotopy type of the groups G_μ^g constant on the intervals $\mu \in (\ell, \ell + 1]$?*

Question 2.6. *The group G_μ^1 is known to be constant for $0 < \mu \le 1$. What is its homotopy type?*

The methods used to prove the above results extend to certain other closely related manifolds. For example, Pinsonnault in thesis [54] studies the symplectomorphism group of the one point blow up of $(S^2 \times S^2, \omega_\mu)$ with $\mu = 1$ and shows it is homotopic to the 2-torus, T^2. As shown by Lalonde–Pinsonnault in [33] this group becomes more complicated when $1 < \mu \le 2$, and its homotopy groups change when the blow up radius r passes through the critical level $\pi r^2 = \mu - 1$. This implies that the homotopy type of the associated space of symplectically embedded balls also changes, the first known example of such a phenomenon.

There has been some attempt to generalize these results to higher dimensions. Le–Ono [34] and Buse [8] use parametric Gromov–Witten invariants to obtain information on the symplectomorphism groups of products $(M_g, \omega_\mu) \times (N, \omega)$, while Seidel studies the case of products of two projective spaces in [68].

2.2. The topology of $\mathrm{Symp}(M)$ for general M. $\pi_0(\mathrm{Symp}(M))$

This group is known as **the symplectic mapping class group**. Seidel has done interesting work here, studying *symplectic Dehn twists* especially on manifolds with boundary. He considers the group $\mathrm{Symp}(M, \partial M)$ of symplectomorphisms that are the

identity near the boundary, detecting quite large subgroups of $\pi_0(\mathrm{Symp}(M,\partial M))$ by using Floer homology to study the effect of Dehn twists on the Lagrangian submanifolds in M: cf [69].

$\pi_1(\mathrm{Symp}(M))$

This is an abelian group and one can try to detect its elements by studying various natural homomorphisms. One such is the Flux homomorphism:

$$\mathrm{Flux}_\omega : \pi_1(\mathrm{Symp}(M)) \to \Gamma_\omega \subset H^1(M,\mathbb{R}),$$

that has kernel equal to $\pi_1(\mathrm{Ham}(M))$. There are several other interesting homomorphisms defined on this kernel, most notably a homomorphism to the units in the quantum homology ring of M known as the Seidel representation [67, 31]: cf. §4.

Very little is known about the higher homotopy groups. Observe, however, that the existence of the diagram (3) implies that the inclusion $\mathrm{Ham}(M) \to \mathrm{Symp}(M)$ induces an isomorphism on $\pi_j, j > 1$.

2.3. **Characteristic classes.** Reznikov shows in [63] how to define classes $\lambda_k \in H^{2k-1}(\mathrm{Ham}(M),\mathbb{R})$ for $k > 1$ that he calls higher Cartan classes using the invariant inner product of (4) and an analog of Chern–Weil theory. However, one can define the corresponding classes

$$c_k^H \in H^{2k}(B\mathrm{Ham}(M),\mathbb{R}), \quad k \geq 2,$$

on the classifying space using the notion of the coupling class of a symplectic fibration. As we will see in §3, given any smooth fibration $\pi : M \to P \to B$ with structural group $\mathrm{Ham}(M,\omega)$ there is a canonical class $u \in H^2(P)$ that extends the symplectic class on the fibers. u is called the Guillemin–Lerman–Sternberg **coupling class** and is characterized by the property that the integral of u^{n+1} over the fibers of π is 0 in $H^2(B)$: cf [18, 61], or [46, Ch 6]. Then we set

$$c_k^H := \int_M u^{k+n}, \quad k > 1.$$

This defines $c_k^H \in H^{2k}(B)$. By naturality this has to come from a class $c_k^H \in H^{2k}(B\mathrm{Ham}(M))$ that we shall call a *Hamiltonian Chern class.*

Other characteristic classes can be constructed using the Chern classes of the vertical tangent bundle $T_{\mathrm{vert}}P \to P$ whose fiber at $x \in P$ is the tangent space to the fiber through x. Denoting these classes by c_k^{vert} we get corresponding elements in $H^*(B\mathrm{Ham}(M,\omega))$ by integrating products of the form

$$u^\ell c_{k_1}^{\mathrm{vert}} \dots c_{k_p}^{\mathrm{vert}}$$

over the fibers of $P \to B$. Rather little is known about these classes, though Januszkiewicz and Kędra [22] have recently calculated them for symplectic toric manifolds. (They appear in slightly different guise in [32]. Note also that the classes c_k^{vert} exist for symplectic bundles, and so when $\ell = 0$ the classes extend to $H^*(B\mathrm{Symp}(M))$.)

Now consider a situation in which a compact Lie group acts on (M,ω) in such a way as to induce an injective homomorphism $G \to \mathrm{Ham}(M)$. (Such an action is often called weakly Hamiltonian.) Then there is a corresponding map of classifying spaces:

$$BG \to B\mathrm{Ham}(M)$$

and one can ask what happens to the Hamiltonian Chern classes c_k^H under pullback. It follows from Reznikov's definitions that if G is semisimple and the action is effective then the pullback of c_2^H is nonzero. Hence

$$H^3(\mathrm{Ham}(M)) \neq 0$$

whenever (M,ω) admits such an action. Moreover he shows by direct calculation that, in the case of the action of the projective unitary group $\mathrm{PU}(n+1)$ on complex projective space $\mathbb{C}P^n$, the pullbacks of the c_k^H to $B\mathrm{PU}(n+1)$ are multiplicatively independent. Thus the inclusion $\mathrm{PU}(n+1) \to \mathrm{Ham}(\mathbb{C}P^n)$ induces an injection on rational homotopy.

Question 2.7. *What conditions imply that such an inclusion $G \to \mathrm{Ham}(M)$ is nontrivial homotopically?*

The above question is deliberately vague: what does one mean precisely by "nontrivial"? Presumably one could extend Reznikov's calculation to other actions of compact semisimple simply connected Lie groups G on their homogeneous Kähler quotients: see Entov [13].

Reznikov's argument is elementary. In contrast, the next result uses fairly sophisticated analytic tools: cf. McDuff–Slimowitz [48].

Proposition 2.8. *Given a semifree Hamiltonian action of S^1 on (M,ω), the associated homomorphism*

$$\pi_1(S^1) \to \pi_1(\mathrm{Ham}(M))$$

is nonzero.

Proof. Recall that an action $\{\phi_t\}_{t\in\mathbb{R}/\mathbb{Z}}$ of S^1 is called semifree if no stabilizer subgroup is proper. Equivalently, the only points fixed by some ϕ_T for $0 < T < 1$ are fixed for all t. Therefore, by the remarks after Question 1.6, both the paths $\{\phi_t\}_{t\in[0,3/4]}$ and $\{\phi_{-t}\}_{0\le t\le 1/4}$ are length minimizing in their respective homotopy classes. Since they have different lengths, these paths cannot be homotopic. □

This result has been considerably extended by McDuff–Tolman [49]. Note also that the semifree hypothesis is crucial: for example the action of S^1 on $M = \mathbb{C}P^2$ given by

$$[z_0 : z_1 : z_2] \mapsto [e^{2\pi i\theta} z_0 : e^{-2\pi i\theta} z_1 : z_2]$$

gives rise to a nullhomotopic loop in $\mathrm{Ham}(M)$. On the other hand the image of $\pi_1(S^1)$ in $\pi_1(\mathrm{Ham}(M))$ might be finite: for example the rotation of S^2 by one turn has order 2 in $\mathrm{Symp}(S^2) \simeq \mathrm{SO}(3)$.

In an ongoing project [25], Kędra and McDuff have recently shown that if a Hamiltonian circle action is inessential (i.e. gives rise to a contractible element in $\pi_1(\mathrm{Ham}(M))$) then there is an associated nontrivial element in $\pi_3(\mathrm{Ham}(M))$. This extends Reznikov's result: in the case when the circle is a subgroup in a semisimple Lie group G then this element is precisely the one he detected.

Finally, in [34] Le–Ono define **Gromov–Witten characteristic classes** on $B\mathrm{Symp}(M)$. If one restricts to $B\mathrm{Ham}(M)$ and fixes the genus g, these are indexed by the elements of $H_2(M;\mathbb{Z})$ and can be defined as follows. Let $M_{\mathrm{Ham}} \to B\mathrm{Ham}(M)$ denote the bundle with fiber M associated to the obvious action of $\mathrm{Ham}(M)$ on M. Then, by Proposition 1.14 $H_2(M_{\mathrm{Ham}})$ splits as the sum $H_2(M) \oplus H_2(B\mathrm{Ham}(M))$. Hence each

$A \in H_2(M;\mathbb{R})$ gives rise to a well defined class in the homology of the fibers of the fibration $M_{\mathrm{Ham}} \to B\mathrm{Ham}(M)$, and there is a class $\mathrm{GW}_A \in H^{\mu(A)}(B\mathrm{Ham}(M))$ whose value on a cycle $f : B \to B\mathrm{Ham}(M)$ is the "number of isolated $\widetilde{J}$-holomorphic genus g curves in class A" in the pullback fiber bundle $f^*(M_{\mathrm{Ham}}) \to B$. Here the almost complex structure $\widetilde{J}$ is compatible with the fibration in the sense that it restricts on each fiber (M_b, ω_b) to a tame almost complex structure, and the index

$$\mu(A) = (g-1)(2n-6) - 2c_1(A)\,.$$

(Of course, to define the invariants correctly one has to regularize the moduli space in the usual way: see for example [64].)

Question 2.9. *When are these characteristic classes nontrivial?*

Unfortunately one cannot get very interesting examples from the groups G_μ^0 discussed above. Le–Ono show that when $\mu > 1$ the 1-dimensional vector space $H^2(BG_\mu^0, \mathbb{R})$ is generated by a Gromov–Witten characteristic class GW_A. However, the nontriviality of this class GW_A can also be proved by purely homotopical means since the corresponding loop in G_μ^0 does not vanish in the group of self homotopy equivalences of $S^2 \times S^2$. (In fact, Le–Ono show that the cohomology ring of the corresponding fibration over S^2 is not a product ring.) It is shown in [3] that the new elements $w_\ell \in H^*(G_\mu^0)$ do not transgress to $H^*(BG_\mu^0)$, but rather give rise to *relations* in this ring.

One can also define classes by evaluating appropriate moduli spaces of J-holomorphic curves at k points for $k > 0$. One interesting fact pointed out in Kędra [24] is that the nontrivial Gromov–Witten classes of dimension $\mu(A) = 0$ constrain the image of the Flux homomorphism.

2.4. **J-holomorphic curves in $S^2 \times S^2$.** In this section we shall give a brief overview of the proof of some of the results on $G_\mu := \mathrm{Symp}(M \times M, \omega^\mu)$ mentioned in §2.1. Fuller details may be found in the survey article [29] as well as in Abreu's beautiful paper [1]. An introduction to some of the technicalities may be found in the lecture notes [39] or the more broadly based survey [38]. Proposition 2.1 is proved in exhaustive detail in [47]. The proofs are based on the behavior of J-holomorphic spheres in 4-manifolds. Here is a summary of their most important properties.

2.5. **Analytic background. Almost complex structures:** An almost complex structure J on M is an automorphism of TM with square -1. It is said to be ω-tame if $\omega(v, Jv) > 0$ for every nonzero $v \in TM$. For every symplectic manifold, the space $\mathcal{J}(\omega)$ of ω -tame almost complex structures is nonempty and contractible. Given $J \in \mathcal{J}(\omega)$ we shall always use the associated Riemannian metric

$$g_J(v,w) := \frac{1}{2}(\omega(v, Jw) + \omega(w, Jv)$$

on M.

J-holomorphic curves: A map $u : (S^2, j) \to (M, J)$ is said to be J-holomorphic if it satisfies the following nonlinear Cauchy–Riemann equation:

$$\overline{\partial}_J u := \frac{1}{2}(du + J \circ du \circ i) = 0\,.$$

If $J \in \mathcal{J}(\omega)$ then ω restricts to an area form at all non singular points in the image of u. Therefore if u represents the homology class $A \in H_2(M)$ we must have $\omega(A) > 0$ unless $A = 0$ and u is constant. Note also that the Möbius group $\mathrm{PSL}(2,\mathbb{C})$ acts by reparametrization $u \mapsto u \circ \gamma$ on the space of J-holomorphic spheres.

In many respects the behaviour of these curves is exactly the same as in the integrable case. In particular, in dimension 4 there is **positivity of intersections**: every intersection point of two distinct J-holomorphic curves contributes positively to their intersection number, with nontransverse intersections contributing > 1. Thus if $A \cdot B = 0$ every J-holomorphic A-curve is disjoint from every J-holomorphic B-curve, while if $A \cdot B = 1$ every J-holomorphic A-curve meets every J-holomorphic B-curve precisely once and transversally. This is relatively easy to prove when one is considering two distinct curves. A rather more subtle result is that this remains true for a single (non multiply covered) curve; in particular any singular point on a curve u (i.e. point where $du = 0$) contributes positively to the self-intersection number. This can be formulated as the **adjunction inequality**:

• *if* $u : (S^2, j) \to (M^4, J)$ *is a* J*-holomorphic sphere in class* A *then* j *is an embedding if and only if*

$$c_1(A) = 2 + A \cdot A .$$

Thus there is a homological criterion for a curve to be embedded. In particular if A can be represented by a J-holomorphic sphere then *every* J'-holomorphic representative of A is embedded for any $J' \in \mathcal{J}(\omega)$.

The moduli space $\mathcal{M}(A; J)$: The main object of interest to us is the moduli space $\mathcal{M}(A; J)$ of all J-holomorphic A-spheres. This has two fundamental properties.

• *There is a set* $\mathcal{J}_{\mathrm{reg}}$ *of second category in* $\mathcal{J}(\omega)$ *such that* $\mathcal{M}(A; J)$ *is a smooth manifold of dimension* $2n + 2c_1(A)$ *for every* $J \in \mathcal{J}_{\mathrm{reg}}$.

The proof involves Fredholm theory.

• Since $\mathrm{PSL}(2,\mathbb{C})$ is noncompact, the space $\mathcal{M}(A; J)$ is never compact except in the trivial case when $A = 0$. However the quotient $\mathcal{M}(A; J)/\mathrm{PSL}(2,\mathbb{C})$ is sometimes compact. Even if it is not, it has a well behaved compactification that is made up of genus 0 stable maps in class A. These objects were called cusp-curves by Gromov and are sometimes known as bubble trees. One shows that $\mathcal{M}(A; J)/\mathrm{PSL}(2,\mathbb{C})$ can be noncompact only if A has a representation by a connected union of two or more J-holomorphic spheres. In particular, this can happen only if the class A decomposes as a a sum $A = A_1 + A_2$ where $\omega(A_1) > 0, \omega(A_2) > 0$. Hence $\mathcal{M}(A; J)/\mathrm{PSL}(2,\mathbb{C})$ is compact in the case when $[\omega]$ is integral and $\omega(A) = 1$.

Fredholm theory and regularity: We now say a little more about Fredholm theory and regularity since this is so crucial to our argument.

Let $\mathcal{M}(A, \mathcal{J})$ be the space of all pairs (u, J), where $u : (S^2, j) \to (M, J)$ is J-holomorphic, $u_*([S^2]) = A \in H_2(M)$, and $J \in \mathcal{J}(\omega)$. One shows that a suitable completion of $\mathcal{M}(A, \mathcal{J})$ is a Banach manifold and that the projection

$$\pi : \mathcal{M}(A, \mathcal{J}) \to \mathcal{J}$$

is Fredholm. In this situation one can apply an infinite dimensional version of Sard's theorem (due to Smale) that states that there is a set $\mathcal{J}_{\mathrm{reg}}$ of second category in $\mathcal{J}$

consisting of regular values of π, i.e. points where $d\pi$ is surjective. Moreover by the implicit function theorem for Banach manifolds the inverse image of a regular value is a smooth manifold of dimension equal to the index of the Fredholm operator π. Thus one finds that for almost every J

$$\pi^{-1}(J) = \mathcal{M}(A; J)$$

is a smooth manifold of dimension $2(c_1(A)+n)$. The index calculation here follows by investigating the linearization of π which turns out to be essentially the same as the linearization

$$D_u : C^\infty(S^2, u^*(TM)) \to \Omega^{0,1}_J(S^2, u^*(TM)) \tag{5}$$

of the operator $\overline{\partial}_J u$. One can check that D_u is a zeroth order perturbation of the usual Dolbeault differential $\overline{\partial}$ from functions to $(0,1)$-forms with values in the bundle $u^*(TM)$. Hence D_u has the same index as $\overline{\partial}$, which in turn is given by the Riemann–Roch theorem.

There is another important point here. When $u : S^2 \to M^4$ is an embedding, the bundle $u^*(TM)$ splits (as a complex bundle) into the sum of the tangent bundle to S^2 with the normal bundle L to Imu. The restriction of D_u to the tangent bundle is always surjective. It follows that D_u is surjective if and only if its restriction to the line bundle L is surjective. In this case, the Riemann–Roch theorem says there is a dichotomy:

- *either* $c_1(L) \geq -1$ *and* D_u *is surjective;*
- or $c_1(L) < -1$ *and the rank of* coker D_u *is constant* (*and equal to* $2(|c_1(L)| - 1)$.)

We will use this fact later.

Next observe that, by a transversality theorem for paths, given any two elements $J_0, J_1 \in \mathcal{J}_{\text{reg}}$ there is a path $J_t, 0 \leq t \leq 1$, such that the union

$$W = \cup_t \mathcal{M}(A; J_t) = \pi^{-1}(\cup_t J_t)$$

is a smooth (and also oriented) manifold with boundary

$$\partial W = \mathcal{M}(A; J_1) \cup -\mathcal{M}(A; J_0)\,.$$

It follows that the evaluation map

$$\text{ev}_J : \mathcal{M}(A; J) \times_{PSL(2,\mathbb{C})} S^2 \to M\,, \quad (u, z) \mapsto u(z)\,,$$

is independent of the choice of (regular) J up to oriented bordism. In particular, *if* we could ensure that everything is compact and *if* we arrange that ev maps between manifolds of the same dimension then the degree of this map would be independent of J.

2.6. **The case $S^2 \times S^2$.** In the case of $(S^2 \times S^2, \omega_\mu)$ we are interested in looking at curves in the classes $A := [S^2 \times pt]$ and $B := [pt \times S^2]$. Thus $c_1(A) = c_1(B) = 2$ and

$$\dim(\mathcal{M}(A; J) \times_G S^2) = 2(c_1(A) + 2) - 6 + 2 = \dim(M)\,.$$

If $\mu = 1$ the above remarks about compactness imply that the moduli spaces $\mathcal{M}(A; J)/PSL(2, \mathbb{C}))$ are always compact and ev$_J$ has degree 1. Moreover, for each J the J-holomorphic A-curves are mutually disjoint (by positivity of intersections), and

one can show that they form the fibers of a fibration of $S^2 \times S^2$. Similar statements holds for the B-curves.

When $\mu > 1$ this remains true for the smaller sphere B (though one needs some extra arguments to prove this). On the other hand, it is now possible for the A-curve to decompose since $\omega_\mu(A - B) > 0$. Moreover the class $A - B$ is represented by the symplectic embedding of S^2 onto the antidiagonal $z \mapsto (z, -z)$ where we think of $z \in S^2 \subset \mathbb{R}^3\}$, and this submanifold can be made J-holomorphic for suitable J.[6] Thus there are $J \in \mathcal{J}(\omega_\mu)$ for which the curve $A - B$ is represented. Since $(A - B) \cdot (A - B) = -2$, positivity of intersections implies that this representative is unique. Moreover, there cannot be any A-curves since $A \cdot (A - B) = -1$. It follows that when $1 < \mu \leq 2$ every $J \in \mathcal{J}(\omega_\mu)$ is of one of two kinds:

(i) if A is represented, $S^2 \times S^2$ has two transverse fiberings, one by A-curves and one by B-curves;
(ii) if A is not represented, $S^2 \times S^2$ is still fibered by the B-curves; however there is a unique $(A - B)$-curve and there are no A-curves.

Thus we may decompose $\mathcal{J} := \mathcal{J}_{\omega_\mu}$ into the disjoint union of two sets: $\mathcal{J}_0$ on which A is represented and $\mathcal{J}_1$ on which $A - B$ is represented. (For $\mu \in (\ell - 1, \ell]$ one defines $\mathcal{J}_k$ for $k \leq \ell$ to be the set of J for which $A - kB$ is represented.)

To go further, we need to use the consequences of the Riemann-Roch theorem that were mentioned above. Since the normal bundle to an A- or B-curve is trivial, every A- and every B- curve is regular, i.e. D_u is always onto in this case. Since regularity is an open condition this means that the set $\mathcal{J}_0$ is open. On the other hand the class $A - B$ cannot be represented by a regular curve u, i.e. D_u can never be surjective, since $c_1(L) = -2$ in this case. But the cokernel of D_u has constant rank, and one can show that this implies that $\mathcal{J}_1$ is a submanifold of $\mathcal{J}$ of codimension 2. (In fact the normal bundle to $\mathcal{J}_1$ in $\mathcal{J}_0$ at a point (u, J) can be identified with $\mathrm{coker} D_u$: see [1].) Thus $\mathcal{J}(\omega_\mu)$ is a stratified space when $1 < \mu \leq 2$ with 2 strata. The picture is similar when $\mu > 2$ except that there are now more strata: see [41]. This complicates the calculations that we present below.

The strata $\mathcal{J}_i$ as homogeneous spaces: In what follows we either suppose that $\mu = 1$ and let $i = 0$ or suppose that $1 < \mu \leq 2$ and let $i = 0, 1$.

Each stratum $\mathcal{J}_i$ contains an integrable element J_i. We may take J_0 to be the product structure $j \times j$ and J_1 to be the Hirzebruch structure obtained by identifying $S^2 \times S^2$ with the projectivization $\mathbb{P}(L_2 \oplus \mathbb{C})$, where $L_2 \to \mathbb{C}P^1$ is a holomorphic line bundle of Chern class 2. (Note that the section given by $\mathbb{P}(L_2 \oplus 0)$ is rigid, with self-intersection number -2: hence it corresonds to the antidiagonal.) Denote by K_i the maximal compact subgroup in the identity component of the complex automorphism group of J_i. Thus $K_0 \cong \mathrm{SO}(3) \times SO(3)$, while $K_1 \cong \mathrm{SO}(3) \times S^1$. Here the SO(3) factor can be identified with the diagonal subgroup of K_0, while the S^1 factor is generated by an S^1 action. This can either be thought of as coming from the action

[6] Given a symplectically embedded submanifold C of (M, ω) one can always choose $J \in \mathcal{J}(\omega)$ so as to preserve TC since the set of choices at each point is contractible. When C is 2-dimensional, every almost complex structure on it is integrable. Hence there is a complex structure j on C for which the obvious inclusion $\iota : (C, J) \to (M, J)$ is J-holomorphic .

$[w_0 : w_1] \mapsto [e^{2\pi i t} w_0 : w_1]$, or can be explicitly described by the formula:

$$\phi_t : S^2 \times S^2 \to S^2 \times S^2, \quad (z, w) \mapsto (z, R_{z,t} w),$$

where $R_{z,t}$ rotates the sphere about the axis through the points $\pm z$ by the angle $e^{2\pi i t}$. Note that the diagonal and antidiagonal are fixed by each ϕ_t.

Now consider the map

$$G_\mu := \mathrm{Ham}(S^2 \times S^2, \omega_\mu) \to \mathcal{J}_i, \quad \phi \mapsto \phi_*(J_i).$$

Since the stabilizer of J_i is precisely K_i this induces a quotient map

$$q_i : G/K_i \to \mathcal{J}_i. \tag{6}$$

The claim is that these maps q_i are homotopy equivalences. When $\mu = 1$ so that $\mathcal{J}_0$ is contractible, this will follow if we produce a map $s_0 : \mathcal{J}_0 \to G/K_0$ such that $s_0 \circ q_0 \sim \mathrm{id}$. The general argument is a little more complicated and may be found in [1].

To construct s_0 we proceed as follows. Fix a point x_0 in $S^2 \times S^2$ and for each $J \in \mathcal{J}_0$ let C_A, C_B be the unique A- and B-curves through x_0. We shall think of these as "coordinate axes" and of the families of A and B curves as corresponding to their parallel translates $w = const$ and $z = const$. More precisely, choose parametrizations $u : S^2 \to C_A, v : S^2 \to C_B$, and denote by $C_{w,A}$ the unique A-curve through the point $v(w) \in C_B$ and by $C_{z,B}$ the unique B-curve through $u(z) \in C_B$. Then define the map $\phi_J \in \mathrm{Diff}(S^2 \times S^2)$ by setting

$$\phi_J(y) = (z, w) \in S^2 \times S^2, \quad \text{where } y = C_{z,B} \cap C_{w,A}.$$

Because the fibrations by A and B curves are transverse, this map ϕ_J is a diffeomorphism. Though it is not quite symplectic, it turns out that it lies in the "Moser neighborhood" of Symp, in other words, it can be canonically isotoped into G_μ. If ϕ'_J denotes the endpoint of this isotopy, we define

$$s_0(J) := \phi'_J K_0 \in G_1/K_0.$$

To check that s_0 is well defined one must investigate the effect of changing the parametrizations u, v. Initially these are defined modulo PSL_*, the subgroup of $\mathrm{PSL}(2, \mathbb{C})$ consisting of elements that fix a point. Since this is homotopy equivalent to S^1, it is not hard to see that there is a consistent choice of u, v modulo $S^1 \subset \mathrm{SO}(3)$, at least over compact families. There are many ways of getting round this point, for example by using balanced maps as in [47].

Proof of Proposition 2.1(ii). When $\mu = 1$ there is just one stratum: $\mathcal{J} = \mathcal{J}_0$. Since $\mathcal{J}$ is contractible, the result is immediate from (6). □

Proof of Proposition 2.3 in the case $1 < \mu \leq 2$. In this case there are two strata so that $\mathcal{J} = \mathcal{J}_0 \cup \mathcal{J}_1$. By Alexander–Spanier duality we know that

$$H^i(\mathcal{J}_1) \cong H^{i+1}(\mathcal{J}_0), \qquad i \geq 0.$$

Abreu also shows that the spectral sequence of the fibration $K_i \to G_\mu \to \mathcal{J}_i$ degenerates for $i = 0, 1$. Therefore

$$H^*(G_\mu; \mathbb{R}) \cong H^*(\mathcal{J}_0) \otimes H^*(\mathrm{SO}(3) \times \mathrm{SO}(3)) \tag{i}$$

$$H^*(G_\mu; \mathbb{R}) \cong H^*(\mathcal{J}_1) \otimes H^*(S^1 \times \mathrm{SO}(3)). \tag{ii}$$

The argument is now a simple algebraic computation. Since G_μ is a Hopf space, its cohomology algebra is free. It follows that $H^*(\mathcal{J}_i)$ is also a free algebra. Let t (resp. $x.y$) be the image in $H^*(G_\mu;\mathbb{R})$ of the generator of $H^1(S^1)$ (resp. the generators of $H^3(\mathrm{SO}(3)\times\mathrm{SO}(3))$.) Comparing the ranks of H^3 in (i) and (ii) we see that $H^3(\mathcal{J}_1)$ must have at least one generator, say x_1. Therefore $H^4(\mathcal{J}_0)$ has a corresponding generator, say w. One now proves that there can be no other new generator: if there were, let k be the minimum dimension of such an element in $H^*(G_\mu)$ and use the isomorphism $H^k(\mathcal{J}_0) \cong H^{k-1}(\mathcal{J}_1)$ to show that there would also have to be a new generator in dimension $k-1$. Hence $H^*(G_\mu;\mathbb{R})$ is generated by t, x, y, w. □

The proof for the other cases $\mu > 2$ is similar, but both its aspects become more complicated because there are more strata $\mathcal{J}_k$. One must work harder to show that the $\mathcal{J}_k$ do form a stratification of $\mathcal{J}$ (see [41]), and the algebraic calculation is also considerably more elaborate.

3. Symplectic geometry of fibrations over S^2

Many of the proofs of the propositions above rely on properties of Hamiltonian fibrations over S^2. In this lecture we consider the geometric properties of such fibrations, relating them to the Hofer norm. The main ideas in this section come from Lalonde–McDuff and Polterovich.

3.1. **Generalities.** Consider a smooth fibration $\pi : P \to B$ with fiber M, where B is either S^2 or the 2-disc D. Here we consider S^2 to be the union $D_+ \cup D_-$ of two copies of D, with the orientation of D_+. We denote the equator $D_+ \cap D_-$ by ∂, oriented as the boundary of D_+, and choose some point $*$ on ∂ as the base point of S^2. Similarly, $B = D$ is provided with a basepoint $*$ lying on $\partial = \partial D$. In both cases, we assume that the fiber M_* over $*$ has a chosen identification with M.

Since every smooth fibration over a disc can be trivialized, we can build any smooth fibration $P \to S^2$ by taking two product manifolds $D_\pm \times M$ and gluing them along the boundary $\partial \times M$ by a based loop $\lambda = \{\lambda_t\}$ in $\mathrm{Diff}(M)$. Thus

$$P = (D_+ \times M) \cup (D_- \times M)/\sim, \quad (e^{it}, \lambda_t(x))_+ \equiv (e^{it}, x)_- .$$

A symplectic fibration is built from a based loop in $\mathrm{Symp}(M)$ and a Hamiltonian fibration from one in $\mathrm{Ham}(M)$. Thus the smooth fibration $P \to S^2$ is symplectic if and only if there is a smooth family of cohomologous symplectic forms ω_b on the fibers M_b. It is shown in [67, 46, 30] that a symplectic fibration $P \to S^2$ is Hamiltonian if and only if the fiberwise forms ω_b have a closed extension Ω. (Such forms Ω are called **ω-compatible**.) Note that, in any of these categories, two fibrations are equivalent if and only if their defining loops are homotopic.

From now on, we restrict to Hamiltonian fibrations. By adding the pullback of a suitable area form on the base we can choose the closed extension Ω to be symplectic. Observe that there is a unique class $u \in H^2(P,\mathbb{R})$ called the **coupling class** that restricts to $[\omega]$ on the fiber M_* and has the property that

$$\int_P u^{n+1} = 0 .$$

(This class will have the form $[\Omega] - \pi^*(a)$ for a suitable $a \in H^2(B, \partial B)$.) Correspondingly we decompose Ω as

$$\Omega = \tau + \pi^*(\alpha) \tag{7}$$

where $[\tau] = u$, and call τ the **coupling form.** (Although we will not need this, there is a canonical choice for the form τ depending only on the connection defined by Ω.)

The closed form Ω defines a connection on π whose horizontal distribution is Ω-orthogonal to the fibers. If γ is any path in B then $\pi^{-1}(\gamma)$ is a hypersurface in P whose characteristic foliation consists of the horizontal lifts of γ, and it is not hard to check that the resulting holonomy is Hamiltonian round every contractible loop, and hence round every loop. (A proof is given in [46, Thm 6.21].)

Thus, given Ω, π can be symplectically trivialized over each disc $D_\pm$ by parallel translation along a suitable set of rays. This means that there is a fiber preserving mapping

$$\Phi_\pm : \pi^{-1}(D) \to M \times D_\pm, \quad \Phi|_{M_*} = \mathrm{id}_M$$

such that the pushforward $(\Phi_\pm)_*\Omega$ restricts to the same form ω on each fiber $M \times pt$. These two trivializations differ by a loop

$$e^{it} \mapsto \phi_t = \Phi_+ \circ (\Phi_-)^{-1}(e^{it}) \in \mathrm{Symp}(M, \omega)$$

where e^{it} is a coordinate round the equator $\partial = D_- \cap D_+$.

Exercise 3.1. Check that this loop is homotopic to the defining loop for the fibration $P \to S^2$. (Since $\pi_1(G)$ is abelian for any group G it does not matter whether or not we restrict to based homotopies.)

Definition 3.2. *The* **monodromy** $\phi = \phi(P) \in \mathrm{Ham}(M)$ *of a fibration* $(P, \Omega) \to B$ *is defined to be the monodromy of the connection determined by* Ω *around the based oriented loop* $(\partial, *)$. *Using the trivialization of* P *over* ∂ *provided by* B *itself if* $B = D$ *or by* D_+ *if* $B = S^2$, *one gets a well defined lift* $\widetilde{\phi}$ *of* ϕ *to the universal cover* $\widetilde{\mathrm{Ham}}$.

Exercise 3.3. Start with a fibration $(P, \Omega) \to S^2$ and break it in half to get two fibrations $(P_+, \Omega) \to D_+$ and $(P_-, \Omega) \to D_-$. Here the inclusion $D_+ \to S^2$ is orientation preserving, while $D_- \to S^2$ is orientation reversing. What is the relation between the monodromies of these fibrations, (a) considered as elements in $\mathrm{Ham}(M)$ and (b) considered as elements in $\widetilde{\mathrm{Ham}}(M)$?

In Section 4.4 we shall consider a fibration over a base B which is the sphere S^2 with some points removed. We assume that near each deleted point (i.e. end of B) the fibration is identified with the product $[0, \infty) \times S^1 \times M$ and that the form Ω is normalized so that in the coordinates (s, t, x) it can be written as $a(s, t)ds \wedge dt + \omega - d_M H_t \wedge dt$, where d_M denotes the exterior derivative on M.

Exercise 3.4. Check that the monodromy of Ω round such an end is precisely the Hamiltonian flow of $H_t := H_{t+1}$. (The sign conventions are given in (1).)

3.2. **The area of a fibration.** We define the *area* of a fibration $(P, \Omega) \to B$ to be:

$$\mathrm{area}\,(P, \Omega) := \frac{\mathrm{vol}\,(P, \Omega)}{\mathrm{vol}\,(M, \omega)} = \frac{\int_P \Omega^{n+1}}{(n+1)\int_M \omega^n}.$$

Thus a product fibration $(B \times M, \alpha_B + \omega)$ has area $\int_B \alpha_B$.

Exercise 3.5. Decompose Ω as $\tau + \pi^*(\alpha)$ as in (7). Show that area $(P,\Omega) = \int_B \alpha$.

The next definition describes ways to use this area to measure the size of elements of $\mathrm{Ham}(M)$ or $\widetilde{\mathrm{Ham}}(M)$.

Definition 3.6. (i) $\widetilde{a}^+(\widetilde{\phi})$ *(resp.* $a^+(\phi)$*) is the infimum of* area (P,Ω) *taken over all* ω*-compatible symplectic forms* Ω *on the fibration* $P \to D$ *with monodromy* $\widetilde{\phi}$ *(resp.* ϕ*).*

(ii) $a(\phi)$ *is the infimum of* area (P,Ω) *taken over all fibrations* $(P,\Omega) \to S^2$ *with monodromy* ϕ*.*

(iii) $\widetilde{a}^-(\widetilde{\phi}) := \widetilde{a}^+(\widetilde{\phi}^{-1})$ *and* $a^-(\phi) := a^+(\phi^{-1})$.

We now show that these area measurements agree with Hofer type measurements. Recall that $\rho^+(\phi)$ is the infimum of $\mathcal{L}^+(H_t)$ over all Hamiltonians H_t whose times 1 map is ϕ. Similarly, we define $\widetilde{\rho}^+(\widetilde{\phi})$ to be the infimum of $\mathcal{L}^+(H_t)$ over all Hamiltonians H_t whose flow over $t \in [0,1]$ is a representative of the element $\widetilde{\phi} \in \widetilde{\mathrm{Ham}}(M)$.

The following lemma is a slightly sharper version of Polterovich's results in [59].

Proposition 3.7. (i) $\widetilde{\rho}^+(\widetilde{\phi}) = \widetilde{a}^+(\widetilde{\phi})$;

(ii) $\rho^+(\phi) + \rho^-(\phi) = a(\phi)$;

We prove (i). The proof of (ii) follows easily (see [43]).

Proof that $\widetilde{\rho}^+(\widetilde{\phi}) \geq \widetilde{a}^+(\widetilde{\phi})$:

This is by direct construction. Suppose that the path ϕ_t^H generated by H_t is $\widetilde{\phi}$. For simplicity let us suppose that the functions $\min(t) = \min_x H_t(x)$ and $\max(t) = \max_x H_t(x)$ are smooth, so that by replacing H_t by $H_t - \max(t)$ we have that $\max_x H_t(x) = 0$ for all t. Suppose also that $H_t(x) = 0$ for all $x \in M$ and all t sufficiently near $0,1$.[7] Then define the graph Γ_H of H_t by

$$\Gamma_H := \{(x,t,H_t(x)) : x \in M, t \in [0,1]\} \subset M \times [0,1] \times \mathbb{R}.$$

For some small $\varepsilon > 0$ choose a smooth function $\mu(t) : [0,1] \to [0,+2\varepsilon]$ such that

$$\int_0^1 \mu(t)\,dt = \varepsilon\,.$$

Consider the following **thickening of the region over** Γ_H:

$$R_H^+(\varepsilon) := \{(x,t,h) \,|\, H_t(x) \leq h \leq \mu(t)\} \subset M \times [0,1] \times \mathbb{R}.$$

Note that if μ is chosen for t near $0,1$ to be tangent to the lines $t = \text{const}$ at $t = 0,1$ we may arrange that $R_H^+(\varepsilon)$ is a manifold with corners along $t = 0,1$. (Recall that $H_t \equiv 0$ for t near $0,1$.)

Let $\Omega_0 = \omega + dt \wedge dh$ be the standard symplectic form on $M \times [0,1] \times \mathbb{R}$. Then the monodromy of the hypersurface Γ_H (oriented as the boundary of $R_H^+(\varepsilon)$) is precisely ϕ_t^H, while the rest of the boundary has trivial monodromy. Further, it is easy to define a projection π from $R_H^+(\varepsilon)$ to the half disc HD whose fibers all lie in the hypersurfaces $t = \text{const}$. Thus, after rounding the corners, we get a fibered space $\pi : R_H^+(\varepsilon) \to D$

[7] This can be arranged without altering the time-1 map or the Hofer length. For this and other technicalities see [27] or [43].

with monodromy $\widetilde{\phi}$. It remains to check that the area of $(R_H^+(\varepsilon), \Omega_0)$ (before rounding corners) is precisely $\mathcal{L}^+(H_t) + \varepsilon$. □

Remark 3.8. Similarly, we can define a manifold with corners $R_H^-(\varepsilon)$ that thickens the region below Γ_H by setting

$$R_H^-(\varepsilon) := \{(x,t,h) \mid \min(t) - \mu(t) \le h \le H_t(x)\} \subset M \times [0,1] \times \mathbb{R}.$$

Note that area $(R_H^-(\varepsilon)) = \mathcal{L}^-(H_t) + \varepsilon$.

To prove the other inequality we combine Polterovich's arguments from [57]§3.3 and [59]§3.3.

Lemma 3.9. $\widetilde{a}^+(\widetilde{\phi}) \ge \widetilde{\rho}^+(\widetilde{\phi})$.

Proof. Suppose we are given a fibration $(P,\Omega) \to D$ with area $< \widetilde{\rho}^+(\widetilde{\phi})$ and monodromy $\widetilde{\phi}$. By Moser's theorem we may isotop Ω so that it is a product in some neighborhood $\pi^{-1}(\mathcal{N})$ of the base fiber M_*. Identify the base D with the unit square $K = \{0 \le x,y \le 1\}$ taking $\mathcal{N}$ to a neighborhood of $\partial' K = \partial K - \{1\} \times (0,1)$, and then identify P with $K \times M$ by parallel translating along the lines $y = \text{const}$. In these coordinates, the form Ω may be written as

$$\Omega \;=\; \omega + d_M F' \wedge dy + L' dx \wedge dy$$

where F', L' are suitable functions on $K \times M$ and d_M denotes the fiberwise exterior derivative. Because Ω is a product near $\pi^{-1}(\partial' K)$, $d_M F' = 0$ there and L' reduces to a function of x,y only. By subtracting a suitable function $c(x,y)$ from F' we can arrange that $F = F' - c(x,y)$ has zero mean on each fiber $\pi^{-1}(x,y)$ and then write $L' + \partial_x c(x,y)$ as $-L + a(x,y)$ where L also has zero fiberwise means. Thus

$$\Omega \;=\; \omega + d_M F \wedge dy - L dx \wedge dy + a(x,y) dx \wedge dy\,, \tag{8}$$

where both F and L vanish near $\pi^{-1}(\partial' K)$ and have zero fiberwise means. Since Ω is symplectic it must be nondegenerate on the 2-dimensional distribution *Hor* formed by the Ω-orthogonals to the fibers. Hence we must have $-L(x,y,z) + a(x,y) > 0$ for all $x,y \in K, z \in M$. Moreover, because L has zero fiberwise means, area $(P,\Omega) = \int a(x,y) dx \wedge dy$. Hence

$$\int \max_{z \in M} L(x,y,z)\, dx \wedge dy < \int a(x,y)\, dx \wedge dy = \text{area}\,(P,\Omega)\,. \tag{9}$$

We claim that $-L$ is the curvature of the induced connection Ω_Γ. To see this, consider the vector fields $X = \partial_x, Y = \partial_y - \text{s.grad} F$ on P that are the horizontal lifts of ∂_x, ∂_y.[8] It is easy to check that their commutator $[X,Y] = XY - YX$ is vertical and that

$$[X,Y] = -\text{s.grad}(\partial_x F) = \text{s.grad} L$$

on each fiber $\pi^{-1}(x,y)$ as claimed. (In fact, the first three terms in (8) make up the coupling form τ_Γ.)

Now let $f_s \in \text{Ham}(M)$ be the monodomy of Ω_Γ along the path $t \mapsto (s,t), t \in [0,1]$. (This is well defined because all fibers have a natural identification with M.) The

[8] Here the symplectic gradient s.grad$F$ is defined by setting $\omega(\text{s.grad}F, \cdot) = -d_M F(\cdot)$.

path $s \mapsto f_s$ is a Hamiltonian isotopy from the identity to $\phi = f_1$, and it is easy to see that it is homotopic to the original path $\widetilde{\phi}$ given by parallel transport along $t \mapsto [1,t]$. (An intermediate path p_T in the homotopy might consist of the path $s \mapsto f_s^T$ for $0 \le s \le T$, where f_s^T is the monodromy along $t \mapsto (s/T, t), t \in [0,T]$, followed by the lift of $\widetilde{\phi}_s, s \in [T,1]$, to $\widetilde{\text{Ham}}$.) Therefore $\mathcal{L}^+(f_s) \ge \widetilde{\rho}^+(\widetilde{\phi})$, and we will derive a contradiction by estimating $\mathcal{L}^+(f_s)$.

To this end, let X^s, Y^t be the (partially defined) flows of the vector fields X, Y on P and set $h_{s,t} = Y^t X^s$. Consider the 2-parameter family of (partially defined) vector fields $v_{s,t}$ on P where $v_{s_0,t_0}(p)$ is tangent to the path $s \mapsto h_{s,t_0}$ at $h_{s_0,y_0}(p)$ for $p = (x,y,z) \in P$. Thus

$$v_{s,t} = \partial_s h_{s,t} = Y^t_*(X) \quad \text{on} \quad \operatorname{Im} h_{s,t}\,.$$

In particular $v_{s,1}(x,y,z)$ is defined when $y = 1, s \le x$. Since $f_s = h_{s,1}$ we are interested in calculating the vertical part of $v_{s,1}(s,1,z)$. Since the points with $y = 1$ are in $\operatorname{Im} h_{s,t}$ for all (s,t) we may write

$$\begin{aligned} v_{s,1} &= v_{s,0} + \int_0^1 \partial_t(v_{s,t})\, dt \\ &= \partial_x + \int_0^1 Y^t_*([X,Y])\, dt\,. \end{aligned}$$

We saw above that $[X,Y] = \text{s.grad}L$. Hence $Y^t_*([X,Y]) = \text{s.grad}(L \circ (Y^t)^{-1})$ and

$$\begin{aligned} v_{s,1}(s,1,z) &= \partial_x + \int_0^1 \text{s.grad}(L((Y^t)^{-1}(s,1,z))\, dt \\ &= \partial_x + \text{s.grad} \int_0^1 L(s, 1-t, (Y^t_v)^{-1}(z))\, dt \end{aligned}$$

where Y^t_v denotes the vertical part of Y^t. Hence the Hamiltonian H_s that generates the path $f_s, s \in [0,1]$, and has zero mean satisfies the inequality

$$H_s(z) \;\le\; \int \left(\max_{z \in M} L(s,t,z) \right) dt \;\le\; \int a(s,t)\, dt$$

since $L(s,t,z) \le a(s,t)$ by (9). Thus $\widetilde{\rho}^+(\widetilde{\phi}) \le \text{area}\, P$, contrary to hypothesis. □

4. The quantum homology of fibrations over S^2

In this lecture we show how to use the Seidel representation

$$\mathcal{S} : \pi_1(\text{Ham}(M,\omega)) \to (\text{QH}_{ev}(M))^\times$$

to estimate areas of fibrations and also get information on the homotopy properties of Hamiltonian fibrations.

4.1. **Quantum and Floer homology.** First of all, what is the small quantum homology ring $\mathrm{QH}_*(M)$? Because it is more efficient, we shall use the formulation in [47], which is slightly different from [32, 40].

Denote by Λ^{univ} the universal Novikov ring formed by all formal power series with rational coefficients λ_κ of the form

$$\lambda = \sum_{\kappa \in \mathbb{R}} \lambda_\kappa t^\kappa, \qquad \#\{\kappa \in \mathbb{R} \,|\, \lambda_\kappa \neq 0, \kappa \geq c\} < \infty \ \text{ for all } \ c \in \mathbb{R}.$$

Thus the power κ of t is allowed to go to $-\infty$. Set $\Lambda := \Lambda^{\mathrm{univ}}[q, q^{-1}]$, where q is a variable of degree 2. Additively, the quantum homology is simply the usual homology with coefficients in Λ:

$$\mathrm{QH}_*(M) := \mathrm{QH}_*(M; \Lambda) = H_*(M) \otimes \Lambda .$$

We may define an $\mathbb{R}$ grading on $\mathrm{QH}_*(M; \Lambda)$ by setting

$$\deg(\alpha \otimes q^d t^\kappa) = \deg(\alpha) + 2d, \qquad \alpha \in H_j(M),$$

but can also think of $\mathrm{QH}_*(M; \Lambda)$ as $\mathbb{Z}/2\mathbb{Z}$-graded with

$$\mathrm{QH}_{\mathrm{ev}} = H_{\mathrm{ev}}(M) \otimes \Lambda, \quad \mathrm{QH}_{\mathrm{odd}} = H_{\mathrm{odd}}(M) \otimes \Lambda .$$

The quantum intersection product is linear over Λ and is defined on classes $\alpha \in H_i(M), \beta \in H_j(M)$ as follows. We abbreviate $c := c_1(M)$.

$$\alpha * \beta = \sum_{B \in H_2^S(M)} (\alpha * \beta)_B \otimes q^{-c(B)} t^{-\omega(B)} \ \in \ \mathrm{QH}_{i+j-2n}(M), \tag{10}$$

where $(\alpha * \beta)_B \in H_{i+j-2n+2c_1(B)}(M)$ is defined by the requirement that

$$(\alpha * \beta)_B \cdot \gamma = \mathrm{GW}^M_{B,3}(\alpha, \beta, \gamma) \quad \text{for all } \ \gamma \in H_*(M). \tag{11}$$

Here $\mathrm{GW}^M_{B,3}(\alpha, \beta, \gamma)$ denotes the Gromov–Witten invariant that counts the number of B-spheres in M meeting representing cycles for the classes $\alpha, \beta, \gamma \in H_*(M)$, and we have written $\cdot$ for the usual intersection pairing on $H_*(M) = H_*(M, \mathbb{Q})$. In good cases, one can compute $\mathrm{GW}^M_{B,3}(\alpha, \beta, \gamma)$ as the intersection number of the class $\alpha \times \beta \times \gamma$ in M^3 with the image of the evaluation map

$$\mathrm{ev} : \mathcal{M}(M, B; J) \to M^3, \quad u \mapsto \big(u(0), u(1), u(\infty)\big),$$

where J is a generic ω-tame almost complex structure and $\mathcal{M}(M, B; J)$ denotes the $(2n + 2c(B))$-dimensional moduli space of J-holomorphic B-spheres in M as discussed in Section 2.4. (In general one needs to use the virtual moduli cycle.) Note that $\alpha \cdot \beta = 0$ unless $\dim(\alpha) + \dim(\beta) = 2n$ in which case it is the algebraic number of intersection points of the cycles. The product $*$ is extended to $\mathrm{QH}_*(M)$ by linearity over Λ, and is associative. Moreover, it preserves the $\mathbb{R}$-grading.

This product $*$ gives $\mathrm{QH}_*(M)$ the structure of a graded commutative ring with unit $\mathbb{1} = [M]$. Further, the invertible elements in $\mathrm{QH}_{\mathrm{ev}}(M)$ form a commutative group $(\mathrm{QH}_{\mathrm{ev}}(M, \Lambda))^\times$ that acts on $\mathrm{QH}_*(M)$ by quantum multiplication.

We shall say very little about Floer homology; basic definitions can be found for example in [38, 65, 45, 47]. It is the Morse complex of the action functional on the loop space of M. Given a Hamiltonian function $H_h := H_{t+1}$ satisfying a suitable nondegeneracy hypothesis, one forms a chain complex that is generated as a Λ-module

by the 1-periodic orbits $\mathbf{x}$ of H. When H is autonomous (i.e. independent of time) and is sufficiently C^2-small, these orbits are simply the critical points of H. The $\mathbf{y}$-coefficient of the boundary $\delta(\mathbf{x})$ is formed by counting isolated **Floer trajectories** from $\mathbf{x}$ to $\mathbf{y}$, i.e. solutions $u : \mathbb{R} \times S^1 \to M$ of the **Floer equation**

$$\partial_s u + J(u)(\partial_t(u) - X_H(u)) = 0\,, \quad \lim_{s\to-\infty} u(s,\cdot) = \mathbf{x}\,, \quad \lim_{s\to\infty} u(s,\cdot) = \mathbf{y}\,. \tag{12}$$

The resulting homology groups are denoted $\mathrm{FH}_*(M;H,J)$. They are all canonically isomorphic. For each (nondegenerate) H, there is a canonical isomorphism Φ from quantum homology $\mathrm{QH}_*(M;\Lambda)$ to Floer homology $\mathrm{FH}_*(M;H,J)$ given by counting maps $u : \mathbb{C} \to M$ that are J-holomorphic near the unit disc D with Floer boundary conditions at infinity; in other words, if we identify $\mathbb{C} \setminus D$ with the cylinder $(0,\infty) \times S^1$ then $u(s,t)$ satisfies the Floer equation (12) for $s > 2$. These isomorphisms are called PSS isomorphisms after Piunikhin–Salamon–Schwarz [55]. This construction is important in the discussion of the ABW inequalities.

4.2. **The Seidel representation $\mathcal{S}$.** Now consider the fibration $P_\lambda \to S^2$ constructed from a loop $\lambda \in \pi_1(\mathrm{Ham}(M))$ as in §2.4. The manifold P_λ carries two canonical cohomology classes, the first Chern class of the vertical tangent bundle

$$c := c_1(TP_\lambda^{\mathrm{vert}}) \in H^2(P_\lambda, \mathbb{Z})\,,$$

and the coupling class u_λ, i.e. the unique class in $H^2(P_\lambda, \mathbb{R})$ such that

$$i^*(u_\lambda) = [\omega]\,, \qquad u_\lambda^{n+1} = 0\,,$$

where $i : M \to P_\lambda$ is the inclusion of a fiber. We denote by $H_2^\sigma(P)$ set of **section classes** in $H_2(P_\lambda, \mathbb{R})$, i.e. the classes that project to the positive generator of $H_2(S^2;\mathbb{Z})$. We then define the **Seidel element**

$$\mathcal{S}(\lambda) := \sum_{\tilde B \in H_2^\sigma(P)} \alpha_{\tilde B} \otimes q^{-c(\tilde B)} t^{-u_\lambda(\tilde B)} \tag{13}$$

where, for all $\gamma \in H_*(M)$,

$$\alpha_{\tilde B} \cdot_M \gamma = \mathrm{GW}^{P_\lambda}_{\tilde B,3}([M],[M],\gamma)\,. \tag{14}$$

Note that $\mathcal{S}(\lambda)$ belongs to the strictly commutative part $\mathrm{QH}_{\mathrm{ev}}$ of $\mathrm{QH}_*(M)$. Moreover $\deg(\mathcal{S}(\lambda)) = 2n$. It is shown in [40] (using ideas from Seidel [67] and Lalonde–McDuff–Polterovich[32]) that for all $\lambda_1, \lambda_2 \in \pi_1(\mathrm{Ham}(M))$

$$\mathcal{S}(\lambda_1\lambda_2) = \mathcal{S}(\lambda_1) * \mathcal{S}(\lambda_2)\,, \qquad \mathcal{S}(0) = \mathbb{1}\,,$$

where 0 denotes the constant loop. Therefore $\mathcal{S}(\lambda)$ is invertible for all λ and we get a representation

$$\mathcal{S} : \pi_1(\mathrm{Ham}(M,\omega)) \;\to\; (\mathrm{QH}_{\mathrm{ev}}(M;\Lambda))^\times\,.$$

Moreover since all ω-compatible forms are deformation equivalent, $\mathcal{S}$ is independent of the choice of Ω. It is often useful to identify $(\mathrm{QH}_{\mathrm{ev}}(M;\Lambda))^\times$ with the space $\mathrm{Aut}(\mathbb{Q}_*(M;\Lambda))$ of automorphisms of $\mathrm{QH}_*(M;\Lambda)$ considered as a (left) module over itself via the correspondence $\alpha \mapsto \alpha * \cdot$. We denote by Ψ the resulting representation:

$$\Psi : \pi_1(\mathrm{Ham}(M,\omega)) \;\to\; \mathrm{Aut}(\mathrm{QH}_*(M;\Lambda))\,, \qquad \Psi(\lambda)(a) := \mathcal{S}(\lambda) * a\,.$$

In this language, $\mathcal{S}(\lambda) = \Psi(\lambda)(\mathbb{1})$.

The fact that $\mathcal{S}(\lambda)$ is a unit means that Hamiltonian fibrations over S^2 always have plenty of holomorphic sections. This has consequences for the cohomology of P_λ. Indeed one deduces that the map $H_*(M;\mathbb{R}) \to H_*(P_\lambda;\mathbb{R})$ is injective by showing that any class β in its kernel must be annihilated by $\Phi(\lambda)$, i.e. $\Phi(\lambda)(\beta) = 0$. Since $\Phi(\lambda)$ is an isomorphism, this implies $\beta = 0$. Since these arguments are sketched in [44], we shall concentrate here on explaining other applications.

4.3. **Using $\mathcal{S}$ to estimate area.** Now let Ω be any ω-compatible symplectic form on P_λ. As in Exercise 3.5, its cohomology class has the form

$$[\Omega] = u_\lambda + \pi^*([\alpha])$$

where $\text{area}\,(P_\lambda, \Omega) = \int_{S^2} \alpha$. The next results are due to Seidel.

Consider the valuation $v : \mathrm{QH}_*(M) \to \mathbb{R}$ defined by

$$v\left(\sum \alpha_{d,\kappa} \otimes q^d t^\kappa\right) = \sup\{\kappa : \alpha_{d,\kappa} \neq 0\}\,. \tag{15}$$

It follows from the definition of the quantum intersection product in (10), (11) that $v(\alpha * \beta) \leq v(\alpha) + v(\beta)$. In fact, the following stronger statement is true.

Denote the usual intersection product by $\cap$, so that $a * b - a \cap b$ is the quantum correction to the usual product. Define

$$\hbar = \hbar(M) = \min\{\omega(B) : B \neq 0\,,\ \text{some}\ \ \mathrm{GW}^M_{B,3}(\alpha,\beta,\gamma) \neq 0\}\,, \tag{16}$$

and note that $\hbar(M) > 0$: standard compactness results imply that for each $c > 0$ there are only finitely many classes B with $\omega(B) \leq c$ that can be represented by a J-holomorphic curve for generic J, and it is only such classes that give rise to nonzero invariants. If all the invariants $\mathrm{GW}^M_{B,3}(\alpha,\beta,\gamma)$ with $\alpha,\beta,\gamma \in H_*(M)$ and $B \neq 0$ vanish, we set $\hbar = \infty$.

Lemma 4.1. *For all $\alpha, \beta \in \mathrm{QH}_*(M)$, $v(\alpha * \beta - \alpha \cap \beta) \leq v(\alpha) + v(\beta) - \hbar(M)$.*

Proof. This follows immediately from the definitions. □

Proposition 4.2 (Seidel). *For each loop λ in* $\mathrm{Ham}(M)$

$$\text{area}\,(P_\lambda, \Omega) \ > \ v(\mathcal{S}(\lambda))\,.$$

Proof. Let $\mathcal{S}(\lambda) = \sum_{d,\kappa} \alpha_{d,\kappa} \otimes q^d t^\kappa$, and let $v(\mathcal{S}(\lambda)) = \kappa_0$. Then $\alpha_{d,\kappa_0} \neq 0$. By definition α_{d,κ_0} is determined by a count of J-holomorphic curves in (P_λ, Ω) in the class $\widetilde{B}$ where $c(\widetilde{B}) = -d, u_\lambda(\widetilde{B}) = -\kappa_0$. Hence this moduli space cannot be empty. Therefore

$$\begin{aligned} 0 \ < \ [\Omega](\widetilde{B}) \ &= \ \pi^*([\alpha])(\widetilde{B}) + u_\lambda(\widetilde{B}) \\ &= \ \textstyle\int_{S^2} \alpha - \kappa_0 \\ &= \ \text{area}\,(P_\lambda, \Omega) - \kappa_0\,. \end{aligned}$$

Thus $\text{area}\,(P_\lambda, \Omega) > \kappa_0$ for all Ω, as claimed. □

Corollary 4.3. *In these circumstances $\widetilde{\rho}^+(\lambda) \geq v(\mathcal{S}(\lambda))$.*

Proof. Combine Proposition 3.7(i) with Proposition 4.2. □

For applications of this estimate, see [43] and Lemma 5.7 below.

4.4. **The ABW inequalities.** We now give a very brief sketch of Entov's explanation of the ABW (Agnihotri–Belkale–Woodward) inequalities concerning the eigenvalues of products of unitary matrices. Let us first consider the simplest case $G = \mathrm{SU}(2)$. The question is: suppose we know the eigenvalues of the elements $A_1, A_2 \in G$. What can we say about the eigenvalues of A_1A_2? Equivalently, suppose we know that $A_1A_2A_3 = \mathbb{1}$. What can we say about their eigenvalues? In this case, A_j has eigenvalues $e^{\pm 2\pi i\zeta_j}$ for a unique $\zeta_j \in [0,1/2]$, and one can prove by computation that the triples $(\zeta_1,\zeta_2,\zeta_3)$ corresponding to solutions of the identity $A_1A_2A_3 = \mathbb{1}$ form the convex polygon in $[0,1/2]\times[0,1/2]\times[0,1/2] \subset \mathbb{R}^3$ described by the inequalities

$$\zeta_1+\zeta_2+\zeta_3 \le 1, \quad \zeta_1 \le \zeta_2+\zeta_3, \quad \zeta_2 \le \zeta_3+\zeta_1, \quad \zeta_3 \le \zeta_1+\zeta_2\,.$$

There are corresponding inequalities that describe the relations among the eigenvalues of matrices $A_j \in \mathrm{SU}(n), j = 1,\dots,N$ such that $\prod_j A_j = \mathbb{1}$. Belkale found a generating set of inequalities for the resulting convex set. Agnihotri–Woodward then observed that one may choose generators that are in bijective correspondence to the nontrivial correlators in the quantum Schubert calculus. (These are the correlators that generate the relations in the quantum cohomology of the Grassmannian.) We now give a brief sketch of Entov's explanation of this fact.

Entov considers the obvious action of $\mathrm{SU}(n)$ on the Grassmannian $M = \mathrm{Gr}(r,n)$ of complex r planes in $\mathbb{C}^n$. This action is Hamiltonian. Moreover each $A \in G \subset \mathrm{Ham}(M)$ is the time-1 map of a Hamiltonian flow generated by a function $H_A : M \to \mathbb{R}$ whose critical points x_I correspond to the fixed points of the action of A on $\mathrm{Gr}(r,n)$ and hence to the r-planes that are spanned by eigenvectors. Thus the label I is a subset of $\{1,\dots,n\}$ of cardinality r. Moreover the critical value of H_A at x_I is just $\sum_{k\in I}\zeta_{Ak}$, where $\zeta_{A1} \ge \cdots \ge \zeta_{An}$ are the eigenvalues of A. These critical values are precisely the terms that occur in the general ABW inequalities. (When $n = 2$ we take $r = 1$ and so are considering the action of $\mathrm{SU}(2)$ on $\mathbb{C}P^1$.) An important point here is that this Hamiltonian H_A is slow in the sense of Section 1.3 so that the path it generates does minimize Hofer length. This is why the methods sketched below give sharp estimates.

Suppose given a product $\prod_{j=1}^N A_j = \mathbb{1}$ of N natrices. Entov constructs a Hamiltonian bundle $M \to (E,\Omega) \to B$ over $B = S^2 \setminus \{z_1,\dots,z_N\}$ which is trivial topologically but supports a symplectic form Ω whose monodromy round the jth puncture is (conjugate to) A_j. When $\prod_j A_j = \mathbb{1}$ it is possible to construct E by cutting and pasting so that the area of (E,Ω) is arbitrarily close to zero. Then one chooses an almost complex structure $\widetilde{J}$ on E so that the projection to B (with its obvious structure) is holomorphic and so that $\widetilde{J}$ is normalized near each puncture to be compatible with the monodromy. One can do this in such a way that, in the obvious product coordinates near each end, the $\widetilde{J}$-holomorphic sections are graphs of solutions of the Floer trajectory equation (12) for the Hamiltonian functions H_j that generate the monodromy A_j. This has several consequences:

- at the jth end each $\widetilde{J}$-holomorphic section $\tilde{u} : B \to E$ of finite energy converges to some critical point $x_{I_j} \in M$ of the appropriate Hamiltonian H_j;

- the symplectic area $\int_B \tilde{u}^*(\Omega)$ of the section (which has to be positive) is now the sum of three terms; the area of (E,Ω), the term $\omega(A)$ where $A \in H_2(M)$ measures the homology class of $\tilde{u}$, and the boundary contribution, which is precisely equal to

the sum of the critical values $H_{A_j}(x_{I_j})$ at the limiting critical points. Thus for each section $\tilde{u}$ one obtains an inequality

$$\sum_{j=1}^{N} -H_{A_j}(x_{I_j}) \leq \omega(A)\,.$$

Entov calls these **action inequalities**. They are generated in the same way as the area inequality of Proposition 4.2. Although they seem rather different, because they come from a section of a trivial bundle over a noncompact space rather than a section of a nontrivial bundle over S^2, the next paragraph explains that there is little essential difference in the set up.

We need to understand the moduli spaces of sections of $(E, \widetilde{J}) \to B$. When are they nonempty? Here we are looking at sections over a punctured sphere with Floer boundary conditions. One can "cap off" these ends (by the same gluing arguments that establish the PSS isomorphism between quantum and Floer homology), constructing from such a section $\tilde{u}$ a section $\tilde{v}$ of the trivial bundle $S^2 \times M \to S^2$ such that $\tilde{v}(z_j) \in z_j \times C_{I_j}$, where C_{I_j} is the Schubert cycle corresponding to the critical point x_{I_j}, i.e. the unstable manifold of x_{I_j} under the downward gradient flow of H_j. Conversely, given a J-holomorphic sphere $v : S^2 \to M$ such that $v(z_j) \in C_{I_j}$ for all j, one can construct from its graph $\tilde{v}$ a $\widetilde{J}$-holomorphic section $\tilde{u}$ of $(E, \widetilde{J}) \to B$ with the corresponding limiting behaviour. Therefore we need to understand when these spheres v exist. Here the points $z_j \in S^2$ are fixed. Therefore, the number of such maps v is measured by the Gromov–Witten invariant

$$\mathrm{GW}_{A,N}^{M,\{1,\dots,N\}}([C_{I_1}], \dots, [C_{I_N}])\,. \tag{17}$$

The superscript $\{1, \dots, N\}$ on GW indicates that the marked points are fixed, so that this invariant measures the number of intersections of $[C_{I_1}] \times \cdots \times [C_{I_N}]$ with the evaluation map

$$\mathrm{ev} : \mathcal{M}(M, A; J) \to M^N\,, \quad u \mapsto u(z_1, \dots, z_N)\,.$$

These invariants (or correlators) are nonzero precisely when the A-component of the quantum product $[C_{I_1}] * \cdots * [C_{I_{N-1}}]$ has nontrivial intersection with the Schubert class $[C_{I_N}]$. These are the nontrivial correlators in quantum Schubert calculus to which Agnihotri–Belkale refer. Whenever one such correlator is nonzero, the corresponding moduli space of sections cannot be empty, and hence one gets an inequality which turns out to be precisely one of the ABW inequalities.

This is just one application of Entov's work. He generalizes Proposition 3.7 and Corollary 4.3 to give an interpretation of the action inequalities in terms of an appropriate Hofer distance between the conjugacy classes in $\mathrm{Ham}(M)$ containing the elements A_i.

5. Existence of length minimizing paths in $\mathrm{Ham}(M)$.

Recall that a Hamiltonian $H_t, t \in [0, 1]$, is said to have a **fixed minimum** at the point p if each function H_t takes its minimum value at p. In this lecture we sketch the proof of the following result from [43]. Oh gives a quite different proof of this in [52].

Theorem 5.1. *If $H_t, t \in [0,1]$, has both a fixed maximum and a fixed minimum and if it is sufficiently small in the C^2-norm, then the path ϕ_t^H that it generates in* $\mathrm{Ham}(M)$ *minimizes the Hofer norm, i.e.*

$$\rho(\phi_1^H) = \mathcal{L}(\phi_t^H).$$

Remarks
(i) The existence of the fixed extrema is necessary: any path without a fixed minimum for example can be altered (keeping the endpoints fixed) so as to preserve $\mathcal{L}^+$ but decrease $\mathcal{L}^-$. Therefore for C^2 small paths the above proposition gives a necessary and sufficient condition for them to realise the Hofer norm.
(ii) It is possible to extract from the proof a precise description of how small H_t must be for the above result to hold. The bound depends only on (M, ω).

5.1. **Idea of the proof.** If $\rho(\phi_1^H) < \mathcal{L}(\phi_t^H)$ then there is another shorter Hamiltonian, say K_t, with the same time-1 map. Therefore either $\mathcal{L}^+(K_t) < \mathcal{L}^+(H_t)$ or $\mathcal{L}^-(K_t) < \mathcal{L}^-(H_t)$; say the former. We then consider the space $(R_{K,H}(2\varepsilon), \Omega_0)$ formed by gluing the thickened region $(R_H^-(\varepsilon), \Omega_0)$ under the graph of H to the region $(R_K^+(\varepsilon), \Omega_0)$ above the graph of K along the monodromy of the hypersurfaces Γ_K and Γ_H. (For definitions, see the proof of Proposition 3.7.) This gives rise to a space $(R_{K,H}(2\varepsilon), \Omega_0)$ with trivial monodromy round its boundary and that fibers over a disc. Identifying the boundary of this disc to a point, one therefore gets a symplectic fibration

$$(P_{K,H}(2\varepsilon), \Omega_0) \to S^2.$$

By construction,

$$\text{area}\,(P_{K,H}(2\varepsilon), \Omega_0) \;=\; \mathcal{L}^+(K_t) + \mathcal{L}^-(H_t) + 2\varepsilon \;<\; \mathcal{L}(H_t),$$

provided that ε is sufficiently small.

Next we use the following fact from [27] which is proved by a simple geometric construction. Recall that the capacity of a ball of radius r is πr^2.

Lemma 5.2. *If H_t is sufficiently small in the C^2-norm and has a fixed maximum (resp. minimum), then for all $\varepsilon > 0$ it is possible to embed a ball of capacity $\mathcal{L}(H_t)$ in $R_H^-(\varepsilon)$ (resp. $R_H^+(\varepsilon)$).*

Therefore the manifold $(P_{K,H}(2\varepsilon), \Omega_0)$ contains an embedded ball with capacity larger that its area. If the fibration $(P_{K,H}(2\varepsilon), \Omega_0) \to S^2$ were symplectically trivial, this would contradict the nonsqueezing theorem proven in [27] for so-called "quasi-cylinders". As it is, we have no control on the topology of this fibration: it is built from the loop $\lambda = (\phi_t^K) * (\phi_t^H)^{-1}$ which does not have to contract in $\mathrm{Ham}(M)$. Therefore, what we need to do is prove a version of the nonsqueezing theorem that holds in this context.

The rest of the lecture will discuss this question. Full details of the argument outlined above can be found in [43, 27].

5.2. **Nonsqueezing for fibrations of small area.**

Definition 5.3. We say that **the nonsqueezing theorem holds for the fibration** $(P,\Omega) \to S^2$ if area(P,Ω) constrains the radius of any embedded symplectic ball $B^{2n+2}(r)$ in (P,Ω) by the inequality

$$\pi r^2 \leq \text{area}\,(P,\Omega)\,.$$

Here is a question that is still open for arbitrary manifolds (M,ω).

Question 5.4. *Is there an $\varepsilon = \varepsilon(M,\omega) > 0$ such that the nonsqueezing theorem holds for all fibrations $(P_\lambda,\Omega) \to S^2$ whose generating loop λ has length $\widetilde{\rho}^+(\lambda) \leq \varepsilon$? Would this be true if we bounded the length of both sides of λ, i.e. we assumed that both $\widetilde{\rho}^+(\lambda)$ and $\widetilde{\rho}^-(\lambda) = \widetilde{\rho}^+(-\lambda)$ are $\leq \varepsilon$?*

An affirmative answer (to either question) would be enough to finish the proof of Theorem 5.1. For, by choosing H_t so small that $\mathcal{L}(H_t) < \varepsilon/2$ we could ensure that both $(P_{K,H}(2\delta),\Omega_0)$ and $(P_{H,K}(2\delta),\Omega_0)$ had area $< \varepsilon$. But they both contain balls of capacity $= \mathcal{L}(H_t)$ by Lemma 5.2, and one of them has to have area $< \mathcal{L}(H_t)$.

We show in [43] that if (M,ω) is a spherically integral symplectic manifold (i.e. $[\omega] \in H^2(M,\mathbb{Z})$), the nonsqueezing theorem holds for all loops λ in $\text{Ham}(M,\omega)$ with $\widetilde{\rho}^+(\lambda) + \widetilde{\rho}^-(\lambda) < 1/2$. Thus in this case we may take $\varepsilon = 1/2$.

The best result for general manifolds involves the idea of **weighted nonsqueezing**. In other words the nonsqueezing inequality must be modified by a weight κ_0. Now the size of ε is governed by the constant $\hbar$ of equation (16).

Proposition 5.5. *Suppose that λ is a loop in $\pi_1(Ham(M),\omega))$ such that $\widetilde{\rho}^+(\pm\lambda) < \hbar(M)/2$. Then there is $\kappa_0 \in \mathbb{R}$ with $|\kappa_0| \leq \max(\widetilde{\rho}^+(\lambda),\widetilde{\rho}^+(-\lambda))$ such that the radii of all symplectically embedded balls in $(P_{\pm\lambda},\Omega)$ are constrained by the inequalities*

$$\pi r^2 \leq \text{area}\,(P_\lambda,\Omega) + \kappa_0, \quad \pi r^2 \leq \text{area}\,(P_{-\lambda},\Omega) - \kappa_0.$$

Proof of Theorem 5.1 assuming Proposition 5.5.

Suppose as above that $\phi_1^K = \phi_1^H$ has Hofer norm $< \hbar/4$ and that

$$\mathcal{L}^+(K_t) + \mathcal{L}^-(K_t) \;=\; \mathcal{L}(\gamma) - \delta \;<\; \mathcal{L}^+(H_t) + \mathcal{L}^-(H_t) < \hbar/4\,.$$

As before, we may assume that:

$$\mathcal{L}^+(K_t) \;=\; \mathcal{L}^+(H_t) - \delta' \;<\; \mathcal{L}^+(H_t)\,, \qquad \mathcal{L}^-(K_t) \;=\; \mathcal{L}^-(H_t) - \delta + \delta'\,,$$

for some $\delta' > 0$. Let $\lambda = \phi_t^K \circ (\phi_t^H)^{-1}$ as before so that $P_{K,H} = P_\lambda$, $P_{H,K} = P_{-\lambda}$. Then for small ε

$$\begin{aligned} \text{area}\,(P_{K,H}(\varepsilon),\Omega_0) &= \mathcal{L}(H_t) - \delta' + \varepsilon \;<\; \mathcal{L}(H_t) \;<\; \hbar/4\,, \\ \text{area}\,(P_{H,K}(\varepsilon),\Omega_0) &= \mathcal{L}(H_t) - \delta + \delta' + \varepsilon \;\leq\; 2\mathcal{L}(H_t) < \hbar/2\,. \end{aligned}$$

By Proposition 5.5 there is κ_0 with $|\kappa_0| \leq \hbar/2$ such that embedded balls satisfy

$$\pi r^2 \leq \text{area}\,(P_{K,H}(\varepsilon),\Omega) + \kappa_0\,, \quad \pi r^2 \leq \text{area}\,(P_{H,K}(\varepsilon),\Omega) - \kappa_0\,.$$

But, by construction, both $(P_{K,H}(\varepsilon),\Omega)$ and $(P_{H,K}(\varepsilon),\Omega)$ contain embedded balls of capacity $\pi r^2 = \mathcal{L}(\gamma) > \text{area}\,(P_\lambda(\kappa_0),\Omega)$. Hence $\kappa_0 > 0$. Further,

$$\begin{aligned} \mathcal{L}(H_t) &\leq \text{area}\,(P_{K,H}(\varepsilon),\Omega) + \kappa_0 \;=\; \mathcal{L}(H_t) - \delta' + \varepsilon + \kappa_0 \\ \mathcal{L}(H_t) &\leq \text{area}\,(P_{H,K}(\varepsilon),\Omega) - \kappa_0 \;=\; \mathcal{L}(H_t) - \delta + \delta' + \varepsilon - \kappa_0\,. \end{aligned}$$

Adding, we find $0 \le -\delta + 2\varepsilon$. Since δ is positive and ε can be arbitrarily small, this is impossible. Hence result. □

It therefore remains to prove Proposition 5.5. Recall the definition of $\mathcal{S}(\lambda)$ from (13) It will be convenient to separate out the terms in $\mathcal{S}(\lambda)$ with $\alpha_{d,\kappa} \in H_{2n}(M)$. Since $\mathcal{S}(\lambda)$ has degree $2n$, such terms must have $d = 0$. Thus we write

$$\mathcal{S}(\lambda) = \sum_{0,\kappa} r_\kappa \mathbb{1} \otimes t^\kappa + x',$$

where $\mathbb{1} = [M]$ is the unit element, $r_\kappa \in \mathbb{Q}$ and

$$x' \in \mathrm{QH}^+ := \sum_{i>0} H_{2n-i}(M) \otimes \Lambda.$$

Definition 5.6. We say that the fibration $(P,\Omega) \to S^2$ with fiber M has a **good section of weight** κ_0 if there is a class $\widetilde{B} \in H_2(P)$ such that

(i) $\mathrm{GW}^P_{\widetilde{B},3}([M],[M],pt) \neq 0$;

(ii) $u(\widetilde{B}) = -\kappa_0$ where u is the coupling class.

Note that κ_0 could be positive or negative.

In particular, in the language of Lemma 4.1, we can take

$$\kappa_0 = v\Big(\sum r_\kappa \mathbb{1} \otimes t^\kappa\Big).$$

Lemma 5.7. *Suppose that (P_λ,Ω) has a good section of weight κ_0. Then the radius r of an embedded ball in (P,Ω) is constrained by the inequality:*

$$\pi r^2 \le \text{area } (P,\Omega) - \kappa_0.$$

Proof. The hypotheses imply that for some section class $\widetilde{B}$ with $u_\lambda(\widetilde{B}) = -\kappa_0$ we have

$$\mathrm{GW}^P_{\widetilde{B},3}([M],[M],pt) = r_\kappa \neq 0.$$

Since this invariant counts perturbed J-holomorphic stable maps in class $\widetilde{B}$ through an arbitrary point, it follows that there is such a curve through every point in P. Since the perturbation can be taken arbitrarily small, it follows from Gromov's compactness theorem that there has to be some J-holomorphic stable map in this class through every point in P. Hence the usual arguments (cf. [19] or [26]) imply that the radius r of any embedded ball satisfies the inequality:

$$\pi r^2 \;\le\; [\Omega](\widetilde{B}) \;\le\; \text{area}\,(P_\lambda,\Omega) - \kappa_0,$$

where the last inequality follows as in Proposition 4.2. The result follows. □

Proof of Proposition 5.5

By hypothesis there are fibrations $(P_{\pm\lambda},\Omega)$ each with area $< \hbar/2$. Choose $\delta > 0$ so that

$$\widetilde{\rho}^+(\lambda) + \widetilde{\rho}^-(\lambda) < \hbar - 2\delta. \tag{18}$$

By Proposition 3.7, there is a ω-compatible symplectic form Ω_λ on P_λ with area $< \widetilde{\rho}^+(\lambda) + \delta$, and a similar form $\Omega_{-\lambda}$ on $P_{-\lambda}$ with area $< \widetilde{\rho}^+(-\lambda) + \delta$. Write

$$\mathcal{S}(\lambda) = \sum_\kappa \mathbb{1} \otimes r_\kappa t^\kappa + x\,, \quad \mathcal{S}(-\lambda) = \sum_\kappa \mathbb{1} \otimes r'_\kappa t^\kappa + x'$$

where $r_\kappa, r'_\kappa \in \mathbb{Q}$ and $x, x' \in \mathrm{QH}^+$. Proposition 4.2 implies that

$$r_\kappa \neq 0 \implies \kappa < \widetilde{\rho}^+(\lambda) + \delta\,, \qquad r'_\kappa \neq 0 \implies \kappa < \widetilde{\rho}^+(-\lambda) + \delta\,.$$

Next apply the valuation v in (15) to the identity

$$\mathcal{S}(\lambda) * \mathcal{S}(-\lambda) = \mathcal{S}(0) = \mathbb{1}\,.$$

We claim that at least one of $\mathcal{S}(\lambda), \mathcal{S}(-\lambda)$ has a nonzero term $\mathbb{1} \otimes r_\kappa t^\kappa$ with $\kappa \geq 0$. For otherwise since $\mathcal{S}(\lambda) * \mathcal{S}(-\lambda) = \mathbb{1}$, the product $x * x'$ must contain the term $\mathbb{1} \otimes t^0$ with a nonzero coefficient. Because this term appears in $x * x' - x \cap x'$ we find from Lemma 4.1 that

$$0 \;=\; v(\mathbb{1} \otimes t^0) \;\leq\; v(\mathcal{S}(\lambda)) + v(\mathcal{S}(-\lambda)) - \hbar(M) \;\leq\; \widetilde{\rho}^+(\lambda) + \widetilde{\rho}^+(-\lambda) + 2\delta - \hbar \;<\; 0\,,$$

contradicting (18). Therefore for λ' equal to at least one of λ or $-\lambda$, $\mathcal{S}(\lambda')$ has a term $r_\kappa \mathbb{1} \otimes t^\kappa$ with $r_\kappa \neq 0$ and $0 \leq \kappa < \mathrm{area}\,(P_{\lambda'}, \Omega)$.

For $\lambda' = \pm\lambda$ set

$$\kappa(\lambda') = \max\{\kappa : r_\kappa \neq 0 \text{ in } \mathcal{S}(\lambda')\}\,.$$

The equation $\mathcal{S}(\lambda) * \mathcal{S}(-\lambda) = \mathbb{1}$ implies that $\kappa(\lambda) = -\kappa(-\lambda)$. Moreover, by Lemma 5.7, the radius r of any embedded ball in $(P_{\lambda'}, \Omega)$ satisfies

$$\pi r^2 \leq \mathrm{area}\,(P_{\lambda'}, \Omega) - \kappa(\lambda')\,.$$

Hence we may take $\kappa_0 = -\kappa(\lambda)$.

This completes the proof of Theorem 5.1. □

References

[1] M. Abreu, *Topology of symplectomorphism groups of $S^2 \times S^2$*, Invent. Math. **131** (1998), 1–23.

[2] M. F. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), 1–28.

[3] M. Abreu and D. McDuff, *Topology of symplectomorphism groups of rational ruled surfaces*, SG/9910057, J. Amer. Math. Soc. **13** (2000), 971–1009.

[4] A. Banyaga, *Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique*, Commun. Math. Helv. **53** (1978), 174–227.

[5] A. Banyaga, *The structure of classical diffeomorphism groups*, Math. Appl. **400**, Kluwer.

[6] M. Barsamian, Ph. D. thesis, Stony Brook, (2002).

[7] P. Biran, L. Polterovich and D. Salamon, *Propagation in Hamiltonian dynamics and relative symplectic homology*, SG/0108134.

[8] O. Buse, *Relative family Gromov–Witten invariants and symplectomorphisms*, SG/01110313.

[9] E. Calabi, *On the group of automorphisms of a symplectic manifold.* In Problems in Analysis (ed. R. Gunning), Princeton University Press (1970), 1–26.

[10] I. Ekeland and H. Hofer, *Symplectic topology and Hamiltonian dynamics*, Math. Z. **200** (1989), 355–78.

[11] Y. Eliashberg, *A theorem on the structure of wave fronts and its applications in symplectic topology*, Funct. Anal. Appl. **21**, (1987), 65–72.

[12] Y. Eliashberg and L. Polterovich, *Bi-invariant metrics on the group of Hamiltonian diffeomorphisms*, Internat. J. Math. **4** (1993), 727–38.

[13] M. Entov, *K-area, Hofer metric and geometry of conjugacy classes in Lie groups*, SG/0009111, Invent. Math. **146** (2001), 93–141.

[14] M. Entov, *Commutator length of symplectomorphisms*, SG/0112012.

[15] M. Entov and L. Polterovich, *Calabi quasimorphism and quantum homology*, preprint (2002), SG/0205247. Internat. Math. Res. Notices (2003).

[16] D. B. A. Epstein, *The simplicity of certain groups of homeomorphisms*, Compos. Math. **22** (1970), 165–173.

[17] A. Floer, H. Hofer and D. Salamon, *Transversality in Elliptic Morse Theory for the Symplectic Action*, Duke Math. J. **80** (1995), 251–292.

[18] V. Guillemin, E. Lerman, and S. Sternberg, *Symplectic fibrations and multiplicity diagrams*, Cambridge University Press, (1994).

[19] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307–347.

[20] H. Hofer, *On the topological properties of symplectic maps*, Proc. Roy. Soc. Edinburgh **115** (1990), 25–38.

[21] H. Hofer and C. Viterbo, *The Weinstein Conjecture in the Presence of Holomorphic Spheres*, Comm. Pure Appl. Math. **XLV** (1992), 583–622.

[22] T. Januszkiewicz and J. Kędra, *Characteristic classes of smooth fibrations*, SG/0209288.

[23] J. Kędra, *Remarks on the flux groups*, Math. Res. Lett. (2000).

[24] J. Kędra, *Restrictions on symplectic fibrations*, SG/0203232.

[25] J. Kędra and D. McDuff, *Homotopy properties of Hamiltonian group actions*, in preparation.

[26] F. Lalonde and D. McDuff, *The geometry of symplectic energy*, Ann. of Math. **141** (1995), 349–371.

[27] F. Lalonde and D. McDuff, *Hofer's L^∞-geometry: energy and stability of Hamiltonian flows*, parts I and II, Invent. Math. **122** (1995), 1–33 and 35–69.

[28] F. Lalonde and D. McDuff, *The classification of ruled symplectic 4-manifolds*, Math. Res. Lett. **3** (1996), 769–778.

[29] F. Lalonde and D. McDuff, *J-curves and the classification of rational and ruled symplectic 4-manifolds*, Proc. of the 1994 Newton Institute Symplectic Geometry Conference, ed. C. Thomas, Camb. Univ. Press (1996), 3–42.

[30] F. Lalonde and D. McDuff, *Symplectic structures on fiber bundles*, SG/0010275, Topology **42** (2003), 309–347.

[31] F. Lalonde, D. McDuff and L. Polterovich, *On the Flux conjectures*, CRM Proc. and Lect. Not. **15** (1998), 69–85.

[32] F. Lalonde, D. McDuff and L. Polterovich, *Topological rigidity of Hamiltonian loops and quantum homology*, Invent. Math. **135** (1999), 369–385.

[33] F. Lalonde and M. Pinsonneault, *The topology of the space of symplectic balls in rational 4-manifolds*, SG/0207096.

[34] H. V. Le and K. Ono, *Parameterized Gromov–Witten invariants and topology of symplectomorphism groups*, preprint #28, MPIM Leipzig (2001).

[35] F. Mascaro, *Some results on* $\mathrm{Diff}^\Omega(\mathbb{R}^n)$, Differential Geometry, Pensicola, Springer Lecture Notes in Math. **1045**, (1984), 125–133.

[36] D. McDuff, *On the group of volume preserving diffeomorphisms of* $\mathbb{R}^n$, Trans. Amer. Math. Soc. **261** (1980), 103–113.

[37] D. McDuff, *Examples of symplectic structures*, Invent. Math. **89** (1987), 13–36.

[38] D. McDuff, *Elliptic methods in symplectic geometry*, Bull. Amer. Math. Soc. **23** (1990), 311–58.

[39] D. McDuff, *Introduction to Symplectic Topology*, In IAS/Park City Math. Series, vol. **7** ed. Eliashberg, AMS (1998).

[40] D. McDuff, *Quantum homology of Fibrations over S^2*, Internat. J. Math. **11** (2000), 665–721.

[41] D. McDuff, *Almost complex structures on $S^2 \times S^2$*, Duke Math. J. **101** (2000), 135–177.

[42] D. McDuff, *Symplectomorphism Groups and Almost Complex Structures*, SG/0010274, Enseign. Math. **38** (2001), 1–30.

[43] D. McDuff, *Geometric variants of the Hofer norm*, SG/0103089, J. Symplectic Geometry **1** (2002), 197–252.

[44] D. McDuff, *A survey of topological properties of groups of symplectomrophisms*, to appear in *Topology, Geometry and Quantum Field Theory*, Proc. of 2002 Symposium in honor of G.B. Segal, ed. U.L. Tillmann, Cambridge Univ. Press (2004).

[45] D. McDuff and D.A. Salamon, *J-holomorphic curves and quantum cohomology*, Amer. Math. Soc. Lecture Series **6**, (1994).

[46] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, 2nd edition (1998) OUP, Oxford, UK.

[47] D. McDuff and D. Salamon, *J-holomorphic curves and symplectic Topology*, Amer. Math. Soc. Colloq. Publ., to appear (2004).

[48] D. McDuff and J. Slimowitz, *Hofer–Zehnder capacity and length minimizing Hamiltonian paths*, SG/0101085, Geom. Topol. **5** (2001), 799–830.

[49] D. McDuff and S. Tolman, *Topological properties of Hamiltonian circle actions*, in preparation.

[50] J. Milnor, *Remarks on infinite dimensional Lie groups*, in Relativité, groupes at topologie, II ed. B.S. DeWitt and R. Skora, Les Houches session XL, Elsevier (1984), 1010–1057.

[51] Yong-Geun Oh, *Chain level Floer theory and Hofer's geometry on the Hamiltonian diffeomorphism group*, SG/0104243.

[52] Yong-Geun Oh, *Minimax theory, spectral invariants and geometry of the Hamiltonian diffeomorphism group*, SG/0206092.

[53] Yong-Geun Oh, *Spectral invariants and length minimizing property of Hamiltonian paths*, SG/0212337.

[54] M. Pinsonnault, *Remarques sur la Topologie du Groupe des Automorphismes Symplectiques de l'Éclatment de $S^2 \times S^2$*, Ph. D. Thesis, UQAM Montreal (2001).

[55] S. Piunikhin, D. Salamon and M. Schwarz, *Symplectic Floer–Donaldson theory and Quantum Cohomology*, Contact and Symplectic Geometry ed. C. Thomas, Proc. of the 1994 Newton Institute Conference, CUP, Cambridge (1996).

[56] L. Polterovich, *Symplectic displacement energy for Lagrangian submanifolds*, Ergodic Theory Dynam. Systems **13** (1993), 357–367.

[57] L. Polterovich, *Gromov's K-area and symplectic rigidity*, Geom. Funct. Anal. **6** (1996), 726–739.

[58] L. Polterovich, *Hamiltonian loops and Arnold's principle*, Amer. Math. Soc. Transl. (2) **180** (1997), 181–187.

[59] L. Polterovich, *Symplectic aspects of the first eigenvalue*, J. Reine Angew. Math. **502** (1998), 1–17.

[60] L. Polterovich, *Hofer's diameter and Lagrangian intersections*, Internat. Math. Res. Notices **4** (1998), 217–223.

[61] L. Polterovich, *The Geometry of the Group of Symplectic Diffeomorphisms*, Lectures Math. ETH, Birkhauser (2001)

[62] L. Polterovich, *Growth of maps, distortion in groups and symplectic geometry*, DS/0111050.

[63] A. G. Reznikov, *Characteristic classes in symplectic topology*, Selecta Math. **3** (1997), 601–642.

[64] Y. Ruan and Gang Tian, *Higher genus symplectic invariants and sigma model coupled with gravity*, Invent. Math. **130** (1997), 455–516.

[65] D. Salamon, *Lectures on Floer homology*, IAS/Park City Math. Ser. Vol. **7**, AMS 1999, 145–229.

[66] M. Schwarz, *On the action spectrum for closed symplectially aspherical manifolds*, Pacific J. Math. **193** (2000), 419–461.

[67] P. Seidel, π_1 *of symplectic automorphism groups and invertibles in quantum cohomology rings*, Geom. Funct. Anal. **7** (1997), 1046–1095.

[68] P. Seidel, *On the group of symplectic automorphisms of* $\mathbb{C}P^m \times \mathbb{C}P^n$, Amer. Math. Soc. Transl. (2) **196** (1999), 237–250.

[69] P. Seidel, *Graded Lagrangian submanifolds*, Bull. Soc. Math. France **128** (2000), 103–149.

STATE UNIVERSITY OF NEW YORK AT STONY BROOK
E-MAIL: dusa@math.sunysb.edu