

Michael Eastwood

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PROLONGATIONS OF LINEAR OVERDETERMINED SYSTEMS ON AFFINE AND RIEMANNIAN MANIFOLDS

MICHAEL EASTWOOD

According to folklore (a precise criterion in the language of exterior differential systems may be found in [2]), a generic overdetermined partial differential equation may be rewritten as a first order ‘closed system’ in which all first partial derivatives of the dependent variables are expressed in terms of the variables themselves. To do this, one must introduce extra dependent variables for unknown derivatives until all derivatives of the original and extra variables can be determined as consequences of the original equation. This is the well-known procedure of ‘prolongation’. Particular prolongations, however, are usually derived *ad hoc*.

Recent joint work with Thomas Branson, Andreas Čap, and Rod Gover [1] shows how to implement this prolongation procedure for a wide class of geometrically defined equations on manifolds with a suitable differential geometric structure. This article presents a special case of our results. Specifically, we consider only linear equations on affine or Riemannian manifolds. By restricting to these special cases, the proofs are considerably simplified. Several further examples are given to complement [1].

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The ingredients for this work are now known informally as the Bernstein-Gelfand-Gelfand (BGG) machinery. This machinery is obtained by interpreting suitable Lie algebra cohomology as providing geometric constructions on manifolds. It has been a common theme at previous Czech Winter Schools.

The advantages of a closed system are considerable. In the linear case, one obtains a bound on the dimension of the solution space (namely, the final number of dependent variables) whilst, in the semilinear case, constraints on solutions may be derived by cross differentiation and back substitution.

1. EXAMPLES ON AN AFFINE MANIFOLD

In this section we shall suppose that we are working on a smooth manifold equipped with a torsion-free connection. We shall adopt Penrose’s abstract index notation [7]

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and write ∇_a for this connection. In general, indices act as markers to specify the type of a tensor and to record symmetries and contractions. Round brackets, as in $\nabla_{(a}\sigma_b)$, mean that the indices they enclose have been symmetrised, square brackets $\phi_{[ab]c}$ take the skew part, and a repeated index ϕ^a_{ab} denotes contraction. The curvature tensor $R_{ab}{}^c{}_d$ of ∇_a is defined by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)V^c = R_{ab}{}^c{}_d V^d.$$

In particular,

$$\nabla_b \nabla_a V^b = \nabla_a \nabla_b V^b + R_{ab} V^b,$$

where $R_{ab} = R_{ca}{}^c{}_b$ is the Ricci tensor.

1.1. Example. By setting $\mu_b = \nabla_b \sigma$, the differential equation

$$(1) \quad \nabla_a \nabla_b \sigma = 0$$

is manifestly equivalent to

$$\left. \begin{array}{l} \nabla_a \sigma = \mu_a \\ \nabla_a \mu_b = 0 \end{array} \right\} \text{ a closed system.}$$

It is worth noting that, in this particular case, $\nabla_a \nabla_b \sigma$ is already symmetric so (1) can, alternatively, be written as $\nabla_{(a} \nabla_{b)} \sigma = 0$. We can express the closed system as $\widehat{\nabla} \Sigma = 0$ where $\widehat{\nabla}$ is a connection:-

$$\widehat{\nabla}_a \Sigma = \widehat{\nabla}_a \left(\begin{array}{c} \sigma \\ \mu_b \end{array} \right) = \left(\begin{array}{c} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b \end{array} \right).$$

1.2. Example. By setting $\mu_{ab} = \nabla_a \sigma_b$, it is manifest that the differential equation

$$(2) \quad \nabla_{(a} \sigma_{b)} = 0$$

is equivalent to requiring that μ_{ab} be skew. To obtain a closed system we should try to express $\nabla_a \mu_{bc}$ in terms of σ_a and μ_{ab} . By exploiting differential consequences of (2), this is possible. Specifically, by noting that $\nabla_{[a} \mu_{bc]} = 0$, we find

$$\nabla_a \mu_{bc} = \nabla_c \mu_{ba} - \nabla_b \mu_{ca} = \nabla_c \nabla_b \sigma_a - \nabla_b \nabla_c \sigma_a = R_{bc}{}^d{}_a \sigma_d.$$

Therefore, the differential equation (2) is equivalent to

$$\left. \begin{array}{l} \nabla_a \sigma_b = \mu_{ab} \\ \nabla_a \mu_{bc} = R_{bc}{}^d{}_a \sigma_d \end{array} \right\} \text{ a closed system.}$$

Again, we can regard this closed system as covariant constancy under a connection

$$\widehat{\nabla}_a \left(\begin{array}{c} \sigma_b \\ \mu_{bc} \end{array} \right) = \left(\begin{array}{c} \nabla_a \sigma_b - \mu_{ab} \\ \nabla_a \mu_{bc} - R_{bc}{}^d{}_a \sigma_d \end{array} \right)$$

on an appropriate vector bundle.

1.3. **Example.** Even for relatively simple equations, the corresponding prolongations can be quite complicated. The detailed derivation of the following example is confined to an appendix, wherein two other examples can be found. Here, it is useful merely to note the form that the prolongation takes. It is shown in §A.2 that

$$(3) \quad \nabla_{(a} \nabla_b \sigma_c) = 0$$

may be prolonged with the aid of further independent variables μ_{ab} and ρ_{abc} , subject to symmetries $\rho_{abc} = \rho_{(ab)c}$ and $\rho_{(abc)} = 0$, to obtain a connection of the form

$$(4) \quad \widehat{\nabla} \begin{pmatrix} \sigma \\ \mu \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla \sigma - \mu \\ \nabla \mu - \rho + R \bowtie \sigma \\ \nabla \rho + R \bowtie \mu + (\nabla R) \bowtie \sigma \end{pmatrix},$$

where $S \bowtie \phi$ stands for some linear combination of contractions of tensors S and ϕ . At this point, it is worthwhile to introduce a notation for various bundles on our manifold. It will be further explained in §3 that irreducible tensor bundles can be denoted by Young tableau. In particular $\square = \Lambda^1 =$ the bundle of 1-forms,

$$\square \longleftrightarrow \mu_{ab} \quad \text{s.t.} \quad \mu_{ab} = \mu_{[ab]}, \quad \square \longleftrightarrow \mu_{ab} \quad \text{s.t.} \quad \mu_{ab} = \mu_{(ab)},$$

and

$$(5) \quad \square \longleftrightarrow \rho_{abc} \quad \text{s.t.} \quad \rho_{abc} = \rho_{(ab)c} \quad \text{and} \quad \rho_{(abc)} = 0.$$

Hence, our prolonged equations read $\widehat{\nabla} \Sigma = 0$ for a connection $\widehat{\nabla} : V \rightarrow \Lambda^1 \otimes V$ on the vector bundle

$$(6) \quad V = \square \oplus \begin{matrix} \square \\ \square \end{matrix}.$$

2. EXAMPLES ON A RIEMANNIAN MANIFOLD

On a Riemannian manifold, indices may be raised or lowered with the metric in the usual way. For example $\phi_a^c = g_{ab} \phi^{bc}$, where g_{ab} denotes the metric. We shall take ∇_a to be the metric connection and suppose that our manifold has dimension $n \geq 3$.

2.1. **Example.** Consider the differential equation

$$\nabla_a \nabla_b \sigma = \frac{1}{n} g_{ab} \nabla^c \nabla_c \sigma.$$

Manifestly this is equivalent to the system

$$\begin{aligned} \nabla_a \sigma &= \mu_a \\ \nabla_a \mu_b &= g_{ab} \rho \end{aligned}$$

but this is not yet closed since we do not know the derivatives of ρ . To find them, we may substitute the second equation into the Ricci identity

$$\nabla_b \nabla_a \mu^b = \nabla_a \nabla_b \mu^b + R_{ab} \mu^b \quad \implies \quad \nabla_a \rho = n \nabla_a \rho + R_{ab} \mu^b$$

to conclude that

$$\nabla_a \rho = -\frac{1}{n-1} R_a^b \mu_b$$

and now the system has closed.

2.2. **Examples.** Consider the differential equations

$$\text{trace free part of } \nabla_{(a}\sigma_{bc)} = 0 \quad \text{or} \quad \text{trace free part of } \nabla_{(a}\nabla_b\sigma_c) = 0,$$

where, in the first case, σ_{bc} is supposed symmetric and trace-free. Explicitly to prolong these relatively simple equations is already a fearsome task but we shall see that each is equivalent to having a parallel section with respect to a certain connection on a bundle V having the form

$$\begin{array}{ccc} \square_{\circ} & & \square \\ \oplus & & \oplus \\ \square_{\circ} \oplus \square & & \square_{\circ} \oplus \square \oplus \mathbb{R} \\ \oplus & & \oplus \\ \square_{\circ} \oplus \square_{\circ} \oplus \mathbb{R} \oplus \square & \text{or} & \square_{\circ} \oplus \square \oplus \square \\ \oplus & & \oplus \\ \square_{\circ} \oplus \square & & \square_{\circ} \oplus \square \oplus \mathbb{R} \\ \oplus & & \oplus \\ \square_{\circ} & & \square \end{array}$$

respectively, where \mathbb{R} denotes the trivial bundle and ‘ \circ ’ means to take tensors with the specified symmetry that are, in addition, totally trace-free. Without knowing anything about the connection, we can immediately deduce that the dimension of the solution spaces are bounded by

$$\frac{(n-1)(n+2)(n+3)(n+4)}{12} \quad \text{or} \quad \frac{n(n+2)(n+4)}{3},$$

respectively.

3. FORMULATION OF THE RESULTS

3.1. **The affine case.** Firstly, we must say to which equations our prolongation algorithm will apply. Let us work on a smooth manifold M equipped with a torsion-free affine connection ∇_a . We shall regard the bundle of 1-forms on M as the vector bundle tautologically induced from the co-frame bundle by the defining representation of $GL(n, \mathbb{R})$.

Definition. An irreducible tensor bundle on M is one induced from the co-frame bundle by an irreducible representation of $GL(n, \mathbb{R})$.

The irreducible representations corresponding to covariant tensors may be specified by Young tableau in the usual way (see [7] for a discussion of Young tableau well suited to this article and [4] for proofs). There is a choice of realisation for these tensors and it is convenient to take the one in which symmetry is visible and skewing is hidden. We shall use (5), for example, rather than the tensors

$$\{\phi_{abc} = \phi_{a[bc]} \quad \text{s.t.} \quad \phi_{[abc]} = 0\}, \quad \text{isomorphic by} \quad \begin{cases} \phi_{abc} = \rho_{a[bc]} \\ \rho_{abc} = \frac{4}{3}\phi_{(ab)c} \end{cases}.$$

As further typical examples,

$$\begin{array}{l} \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array} \longleftrightarrow \rho_{abcde} \quad \text{s.t.} \quad \rho_{abcde} = \rho_{(abcd)e} \quad \text{and} \quad \rho_{(abcde)} = 0, \\ \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} \longleftrightarrow \rho_{abcde} \quad \text{s.t.} \quad \rho_{abcde} = \rho_{(abc)[de]} \quad \text{and} \quad \rho_{(abcd)e} = 0. \end{array}$$

For reasons that will soon become clear, let us denote by \mathfrak{g}_1 the defining representation of $GL(n, \mathbb{R})$. The corresponding tensor bundle is Λ^1 . The symmetric tensor power $\odot^k \mathfrak{g}_1$ is irreducible and the corresponding vector bundle is the symmetric covariant tensors of valence k . Suppose \mathbb{E} is an irreducible representation of $GL(n, \mathbb{R})$ and denote by E the corresponding irreducible tensor bundle on M . The tensor product $\odot^k \mathfrak{g}_1 \otimes \mathbb{E}$ decomposes into irreducibles under action of $GL(n, \mathbb{R})$ amongst which the one whose highest weight is simply the sum of the highest weights for $\odot^k \mathfrak{g}_1$ and \mathbb{E} occurs with multiplicity one. It is called the Cartan product [3] of $\odot^k \mathfrak{g}_1$ and \mathbb{E} and will be denoted $\odot^k \mathfrak{g}_1 \odot \mathbb{E}$. Since it occurs without multiplicity there is a well-defined projection $\odot^k \mathfrak{g}_1 \otimes \mathbb{E} \rightarrow \odot^k \mathfrak{g}_1 \odot \mathbb{E}$, which we shall also denote by \odot . If $\phi \in \odot^k \mathfrak{g}_1$ and $\psi \in \mathbb{E}$, we shall also denote by $\phi \odot \psi \in \odot^k \mathfrak{g}_1 \odot \mathbb{E}$, the image of $\phi \otimes \psi$ under \odot . Similar abuses of notation occur for the symmetric product of symmetric tensors, which is a special case of Cartan product. These constructions have an immediate interpretation on M . Thus, there is a canonical homomorphism of vector bundles

$$\odot^k \Lambda^1 \otimes E \rightarrow \odot^k \Lambda^1 \odot E,$$

which, of course, we shall denote by \odot . At this point, there is a slight conflict with our previous agreement always to write irreducible tensors with symmetry visible. If we resolve this conflict with suitable Young projectors, then the Cartan product is much more easily expressed. For example, it is more convenient to write

$$\odot^2 \Lambda^1 \otimes \square = \square \otimes \square \ni \theta_{ab} \otimes \sigma_{cd} \longmapsto \theta_{(ab}\sigma_{c)d} \in \square\square$$

rather than what is literally true:-

$$\theta_{ab} \otimes \sigma_{cd} \xrightarrow{\odot} \frac{3}{4} (\theta_{(ab}\sigma_{c)d} - \theta_{(ab}\sigma_{d)c}).$$

From now on we shall suppress this conflict and it is clear that all the affine examples considered above are concerned with differential equations of the form $D\sigma = 0$, where E is an irreducible tensor bundle and $D : E \rightarrow F$ is a differential operator obtained as a composition

$$E \xrightarrow{\nabla \odot^k} \odot^k \Lambda^1 \otimes E \xrightarrow{\odot} \odot^k \Lambda^1 \odot E = F.$$

In fact, it is only the symbol of $D : E \rightarrow F$ that is important in order that our prolongation method succeed. Let us see this by revisiting Example 1.2 with the addition of a general linear term, namely

$$(7) \quad \nabla_{(a}\sigma_{b)} - \Gamma_{ab}{}^c \sigma_c = 0$$

where $\Gamma_{ab}{}^c$ is a tensor that is symmetric in ab . By setting $\mu_{ab} = \nabla_a \sigma_b - \Gamma_{ab}{}^c \sigma_c$ and following exactly the same steps as in Example 1.2,

$$\begin{aligned} \nabla_a \mu_{bc} &= \nabla_c \mu_{ba} - \nabla_b \mu_{ca} = \nabla_c \nabla_b \sigma_a - \nabla_b \nabla_c \sigma_a - \nabla_c (\Gamma_{ba}{}^d \sigma_d) + \nabla_b (\Gamma_{ca}{}^d \sigma_d) \\ &= R_{bc}{}^d{}_a \sigma_d + 2(\nabla_{[b} \Gamma_{c]a}{}^d) \sigma_d - 2\Gamma_{a[b}{}^d \nabla_{c]} \sigma_d \\ &= R_{bc}{}^d{}_a \sigma_d + 2(\nabla_{[b} \Gamma_{c]a}{}^d) \sigma_d - 2\Gamma_{a[b}{}^d \Gamma_{c]d}{}^e \sigma_e - 2\Gamma_{a[b}{}^d \mu_{c]d} \end{aligned}$$

we obtain the closed system

$$\left. \begin{aligned} \nabla_a \sigma_b &= \mu_{ab} + \Gamma_{ab}{}^c \sigma_c \quad \text{where } \mu_{ab} \text{ is skew} \\ \nabla_a \mu_{bc} &= R_{bc}{}^d{}_a \sigma_d + 2(\nabla_{[b} \Gamma_{c]a}{}^d - \Gamma_{a[b}{}^e \Gamma_{c]e}{}^d) \sigma_d - 2\Gamma_{a[b}{}^d \mu_{c]d} \end{aligned} \right\}.$$

Of course, in this particular case, further confirmation is provided by noting that this new closed system may be also be obtained by regarding (7) as a general change of torsion-free affine connection. However, we shall see that adding lower order linear terms is always permissible in our prolongation procedure.

We are almost in a position to formulate the main result in the affine case. It remains only to recall some basic notions on linear differential operators as detailed, for example, in [8]. To every smooth vector bundle E on a smooth manifold M there are the canonically associated jet bundles $J^k E$ on M and short exact sequences of homomorphisms of vector bundles

$$0 \rightarrow \odot^k \Lambda^1 \otimes E \rightarrow J^k E \rightarrow J^{k-1} E \rightarrow 0.$$

A k^{th} order linear differential operator $D : E \rightarrow F$ between vector bundles E and F is equivalent to a homomorphism of vector bundles $J^k E \rightarrow F$ and the symbol $\sigma(D)$ of D is defined as the composition

$$\odot^k \Lambda^1 \otimes E \hookrightarrow J^k E \rightarrow F.$$

Theorem 1. *Suppose E is an irreducible tensor bundle on a smooth manifold M . Fix $k \geq 1$ and let $F = \odot^k \Lambda^1 \otimes E$. Then there are canonical constructions*

- from E and k , a graded vector bundle

$$(8) \quad V = V_0 \oplus V_1 \oplus V_2 \oplus \cdots \oplus V_N$$

on M with $V_0 = E$;

- from any torsion-free affine connection ∇ on M , a connection $\tilde{\nabla}$ on V and an N^{th} order linear differential operator $L : E \rightarrow V$ such that, if we denote by Σ_0 the component in V_0 of $\Sigma \in \Gamma(V)$, then $(L\sigma)_0 = \sigma$ for any $\sigma \in \Gamma(E)$;

and they have the following property. For every k^{th} order linear differential operator $D : E \rightarrow F$ whose symbol

$$\sigma(D) : \odot^k \Lambda^1 \otimes E \rightarrow F = \odot^k \Lambda^1 \otimes E$$

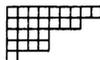
is the Cartan product, there is a canonically constructed homomorphism of vector bundles $\Phi : V \rightarrow \Lambda^1 \otimes V$ such that

$$(9) \quad \{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\} \cong \{\Sigma \in \Gamma(V) \text{ s.t. } \tilde{\nabla}\Sigma + \Phi(\Sigma) = 0\},$$

the isomorphism being given by $\Sigma = L\sigma$ and, conversely, $\sigma = \Sigma_0$.

Of course, the isomorphism (9) gives the prolongation we desire: the homomorphism Φ may be incorporated into a new connection $\hat{\nabla} \equiv \tilde{\nabla} + \Phi : V \rightarrow \Lambda^1 \otimes V$ whose covariant constant sections correspond to solutions of $D\sigma = 0$.

The bundle V is constructed from E and k as follows. Recall that E is induced from an irreducible representation \mathbb{E} of $GL(n, \mathbb{R})$ and so corresponds to a Young tableau, typically



3.2. The Riemannian case. The discussion of the affine case needs only minor modification to be valid in the Riemannian case. Firstly, $GL(n, \mathbb{R})$ should be replaced by $O(n)$ or by $SO(n)$ in the case of an oriented manifold. If the manifold has a spin structure, then we can use $Spin(n)$ and include spin bundles in addition to tensor bundles (as is done in [1]). For simplicity, let us stick with tensor bundles on an oriented Riemannian manifold M .

Definition. An irreducible tensor bundle on M is one induced from the co-frame bundle by an irreducible representation of $SO(n)$.

Most such irreducible representations can be specified by Young tableau, namely they have the symmetries of an irreducible $GL(n, \mathbb{R})$ representation but, in addition, are totally trace-free. As we did already in §2.2, we shall adorn such tableau with an additional ‘o’. In this case, the construction of V from E and k is exactly parallel to the affine case:-

$$E = \begin{array}{cccccccc} \square & \square \\ \square & \square \\ \square & \square \\ \square & \square \\ \square & \square \\ \square & \square \\ \square & \square \\ \square & \square \\ \square & \square \\ \square & \square \end{array} \implies V = \begin{array}{cccccccc} \square & \square \\ \square & \square \\ \square & \square \\ \square & \square \\ \square & \square \\ \square & \square \\ \square & \square \\ \square & \square \\ \square & \square \\ \square & \square \\ \square & \square \end{array} \underbrace{\hspace{10em}}_{k-1}$$

where V is now induced from a representation of $SO(n + 1, 1)$ restricted to $SO(n)$ under the embedding

$$A \mapsto \left(\begin{array}{c|ccc|c} 1 & 0 & \dots & 0 & 0 \\ \hline 0 & & & & 0 \\ \vdots & & A & & \vdots \\ \hline 0 & & & & 0 \\ \hline 0 & 0 & \dots & 0 & 1 \end{array} \right)$$

where $SO(n + 1, 1)$ is realised as preserving the quadratic form $2x_0x_{n+1} + \sum_{i=1}^n x_i^2$. Occasionally, the representations specified in this way decompose into two irreducibles. The following theorem is still valid in these cases but a more precise construction of irreducible V from irreducible E is given in [1].

Theorem 2. Let E be an irreducible tensor bundle on a Riemannian manifold M . Fix $k \geq 1$ and let $F = \odot_o^k \Lambda^1 \otimes E$. Then there are canonical constructions

- from E and k , a graded vector bundle

$$V = V_0 \oplus V_1 \oplus V_2 \oplus \dots \oplus V_N$$

on M with $V_0 = E$;

- a connection $\tilde{\nabla}$ on V and an N^{th} order linear differential operator $L : E \rightarrow V$ such that, if we denote by Σ_0 the component in V_0 of $\Sigma \in \Gamma(V)$, then $(L\sigma)_0 = \sigma$ for any $\sigma \in \Gamma(E)$;

and they have the following property. For every k^{th} order linear differential operator $D : E \rightarrow F$ whose symbol

$$\sigma(D) : \odot_o^k \Lambda^1 \otimes E \rightarrow F = \odot_o^k \Lambda^1 \otimes E$$

is the Cartan product, there is a canonically constructed homomorphism of vector bundles $\Phi : V \rightarrow \Lambda^1 \otimes V$ such that

$$\{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\} \cong \{\Sigma \in \Gamma(V) \text{ s.t. } \tilde{\nabla}\Sigma + \Phi(\Sigma) = 0\},$$

the isomorphism being given by $\Sigma = L\sigma$ and, conversely, $\sigma = \Sigma_0$.

Here, the ‘Cartan product’ is with respect to representations of $SO(n)$ and $\odot_o^k \Lambda^1$ denotes the trace-free part of $\odot^k \Lambda^1$, in other words the k -fold Cartan product. There is no longer any freedom in torsion-free affine connection ∇ : the connection should be compatible with the metric and the Levi-Civita connection is the only possibility. (A more closely aligned statement is obtained by starting with a conformal manifold rather than Riemannian.) Finally, the grading on V is induced by the element

$$H = \left(\begin{array}{c|ccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & & & & 0 \\ \vdots & & & & \vdots \\ 0 & & & & 0 \\ \hline 0 & 0 & \cdots & 0 & -1 \end{array} \right)$$

and $\mathfrak{g} = \mathfrak{so}(n + 1, 1)$ itself is $|1|$ -graded by this element as follows:-

$$\mathfrak{g}_{-1} \ni \left(\begin{array}{c|ccc|c} 0 & 0 & \cdots & 0 & 0 \\ -x_1 & & & & 0 \\ \vdots & & & & \vdots \\ -x_n & & & & 0 \\ \hline 0 & x_1 & \cdots & x_n & 0 \end{array} \right) \quad \mathfrak{g}_0 \ni \left(\begin{array}{c|ccc|c} * & 0 & \cdots & 0 & 0 \\ 0 & & & & 0 \\ \vdots & & & & \vdots \\ 0 & & & & 0 \\ \hline 0 & 0 & \cdots & 0 & * \end{array} \right) \quad \mathfrak{g}_1 \ni \left(\begin{array}{c|ccc|c} 0 & -z_1 & \cdots & -z_n & 0 \\ 0 & & & & z_1 \\ \vdots & & & & \vdots \\ 0 & & & & z_n \\ \hline 0 & 0 & \cdots & 0 & 0 \end{array} \right)$$

Again, \mathfrak{g}_{-1} and \mathfrak{g}_1 are canonically dual under the pairing $X \otimes Z \mapsto \text{trace}(XZ)$.

4. ALGEBRAIC INTERLUDE

Suppose \mathfrak{a} is an Abelian Lie algebra and \mathbb{V} is an \mathfrak{a} -module. The action of \mathfrak{a} on \mathbb{V} defines linear transformations

$$(12) \quad \mathbb{V} \xrightarrow{\partial} \text{Hom}(\mathfrak{a}, \mathbb{V}) \xrightarrow{\partial} \text{Hom}(\Lambda^2 \mathfrak{a}, \mathbb{V})$$

by $(\partial v)(X) = Xv$ and $(\partial \phi)(X \wedge Y) = \frac{1}{2}(X\phi(Y) - Y\phi(X))$, respectively. Then $\partial^2 = 0$ and so we may define the Lie algebra cohomology:-

$$H^0(\mathfrak{a}, \mathbb{V}) = \ker \partial : \mathbb{V} \rightarrow \text{Hom}(\mathfrak{a}, \mathbb{V})$$

and

$$H^1(\mathfrak{a}, \mathbb{V}) = \frac{\ker \partial : \text{Hom}(\mathfrak{a}, \mathbb{V}) \rightarrow \text{Hom}(\Lambda^2 \mathfrak{a}, \mathbb{V})}{\text{im } \partial : \mathbb{V} \rightarrow \text{Hom}(\mathfrak{a}, \mathbb{V})}$$

In the particular case that \mathbb{V} is the finite-dimensional representation of the $|1|$ -graded Lie algebra $\mathfrak{g} = \mathfrak{gl}(n + 1, \mathbb{R})$ or $\mathfrak{g} = \mathfrak{so}(n + 1, 1)$ constructed from \mathbb{E} and k as in §3 but regarded as a representation of the Abelian subalgebra \mathfrak{g}_{-1} , it is easily verified that the linear transformations ∂ are homomorphisms of \mathfrak{g}_0 -modules and a theorem of Kostant [5] (see also [6]) identifies the cohomology as \mathfrak{g}'_0 -modules:-

$$(13) \quad H^0(\mathfrak{g}_{-1}, \mathbb{V}) = \mathbb{E} \quad H^1(\mathfrak{g}_{-1}, \mathbb{V}) = \mathbb{F} \equiv \odot^k \mathfrak{g}_1 \odot \mathbb{E}.$$

The canonical duality between \mathfrak{g}_{-1} and \mathfrak{g}_1 allows us rewrite (12) as

$$(14) \quad \mathbb{V} \xrightarrow{\partial} \mathfrak{g}_1 \otimes \mathbb{V} \xrightarrow{\partial} \Lambda^2 \mathfrak{g}_1 \otimes \mathbb{V}$$

and there are \mathfrak{g}_0 -module homomorphisms in the other direction:-

$$(15) \quad \mathbb{V} \xleftarrow{\partial^*} \mathfrak{g}_1 \otimes \mathbb{V} \xleftarrow{\partial^*} \Lambda^2 \mathfrak{g}_1 \otimes \mathbb{V}$$

defined by $\partial^*(Z \otimes v) = Zv$ and $\partial^*(Z \wedge W \otimes v) = Z \otimes Wv - W \otimes Zv$, respectively. Kostant's theorem includes an algebraic 'Hodge decomposition':-

$$(16) \quad \mathbf{V} = \mathbb{E} \oplus \text{im } \partial^* \quad \text{and} \quad \mathfrak{g}_1 \otimes \mathbf{V} = \text{im } \partial \oplus \mathbb{F} \oplus \text{im } \partial^*,$$

as \mathfrak{g}'_0 -modules. (Of course, the results in [5] are for higher cohomology too and for any $|\ell|$ -graded semisimple Lie algebra.)

5. PROOF OF THE RESULTS

Roughly speaking, the proof is but a diagram chase once a suitable diagram has been constructed. The diagram and its properties are obtained as geometric consequences of the algebraic discussion of §3 and especially the Lie algebra cohomology of §4. Specifically, let us consider the complex (14). It gives rise to a complex of vector bundles and vector bundle homomorphisms on M :-

$$(17) \quad V \xrightarrow{\partial} \Lambda^1 \otimes V \xrightarrow{\partial} \Lambda^2 \otimes V.$$

It is easily verified that these homomorphisms ∂ have degree 1 with respect to the grading (8) on V . (From now on we shall consider only the affine case, Theorem 1. The proof of Theorem 2 is word-for-word identical except that, in this Riemannian case, there is no need to choose a torsion-free connection ∇ since there is only one.) In other words, the series of homomorphisms (11) is extended to

$$(18) \quad 0 \rightarrow V_j \xrightarrow{\partial} \Lambda^1 \otimes V_{j-1} \xrightarrow{\partial} \Lambda^2 \otimes V_{j-2}.$$

The geometric interpretation of (13) is that this complex is exact except for

$$(19) \quad \frac{\ker \partial : \Lambda^1 \otimes V_{k-1} \rightarrow \Lambda^2 \otimes V_{k-2} = F,}{\text{im } \partial : V_k \rightarrow \Lambda^1 \otimes V_{k-1}}$$

where we have used the grading element (10) to locate $\mathbb{F} \subset \mathfrak{g}_1 \otimes \mathbf{V}$ as residing within $\mathfrak{g}_1 \otimes \mathbf{V}_{k-1}$. It is also useful to know that, as a \mathfrak{g}_0 -module, \mathbb{F} occurs with multiplicity one in $\mathfrak{g}_1 \otimes \mathbf{V}$ whence there is a \mathfrak{g}_0 -invariant projection $\pi : \mathfrak{g}_1 \otimes \mathbf{V} \rightarrow \mathbb{F}$ with geometric import a canonical surjection of vector bundles $\pi : \Lambda^1 \otimes V \rightarrow F$.

In addition to algebraic input as above, the diagram we seek needs some differential input in the form of a torsion-free affine connection ∇ . Such a connection gives rise to an induced connection on all tensor bundles and, in particular, on V . Let us inspect the resulting diagram in a typical case, as arising from equation (3) introduced in §1.3 and completed in §A.2.

$$(20) \quad \begin{array}{ccccccc} V & \xrightarrow{\nabla} & \Lambda^1 \otimes V & \xrightarrow{\nabla} & \Lambda^2 \otimes V & & \\ \parallel & & \parallel & & \parallel & & \\ \square & \xrightarrow{\nabla} & \square \oplus \square & \xrightarrow{\nabla} & \square \oplus \square & & \\ \oplus & \nearrow & \oplus & \nearrow & \oplus & & \\ \square \oplus \square & \xrightarrow{\nabla} & \square \oplus \square \oplus \square \oplus \square & \xrightarrow{\nabla} & \square \oplus \square \oplus \square \oplus \square \oplus \square & & \\ \oplus & \nearrow & \oplus & \nearrow & \oplus & & \\ \square & \xrightarrow{\nabla} & \square \oplus \square \oplus \square & \xrightarrow{\nabla} & \square \oplus \square \oplus \square \oplus \square & & \end{array}$$

In this diagram, the first column is the vector bundle V as previously identified (6). It is graded: $V = V_0 \oplus V_1 \oplus V_2$. The subsequent columns are obtained by decomposing

$\Lambda^i \otimes V_j$ into irreducible tensor bundles, easily accomplished by Littlewood-Richardson rules [4]. The horizontal arrows denote the action of connections induced from a chosen torsion-free affine connection. The sloping arrows denote the vector bundle homomorphisms ∂ as in (18). Kostant's Theorem (13) looks very reasonable in the context of this diagram: the geometric interpretations (18) and (19) say that

$$(21) \quad \begin{array}{ccc} V_1 & \xrightarrow{\partial} & \Lambda^1 \otimes V_0 \\ \parallel & & \parallel \\ \square \oplus \square & \longrightarrow & \square \oplus \square \end{array}$$

is an isomorphism, that the first homomorphism of the complex

$$(22) \quad \begin{array}{ccccc} V_2 & \xrightarrow{\partial} & \Lambda^1 \otimes V_1 & \xrightarrow{\partial} & \Lambda^2 \otimes V_0 \\ \parallel & & \parallel & & \parallel \\ \square & \longrightarrow & \square \oplus \square \oplus \square \oplus \square & \longrightarrow & \square \oplus \square \end{array}$$

is injective and the complex has $F = \square$ as its middle cohomology, and that

$$\begin{array}{ccc} \Lambda^1 \otimes V_2 & \xrightarrow{\partial} & \Lambda^2 \otimes V_1 \\ \parallel & & \parallel \\ \square \oplus \square \oplus \square & \longrightarrow & \square \oplus \square \oplus \square \oplus \square \oplus \square \end{array}$$

is injective.

Another key feature of the diagram

$$(23) \quad \begin{array}{ccccc} V & & \Lambda^1 \otimes V & & \Lambda^2 \otimes V \\ \parallel & & \parallel & & \parallel \\ V_0 & \xrightarrow{\nabla} & \Lambda^1 \otimes V_0 & \xrightarrow{\nabla} & \Lambda^2 \otimes V_0 \\ \oplus & \xrightarrow{\partial} & \oplus & \xrightarrow{\partial} & \oplus \\ V_1 & \xrightarrow{\nabla} & \Lambda^1 \otimes V_1 & \xrightarrow{\nabla} & \Lambda^2 \otimes V_1 \\ \oplus & \xrightarrow{\partial} & \oplus & \xrightarrow{\partial} & \oplus \\ V_2 & \xrightarrow{\nabla} & \Lambda^1 \otimes V_2 & \xrightarrow{\nabla} & \Lambda^2 \otimes V_2 \\ \vdots & & \vdots & & \vdots \end{array}$$

in general is the following:-

Proposition 1. *As operators $V \rightarrow \Lambda^2 \otimes V$, we have $\nabla\partial = -\partial\nabla$.*

Proof. The formulae for ∂ in (12) easily imply that the composition

$$\Lambda^1 \otimes V \xrightarrow{\text{Id} \otimes \partial} \Lambda^1 \otimes \Lambda^1 \otimes V \xrightarrow{\wedge \otimes \text{Id}} \Lambda^2 \otimes V$$

on M coincides with $-\partial$. Being induced from a homomorphism of $\text{GL}(n, \mathbb{R})$ -modules, the vector bundle homomorphism $\partial : V \rightarrow \Lambda^1 \otimes V$ is certainly compatible with ∇ . This means that the diagram

$$(24) \quad \begin{array}{ccc} V & \xrightarrow{\partial} & \Lambda^1 \otimes V \\ \nabla \downarrow & & \nabla \downarrow \\ \Lambda^1 \otimes V & \xrightarrow{\text{Id} \otimes \partial} & \Lambda^1 \otimes \Lambda^1 \otimes V \end{array}$$

commutes. As ∇ is torsion-free, the operator $\Lambda^1 \otimes V \xrightarrow{\nabla} \Lambda^2 \otimes V$ is unambiguously defined either as the composition

$$\Lambda^1 \otimes V \xrightarrow{\nabla} \Lambda^1 \otimes \Lambda^1 \otimes V \xrightarrow{\wedge \otimes \text{Id}} \Lambda^2 \otimes V$$

or, as is more usually done, by the formula $\nabla(\omega \otimes \Sigma) = d\omega \otimes \Sigma - \omega \wedge \nabla \Sigma$. Our result is now obtained by following the commutative diagram (24) with the homomorphism $\Lambda^1 \otimes \Lambda^1 \otimes V \xrightarrow{\wedge \otimes \text{Id}} \Lambda^2 \otimes V$. \square

We may now define the connection $\tilde{\nabla}$ that appears in Theorem 1:-

$$\tilde{\nabla} \equiv \nabla - \partial : V \longrightarrow \Lambda^1 \otimes V.$$

Our chosen torsion-free connection acts on all tensor bundles and therefore has an associated curvature on each. We shall abuse notation and write κ for any of these curvatures. In particular, the homomorphism $\kappa : V \rightarrow \Lambda^2 \otimes V$ respects the grading of V but each component $V_j \rightarrow \Lambda^2 \otimes V_j$ will also be denoted κ . Now, with reference to diagram (23), Proposition 1 implies that

$$(25) \quad (\nabla - \partial)(\nabla - \partial) = \nabla^2 = \kappa : V \rightarrow \Lambda^1 \otimes V.$$

In effect, this says that the curvature of $\tilde{\nabla}$ is the same as that of ∇ on V .

We are now in a position to present a proof of Theorem 1 as it applies to linear operators with symbol as in (3). Specifically, by chasing the diagram (20) we will be able to find the form of the closed system (A.2) and prove that it is equivalent to the overdetermined equation (3), without performing any calculation. Having done this, we will be able to manufacture, from lower order linear terms, the homomorphism $\Phi : V \rightarrow \Lambda^1 \otimes V$ predicted in Theorem 1. The proof of the general case is similar except for a few technical details, which we shall discuss later.

Recall (21) that $\partial : V_1 \rightarrow \Lambda^1 \otimes V_0$ is an isomorphism. Let δ denote its inverse. Then $\nabla \delta \nabla : V_0 \rightarrow \Lambda^1 \otimes V_1$ is canonically identified with $\sigma \mapsto \nabla(\nabla \sigma)$ as a differential operator $E \rightarrow \Lambda^1 \otimes \Lambda^1 \otimes E$. (In [1] these identifications are carefully distinguished but here, and from now on, canonical identifications will be written as equality.) Recall that $\pi : \Lambda^1 \otimes V_1 \rightarrow F$ is the canonical projection. We conclude that

$$(26) \quad \pi \nabla \delta \nabla = D : E \rightarrow F$$

where D is the differential operator $\sigma_a \mapsto \nabla_a \nabla_b \sigma_c$ occurring in (3).

To proceed further we must choose splitting homomorphisms

$$(27) \quad V_j \xleftarrow{\delta} \Lambda^1 \otimes V_{j-1} \xleftarrow{\delta} \Lambda^2 \otimes V_{j-2}, \quad \text{for } j \geq 2.$$

The \mathfrak{g}_0 -module homomorphisms ∂^* as in (15) are easily modified for this purpose. Though ∂^* is not necessarily a left inverse to the injection

$$V_j \xrightarrow{\partial} \mathfrak{g}_1 \otimes V_{j-1}, \quad \text{for } j \geq 1,$$

the Hodge decomposition (16) implies that $\ker \partial^*$ provides an invariant complement to $\text{im } \partial$ and so we can define an algebraic splitting δ with the same kernel. The geometric

consequence of these considerations is a canonical construction of complexes (27) of vector bundle homomorphisms so that we have a ‘Hodge decomposition’

$$V = E \oplus \text{im } \delta \quad \text{and} \quad \Lambda^1 \otimes V = \underbrace{\text{im } \partial \oplus F \oplus \text{im } \delta}_{=\ker \partial} \quad \begin{matrix} =\ker \delta \\ \end{matrix}$$

as before but, in addition, $\partial = \partial\delta\partial$ and $\delta = \delta\partial\delta$. We are now in a position to define the differential operator $L : E \rightarrow V$ of Theorem 1 in complete generality:-

$$L_0 : E \rightarrow V_0 \quad \text{is the identity} \quad \text{and} \quad L_j : E \rightarrow V_j \quad \text{is given by} \quad L_j = (\delta\nabla)^j.$$

Having added the splitting homomorphisms δ to the diagram (20), we may proceed by a series of simple steps, as follows. Firstly, we observe that

$$(28) \quad \{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = 0\} \cong \{\Sigma \in \Gamma(V) \text{ s.t. } \tilde{\nabla}\Sigma \in \text{im } \delta\},$$

the isomorphism being given by $\Sigma = L\sigma$ and, conversely, $\sigma = \Sigma_0$. This is just pure thought and diagram chasing. The second step is to rewrite the right hand side of this isomorphism to move towards the right hand side of (9). To this end, let us write $\tilde{\nabla}\Sigma \equiv \Psi \in \text{im } \delta$ and apply $\nabla - \partial$ to this equality. By (25), we conclude that

$$\partial\Psi = \nabla\Psi - \kappa\Sigma \quad \text{and} \quad \Psi \in \text{im } \delta$$

whence

$$\Psi = \delta\partial\Psi = \delta\nabla\Psi - \delta\kappa\Sigma.$$

This allows us inductively to deduce that

$$\begin{aligned} \Psi_0 &= 0 \\ \Psi_1 &= \delta\nabla\Psi_0 - \delta\kappa\Sigma_0 = -\delta\kappa\Sigma_0 \\ \Psi_2 &= \delta\nabla\Psi_1 - \delta\kappa\Sigma_1 = -\delta\nabla\delta\kappa\Sigma_0 - \delta\kappa\Sigma_1. \end{aligned}$$

Recycling this information, we have shown that $\tilde{\nabla}\Sigma \in \text{im } \delta$ if and only if

$$\begin{aligned} \nabla\Sigma_0 &= \partial\Sigma_1 \\ \nabla\Sigma_1 &= \partial\Sigma_2 - \delta\kappa\Sigma_0 \\ \nabla\Sigma_2 &= -\delta\kappa\Sigma_1 - \delta\nabla\delta\kappa\Sigma_0. \end{aligned}$$

This is just one step away from the closed system we seek. The only problem with the right hand side of this system is the term $\delta\nabla\delta\kappa\Sigma_0$, which contains a derivative. It may be rewritten as follows. Because δ is compatible with ∇ (cf. (24)), we have $\delta\nabla\delta\kappa\Sigma_0 = \delta^{(2)}\nabla\kappa\Sigma_0$ where $\delta^{(2)}$ is the composition

$$\Lambda^1 \otimes \Lambda^2 \otimes V_0 \xrightarrow{\text{Id} \otimes \delta} \Lambda^1 \otimes \Lambda^1 \otimes V_1 \xrightarrow{\wedge \otimes \text{Id}} \Lambda^2 \otimes V_1 \xrightarrow{\delta} \Lambda^1 \otimes V_2.$$

Furthermore, by the Leibnitz rule, $\nabla\kappa\Sigma_0 = (\text{Id} \otimes \kappa)(\nabla\Sigma_0) + (\nabla\kappa)\Sigma_0$. But $\nabla\Sigma_0$ is already determined earlier in the system: it is $\partial\Sigma_1$. If we make this replacement, then we have eliminated all derivatives of Σ from the right hand side:-

$$(29) \quad \left. \begin{aligned} \nabla\Sigma_0 &= \partial\Sigma_1 \\ \nabla\Sigma_1 &= \partial\Sigma_2 - \delta\kappa\Sigma_0 \\ \nabla\Sigma_2 &= -\delta\kappa\Sigma_1 - \delta^{(2)}(\text{Id} \otimes \kappa)\partial\Sigma_1 - \delta^{(2)}(\nabla\kappa)\Sigma_0 \end{aligned} \right\}.$$

In other words, the system has closed! Referring back to Theorem 1, we see that (9) is established for D the differential operator $\sigma_a \mapsto \nabla_{(a}\nabla_b\sigma_c)$ as in (3). More precisely, we should take $\Phi : V \rightarrow \Lambda^1 \otimes V$ given by

$$(30) \quad \Phi \begin{pmatrix} \Sigma_0 \\ \Sigma_1 \\ \Sigma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \delta\kappa\Sigma_0 \\ \delta\kappa\Sigma_1 + \delta^{(2)}(\text{Id} \otimes \kappa)\partial\Sigma_1 + \delta^{(2)}(\nabla\kappa)\Sigma_0 \end{pmatrix}.$$

The construction of Φ is completely canonical, as asserted in Theorem 1. Notice that very little detail was used concerning the operator D . Exactly the same argument goes through for any operator

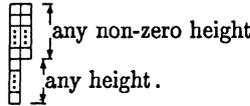
$$\sigma_{cd\dots e} \mapsto \nabla_{(a}\nabla_b\sigma_c)_{d\dots e}, \quad \text{where} \quad \sigma_{cd\dots e} = \sigma_{[cd\dots e]}$$

simply because it's a second order operator with $N = 2$, whence the form of the corresponding diagram is unchanged. Though the detailed meaning of ∂ , δ , and κ , therefore of Φ , will change a lot, the form (4) of the final connection $\widehat{\nabla} = \widetilde{\nabla} + \Phi$ will also be unchanged—it can be seen in (29). In any particular case, the tensorial meaning of δ , for example, can be quite complicated. This is already beginning to show itself in §A.2 below.

The case of first order operators $D : E \rightarrow F$ of the form

$$(31) \quad E \xrightarrow{\nabla} \Lambda^1 \otimes E \xrightarrow{\pi} \Lambda^1 \odot E = F$$

having $N = 2$ is also covered by exactly the same calculations as above. The formula $D = \pi\nabla$ replaces (26) but the first observation (28) is still valid and from then on the argument is unchanged. The E for which $N = 2$ have the shape



As a variation on argument above, let us consider the inhomogeneous equation

$$\nabla_{(a}\nabla_b\sigma_c) = \theta_{abc}, \quad \text{where} \quad \theta_{abc} = \theta_{(abc)}.$$

So, θ is a given section of F and we consider the equation $D\sigma = \theta$. Since F occurs with multiplicity one as an irreducible tensor subbundle of $\Lambda^1 \otimes V$, we may consider θ as a section of $\Lambda^1 \otimes V$. Its only component is in $\Lambda^1 \otimes V_1$. If we follow the same steps as above, then the first observation is that

$$\{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = \theta\} \cong \{\Sigma \in \Gamma(V) \text{ s.t. } \widetilde{\nabla}\Sigma - \theta \in \text{im } \delta\}.$$

In the second step, as before, we write $\widetilde{\nabla}\Sigma - \theta \equiv \Psi$ and find, bearing in mind that $\partial\theta = 0$ (since $F \subset \ker \partial$),

$$\Psi = \delta\nabla\Psi - \delta\kappa\Sigma + \delta\nabla\theta.$$

Inductively, we deduce that

$$\begin{aligned} \Psi_0 &= 0 \\ \Psi_1 &= \delta\nabla\Psi_0 - \delta\kappa\Sigma_0 + \delta\nabla\theta_0 = -\delta\kappa\Sigma_0 \\ \Psi_2 &= \delta\nabla\Psi_1 - \delta\kappa\Sigma_1 + \delta\nabla\theta_1 = -\delta\nabla\delta\kappa\Sigma_0 - \delta\kappa\Sigma_1 + \delta\nabla\theta. \end{aligned}$$

The only change to previous calculations is the extra terms θ and $\delta\nabla\theta$. The final outcome is that $\sigma \mapsto L\sigma$ defines an isomorphism

$$\{\sigma \in \Gamma(E) \text{ s.t. } D\sigma = \theta\} \cong \{\Sigma \in \Gamma(V) \text{ s.t. } \tilde{\nabla}\Sigma + \Phi(\Sigma) = \Theta\},$$

where $\Phi : V \rightarrow \Lambda^1 \otimes V$ is exactly as before and Θ is the section

$$\begin{pmatrix} 0 \\ \theta \\ \delta\nabla\theta \end{pmatrix}$$

of $\Lambda^1 \otimes V$. If, instead, we had started with D being a first order operator (31) with $N = 2$, then we would conclude that the inhomogeneous equation $D\sigma = \theta$ can be prolonged to the equivalent closed system $\hat{\nabla}\Sigma = \Theta$ where $\hat{\nabla} = \tilde{\nabla} + \Phi$ with Φ given by exactly the same formula (30) but

$$\Theta = \begin{pmatrix} \theta \\ \delta\nabla\theta \\ (\delta\nabla)^2\theta \end{pmatrix}.$$

To finish with the case of operators with symbol as in (3) we must consider the effect of lower order terms in the diagram chases presented above. Having fixed a torsion-free connection, the general operator under consideration has the form

$$\sigma_a \mapsto \nabla_{(a} \nabla_{b} \sigma_{c)} + \Gamma_{abc}{}^{de} \nabla_d \sigma_e + \Gamma_{abc}{}^d \sigma_d \quad \text{where} \quad \Gamma_{abc}{}^{de} = \Gamma_{(abc)}{}^{de} \quad \text{and} \quad \Gamma_{abc}{}^d = \Gamma_{(abc)}{}^d.$$

In other words, the general equation has the form

$$(32) \quad D\sigma + \Gamma_1 \nabla\sigma + \Gamma_0 \sigma = 0,$$

where $\Gamma_1 : \Lambda^1 \otimes E \rightarrow F$ and $\Gamma_0 : E \rightarrow F$ are given homomorphisms. Let us pause to note that the isomorphisms $V_0 = E$ and $\partial : V_1 \rightarrow \Lambda^1 \otimes E$ allow us to identify Γ_j as homomorphisms $\Gamma_j : V_j \rightarrow F \hookrightarrow V_1$. Now let us reconsider the result just obtained for the prolongation of the inhomogeneous equation $D\sigma = \theta$. In detail, we obtained the closed system

$$\left. \begin{aligned} \nabla\Sigma_0 &= \partial\Sigma_1 \\ \nabla\Sigma_1 &= \partial\Sigma_2 - \delta\kappa\Sigma_0 + \theta \\ \nabla\Sigma_2 &= -\delta\kappa\Sigma_1 - \delta^{(2)}(\text{Id} \otimes \kappa)\partial\Sigma_1 - \delta^{(2)}(\nabla\kappa)\Sigma_0 + \delta\nabla\theta \end{aligned} \right\}.$$

This system is equivalent to (32) if we take $\theta = -\Gamma_1 \nabla\sigma - \Gamma_0 \sigma$. The first equation, since it does not involve θ , allows us to write $\theta = -\Gamma_1 \Sigma_1 - \Gamma_0 \Sigma_0$ in the second equation:-

$$\nabla\Sigma_1 = \partial\Sigma_2 - \Gamma_1 \Sigma_1 - \delta\kappa\Sigma_0 - \Gamma_0 \Sigma_0.$$

This, in turn, allows us compute

$$\begin{aligned} \nabla\theta &= -\Gamma_1 \wedge \nabla\Sigma_1 - (\nabla\Gamma_1) \wedge \Sigma_1 - \Gamma_0 \wedge \nabla\Sigma_0 - (\nabla\Gamma_0) \wedge \Sigma_0 \\ &= -\Gamma_1 \wedge (\partial\Sigma_2 - \Gamma_1 \Sigma_1 - \delta\kappa\Sigma_0 - \Gamma_0 \Sigma_0) \\ &\quad - (\nabla\Gamma_1) \wedge \Sigma_1 - \Gamma_0 \wedge \partial\Sigma_1 - (\nabla\Gamma_0) \wedge \Sigma_0 \\ &= -\Gamma_1 \wedge \partial\Sigma_2 + \Gamma_1 \wedge \Gamma_1 \Sigma_1 - (\nabla\Gamma_1) \wedge \Sigma_1 - \Gamma_0 \wedge \partial\Sigma_1 \\ &\quad + \Gamma_1 \wedge \delta\kappa\Sigma_0 + \Gamma_1 \wedge \Gamma_0 \Sigma_0 - (\nabla\Gamma_0) \wedge \Sigma_0, \end{aligned}$$

where $\Gamma_j \wedge _$ is interpreted as a homomorphism $\Lambda^1 \otimes V_j \rightarrow \Lambda^2 \otimes V_1$ and $\nabla \Gamma_j \wedge _$ as a homomorphism $V_j \rightarrow \Lambda^2 \otimes V_1$. When substituted into the third equation, this gives a closed system as predicted by Theorem 1. More specifically, we have established Theorem 1 for second order linear differential operators with symbol

$$\odot : \odot^2 \Lambda^1 \otimes \Lambda^1 \rightarrow \odot^3 \Lambda^1$$

so that, having chosen a torsion-free affine connection ∇ and having written the equation $D\sigma = 0$ in the form (32), we have constructed the required $\Phi : V \rightarrow \Lambda^1 \otimes V$ by means of the formula

$$\Phi \begin{pmatrix} \Sigma_0 \\ \Sigma_1 \\ \Sigma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \Gamma_1 \Sigma_1 + \Gamma_0 \Sigma_0 + \delta \kappa \Sigma_0 \\ \delta(\Gamma_1 \wedge \partial \Sigma_2) - \delta(\Gamma_1 \wedge \Gamma_1 \Sigma_1) + \delta((\nabla \Gamma_1) \wedge \Sigma_1) + \delta(\Gamma_0 \wedge \partial \Sigma_1) \\ \quad + \delta \kappa \Sigma_1 + \delta^{(2)}(\text{Id} \otimes \kappa) \partial \Sigma_1 \\ - \delta(\Gamma_1 \wedge \delta \kappa \Sigma_0) - \delta(\Gamma_1 \wedge \Gamma_0 \Sigma_0) + \delta((\nabla \Gamma_0) \wedge \Sigma_0) + \delta^{(2)}(\nabla \kappa) \Sigma_0 \end{pmatrix}$$

as a generalisation of (30). Though this looks a little complicated, it is simply a result of following a well-defined series of simple steps and one's nose. The entire rewriting procedure is canonical, once ∇ has been chosen.

Save for one technical point, the general case is only notationally more complicated. There are always three steps to be taken:-

- express the equation to be prolonged in terms of $\tilde{\nabla}$ and an unknown $\Psi \in \text{im } \delta$;
- apply $\nabla - \partial$ and formula (25) inductively to determine Ψ ;
- pass down the resulting system, using the Leibnitz rule inductively to eliminate any derivatives from the right hand side.

The technical point is concerned with making sure that any derivatives on the right hand side can, indeed, be eliminated. We shall conclude this article by explaining this technicality. Incorporating it into a formal proof with careful inductions is done in [1] (which also covers the somewhat more awkward semilinear case and also some more general G -structures (which necessarily involve using affine connections with torsion, again increasing the level of awkwardness)).

Let us recall the complex (18). Knowing its cohomology immediately yields:-

Proposition 2. *If $k \geq 2$, then $\partial : V_1 \rightarrow \Lambda^1 \otimes V_0$ is an isomorphism. If $k \geq 3$, then*

$$0 \rightarrow V_j \xrightarrow{\partial} \Lambda^1 \otimes V_{j-1} \xrightarrow{\partial} \Lambda^2 \otimes V_{j-2}$$

is exact for $2 \leq j \leq k - 1$.

We used the first of these statements when $k = 2$ to identify D as $\pi \nabla \delta \nabla$ in (26). To generalise this identification we can employ the second statement as follows.

Proposition 3. *For $j \leq k - 1$ there is a canonical identification $V_j = \odot^j \Lambda^1 \otimes E$.*

Proof. The case $j = 0$ is true by choice of V . The case $j = 1$ is the first statement in Proposition 2. Higher j are proved by induction from the second statement of Proposition 2, which now proclaims the exactness of

$$0 \rightarrow V_j \xrightarrow{\partial} \Lambda^1 \otimes \odot^{j-1} \Lambda^1 \otimes E \xrightarrow{\partial} \Lambda^2 \otimes \odot^{j-2} \Lambda^1 \otimes E.$$

As in the proof of Proposition 1, the second ∂ -operator is (save for sign) given by skewing over the first two tensor indices with E simply going along for the ride. Its kernel is clearly $\odot^j \Lambda^1 \otimes E$, as required. \square

Having introduced the splitting homomorphisms δ :-

$$\begin{array}{ccccccc} V_j & \xleftarrow{\delta} & \Lambda^1 \otimes V_{j-1} & \xleftarrow{\delta} & \Lambda^2 \otimes V_{j-2} & & \\ \parallel & & \parallel & & \parallel & & \\ \odot^j \Lambda^1 \otimes E & \xleftarrow{\delta} & \Lambda^1 \otimes \odot^{j-1} \Lambda^1 \otimes E & \xleftarrow{\delta} & \Lambda^2 \otimes \odot^{j-2} \Lambda^1 \otimes E & \text{if } 2 \leq j \leq k-1, & \end{array}$$

it is tempting to believe that the differential operator

$$(\delta \nabla)^j : E \longrightarrow V_j = \odot^j \Lambda^1 \otimes E$$

coincides with the composition

$$E \xrightarrow{\nabla^{\otimes j}} \otimes^j \Lambda^1 \otimes E \xrightarrow{\otimes \text{Id}} \odot^j \Lambda^1 \otimes E$$

for $j = 2, 3, \dots, k-1$, as it does when $j = 1$. Were this to be true, then (26) would be generalised, namely $D \equiv \pi \nabla (\delta \nabla)^{k-1}$ would be the composition

$$E \xrightarrow{\nabla^{\otimes k}} \odot^k \Lambda^1 \otimes E \xrightarrow{\otimes} \odot^k \Lambda^1 \otimes E$$

and we would be able to proceed just as before. Unfortunately, this is not true. In general, it is only true that the symbol of the operator $D \equiv \pi \nabla (\delta \nabla)^{k-1}$ is the Cartan product. The problem is that the splitting operators δ determined by ∂^* as in (15) do not necessarily coincide with the 'naïve' splittings

$$\begin{array}{ccccccc} \odot^j \Lambda^1 & \xleftarrow{\delta} & \Lambda^1 \otimes \odot^{j-1} \Lambda^1 & \xleftarrow{\delta} & \Lambda^2 \otimes \odot^{j-2} \Lambda^1 & & \\ \cup & & \cup & \cup & \cup & & \\ \omega_{(ab\dots c)} & \xleftarrow{\delta} & \omega_{ab\dots c} & \omega_{a(b\dots c)} & \xleftarrow{\delta} & \omega_{ab\dots c} = \omega_{[ab]c\dots d} & \end{array}$$

tensoring with E . That there is a choice in δ , simply as a splitting of ∂ , can be seen already in (22) because the bundle \square occurs with multiplicity two. There are two possible workarounds. One is to choose the naïve splittings for $2 \leq j \leq k-1$ and perhaps stick with those constructed from ∂^* for $j \geq k$ (though, in fact, any \mathfrak{g} -invariant splittings of the representation complexes will do). The other possibility (the one adopted in [1]) is to use the naïve splittings to conclude that any operators $D \equiv \pi \nabla (\delta \nabla)^{k-1}$ and, in particular, those constructed from ∂^* , have the Cartan product as symbol but that this is good enough in the prolongation procedure because the final pass down the system with the Leibnitz rule will, in any case, eliminate any derivatives from the right hand side. With regard to this final pass, Proposition 3 is crucial: the zeroth component of the equation

$$\tilde{\nabla} \Sigma \equiv (\nabla - \partial) \Sigma = \Psi \in \text{im } \delta$$

says that $\Sigma_1 = \delta(\nabla \Sigma_0 - \Psi_0) = \delta \nabla \Sigma_0 = \delta \nabla \sigma$, which Proposition 3 allows us simply to view as saying that $\Sigma_1 = \nabla \sigma$. Hence, (Σ_0, Σ_1) records the first jet of σ . The first component of such an equation is similarly viewed using Proposition 3 as saying that

$$\Sigma_2 = \delta(\nabla \Sigma_1 - \Psi_1) = \delta \nabla \Sigma_1 = \delta \nabla \delta \nabla \Sigma_0 = \nabla \odot \nabla \sigma + \kappa \bowtie \sigma,$$

the 'curvature correction terms' being absent when the naïve splittings are used. In any case, $(\Sigma_0, \Sigma_1, \Sigma_2)$ records the second jet of σ . By induction, we reach the entire

$(k-1)^{\text{st}}$ jet of σ as recorded by $(\Sigma_0, \Sigma_1, \dots, \Sigma_{k-1})$. But this is exactly what is needed to prolong an arbitrary linear (or even semilinear) k^{th} order operator.

APPENDIX A. FURTHER EXAMPLES ON AN AFFINE MANIFOLD

A.1. **Example.** Suppose σ_{bc} is skew and consider the overdetermined differential equation

$$(33) \quad \nabla_{(a}\sigma_{b)c} = 0.$$

We can rewrite it as

$$\nabla_a \sigma_{bc} = \mu_{abc} \quad \text{where } \mu_{abc} \text{ is skew.}$$

Consider the tensors $\mu_{abcd} = \nabla_a \mu_{bcd}$ and $\sigma_{abcd} = \mu_{[ab]cd}$. It is an algebraic consequence of being skew in bcd that μ_{abcd} is determined by σ_{abcd} :-

$$\begin{aligned} 3\sigma_{a[bcd]} - 2\sigma_{[abcd]} &= \frac{3}{2}\mu_{abcd} - \frac{1}{2}(\mu_{bacd} + \mu_{cadb} + \mu_{dabc}) - 2\mu_{[abcd]} \\ &= \mu_{abcd} - \frac{1}{2}(\mu_{bacd} + \mu_{cadb} + \mu_{dabc} - \mu_{bacd} + \mu_{cabd} - \mu_{dabc}) \\ &= \mu_{abcd}. \end{aligned}$$

However,

$$\sigma_{abcd} = \nabla_{[a}\mu_{b]cd} = \nabla_{[a}\nabla_{b]}\sigma_{cd} = R_{ab}{}^e{}_{[c}\sigma_{d]e}$$

and so

$$\nabla_a \mu_{bcd} = 3R_{a[b}{}^e{}_{c}\sigma_{d]e}.$$

Therefore, the differential equation (33) is equivalent to the closed system

$$\left. \begin{aligned} \nabla_a \sigma_{bc} &= \mu_{abc} \\ \nabla_a \mu_{bcd} &= 3R_{a[b}{}^e{}_{c}\sigma_{d]e} \end{aligned} \right\}.$$

This system of equations may be written as $\widehat{\nabla}\Sigma = 0$ where

$$\Sigma = \begin{pmatrix} \sigma_{bc} \\ \mu_{bcd} \end{pmatrix} \quad \text{is a section of the vector bundle } V = \begin{matrix} \Lambda^2 \\ \oplus \\ \Lambda^3 \end{matrix}$$

and $\widehat{\nabla}: V \rightarrow \Lambda^1 \otimes V$ is the connection

$$\widehat{\nabla}_a \begin{pmatrix} \sigma_{bc} \\ \mu_{bcd} \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma_{bc} - \mu_{abc} \\ \nabla_a \mu_{bcd} - 3R_{a[b}{}^e{}_{c}\sigma_{d]e} \end{pmatrix}.$$

A.2. **Example.** Consider the overdetermined second order differential equation (3). If we set $\mu_{ab} = \nabla_a \sigma_b$, then

$$\nabla_{[a}\mu_{b]c} = \nabla_{[a}\nabla_{b]}\sigma_c = -\frac{1}{2}R_{ab}{}^d{}_{c}\sigma_d.$$

Hence, we can rewrite (3) as

$$\begin{aligned} \nabla_a \sigma_b &= \mu_{ab} \\ \nabla_a \mu_{bc} &= -\frac{1}{2}R_{ab}{}^d{}_{c}\sigma_d + \rho_{abc} \quad \text{where } \rho_{abc} = \rho_{(ab)c} \quad \text{and } \rho_{(abc)} = 0. \end{aligned}$$

Consider the tensors $\rho_{abcd} = \nabla_a \rho_{bcd}$ and $\mu_{abcd} = \rho_{[ab]cd}$. It is an algebraic consequence of its symmetries that ρ_{abcd} is determined by μ_{abcd} :-

$$\rho_{abcd} = 4\mu_{d(abc)} + \frac{4}{3}\mu_{a(bc)d}.$$

However,

$$\begin{aligned} \mu_{abcd} &= \nabla_{[a}\rho_{b]cd} = \nabla_{[a}\nabla_{b]}\mu_{cd} + \frac{1}{2}\nabla_{[a}(R_{b]c}{}^e{}_d\sigma_e) \\ &= -\frac{1}{2}R_{ab}{}^e{}_c\mu_{ed} - \frac{1}{2}R_{ab}{}^e{}_d\mu_{ce} \\ &\quad + \frac{1}{2}(\nabla_{[a}R_{b]c}{}^e{}_d)\sigma_e + \frac{1}{4}R_{bc}{}^e{}_d\nabla_a\sigma_e - \frac{1}{4}R_{ac}{}^e{}_d\nabla_b\sigma_e \\ &= -\frac{1}{2}R_{ab}{}^e{}_c\mu_{ed} - \frac{1}{2}R_{ab}{}^e{}_d\mu_{ce} \\ &\quad + \frac{1}{2}(\nabla_{[a}R_{b]c}{}^e{}_d)\sigma_e + \frac{1}{4}R_{bc}{}^e{}_d\mu_{ae} - \frac{1}{4}R_{ac}{}^e{}_d\mu_{be}. \end{aligned}$$

Therefore, the differential equation (3) is equivalent to the closed system

$$\left. \begin{aligned} \nabla_a\sigma_b &= \mu_{ab} \\ \nabla_a\mu_{bc} &= \rho_{abc} - \frac{1}{2}R_{ab}{}^d{}_c\sigma_d \\ \nabla_a\rho_{bcd} &= -2R_{d(a}{}^e{}_b\mu_{|e|c)} - \frac{2}{3}R_{a(b}{}^e{}_c)\mu_{ed} - 3R_{d(a}{}^e{}_b\mu_{c)e} - R_{a(b}{}^e{}_d|\mu_{c)e} \\ &\quad + (\nabla_{(a}R_{b|d}{}^e{}_c)\sigma_e + \frac{1}{3}(\nabla_{(b}R_{c)a}{}^e{}_d)\sigma_e \end{aligned} \right\}$$

where $\rho_{bcd} = \rho_{(bc)d}$ and $\rho_{(bcd)} = 0$.

A.3. Example. Consider the overdetermined second order differential equation

$$(34) \quad \text{the trace-free part of } \nabla_{(a}\nabla_{b)}\sigma^c = 0.$$

If we set $\mu_a{}^b = \nabla_a\sigma^b$, then

$$\nabla_{[a}\mu_{b]}{}^c = \nabla_{[a}\nabla_{b]}\sigma^c = \frac{1}{2}R_{ab}{}^c{}_d\sigma^d.$$

Hence, we can rewrite (34) as

$$\begin{aligned} \nabla_a\sigma^b &= \mu_a{}^b \\ \nabla_a\mu_b{}^c &= \frac{1}{2}R_{ab}{}^c{}_d\sigma^d + \rho_{(a}\delta_{b)}{}^c, \end{aligned}$$

where $\delta_a{}^b$ is the Kronecker delta. Consider the tensors

$$\rho_{ab} = \nabla_a\rho_b \quad \text{and} \quad \mu_{abc}{}^d = \frac{1}{2}\rho_{[ab]}\delta_c{}^d - \frac{1}{2}\delta_{[a}{}^d\rho_{b]c}.$$

Then ρ_{ab} is determined by $\mu_{abc}{}^d$:-

$$\rho_{ab} = \frac{4}{3(n+1)(n-1)} \left((n-1)\mu_{abc}{}^c + (2n+1)\mu_{acb}{}^c + (n+2)\mu_{bca}{}^c \right).$$

However,

$$\begin{aligned} \mu_{abc}{}^d &= \nabla_{[a}\nabla_{b]}\mu_c{}^d - \frac{1}{2}\nabla_{[a}(R_{b]c}{}^d{}_e\sigma^e) \\ &= -\frac{1}{2}R_{ab}{}^e{}_c\mu_e{}^d + \frac{1}{2}R_{ab}{}^d{}_e\mu_c{}^e \\ &\quad - \frac{1}{2}(\nabla_{[a}R_{b]c}{}^d{}_e)\sigma^e - \frac{1}{4}R_{bc}{}^d{}_e\nabla_a\sigma^e + \frac{1}{4}R_{ac}{}^d{}_e\nabla_b\sigma^e \\ &= -\frac{1}{2}R_{ab}{}^e{}_c\mu_e{}^d + \frac{1}{2}R_{ab}{}^d{}_e\mu_c{}^e \\ &\quad - \frac{1}{2}(\nabla_{[a}R_{b]c}{}^d{}_e)\sigma^e - \frac{1}{4}R_{bc}{}^d{}_e\mu_a{}^e + \frac{1}{4}R_{ac}{}^d{}_e\mu_b{}^e. \end{aligned}$$

Therefore, the differential equation (34) is equivalent to the closed system

$$\left. \begin{aligned} \nabla_a \sigma^b &= \mu_a^b \\ \nabla_a \mu_b^c &= \rho_{(a} \delta_{b)}^c + \frac{1}{2} R_{ab}{}^c{}_d \sigma^d \\ \nabla_a \rho_b &= \frac{1}{(n+1)(n-1)} \left(2(n+1) R_{d(a}{}^c{}_b) \mu_c^d - (2n+1) R_{ad} \mu_b^d - (n+2) R_{bd} \mu_a^d \right. \\ &\quad \left. - (\nabla_a R_{bc}) \sigma^c - n (\nabla_b R_{ac}) \sigma^c \right). \end{aligned} \right\}$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ADELAIDE
SOUTH AUSTRALIA 5005
E-mail: meastwo@maths.adelaide.edu.au