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In: Zdeněk Frolík (ed.): Proceedings of the 12th Winter School on Abstract Analysis, Section of Analysis. Circolo Matematico di Palermo, Palermo, 1984. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 5. pp. [83]–94.

Persistent URL: http://dml.cz/dmlcz/701817

AN APPLICATION OF FIXED-POINT THEORY TO EQUILIBRIUM ANALYSIS

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INTRODUCTION

It is a well-known experience that fixed-point theorems play a crucial role in equilibrium analysis. For instance, the classical results on the existence of competitive equilibria as presented in [7] are all based on a theorem of Gale, Nikaido, and Debreu, which may be considered as an equivalent version of Brouwer's fixed-point theorem. In spite of several illuminating results, the classical theory suffered from a somewhat artificial treatment of individual preferences. In recent times the equilibrium problem for the case of more general and natural preference structures was taken up in a series of papers [3], [4], [6], [8]. However, most of the results given there and elsewhere deal only with the case of bounded choice sets and stay on a rather abstract level. Let us also note that the beautiful approach of Gale and Mas-Colell [3] implicitely contains a very restrictive continuity assumption on the support function of the technology set so that some easy and natural examples cannot be treated by this method. Hence it seems that some important aspects of classical equilibrium theory like the Arrow-Debreu model [2], [7] have not yet been worked out in a modern context, which is both sufficiently general and simple. The present paper is to fill this gap. In the first part, we extend the Gale-Nikaido-Debreu theorem in order to handle the most general type of preference structures. In the second part, this result is used to derive the existence of equilibria and quasi-equilibria for a rather general Arrow-Debreu model. Our techniques are based on a suitable extension and combination of some arguments in [3], [4], and [5].

FIXED-POINT THEORY

We start with a common generalization of a classical theorem due to

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This paper is in final form and no version of it will be submitted for publication elsewhere.

Gale, Nikaido, and Debreu [7, Thm. 16.6] and a more recent result due to Greenberg [4]. This theorem turns out to be fundamental for the existence of equilibria in abstract economies without ordered preferences. The main ingredient will be Brouwer's fixed-point theorem via Kakutani's extension thereof. Actually, Theorem 1 may be viewed as Brouwer's fixed-point theorem in perfect disguise, since the latter is known to be equivalent to the Gale-Nikaido-Debreu theorem by an observation due to Uzawa [7, Thm. 16.7]. As usual a *correspondence* $\varphi : X \twoheadrightarrow Y$ is a mapping from a set X into the family of all subsets of another set Y.

1. THEOREM. *Let* $\Pi, X^1, \ldots, X^m$ *be convex compact and non-empty subsets of* $\mathbb{R}^n$ *and define* $X := X^1 \times \ldots \times X^m$. *Furthermore, let* $a \in \mathbb{R}^n$ *and for* $i=1,\ldots,m$ *consider an upper semicontinuous correspondence* $\varphi^i : \Pi \times X \twoheadrightarrow X^i$, *a correspondence* $\rho^i : \Pi \times X \twoheadrightarrow X^i$ *having an open graph in* $\Pi \times X \times X^i$, *and finally a continuous function* $\psi^i : \Pi \times X \to [0,\infty[$ *such that for all* $p \in \Pi$ *and all* $x = (x^1,\ldots,x^m) \in X$ *the following four conditions are fulfilled:*

(1) $\varphi^i(p,x)$ *is convex compact and non-empty.*

(2) *If* $\psi^i(p,x) > 0$, *then* φ^i *is lower semicontinuous at* (p,x).

(3) $\langle p, y^1 + \ldots + y^m + a\rangle \geq 0$ *whenever* $y^i \in \varphi^i(p,x)$ *for* $i=1,\ldots,m$.

(4) $x^i \notin \mathrm{co}\, \rho^i(p,x)$.

Then there exists a pair $(p,x) \in \Pi \times X$ *such that:*

(5) $x^i \in \varphi^i(p,x)$ *for* $i=1,\ldots,m$.

(6) $\psi^i(p,x) = 0$ *or* $\varphi^i(p,x) \cap \rho^i(p,x) = \emptyset$ *for* $i=1,\ldots,m$.

(7) $\langle q, x^1 + \ldots + x^m + a\rangle \geq 0$ *for all* $q \in \Pi$.

Proof. First, let $F^o : \Pi \times X \twoheadrightarrow \Pi$ be given by

$$F^o(p,x) := \{q \in \Pi : \langle q, x^1 + \ldots + x^m + a\rangle = \inf \langle \Pi, x^1+\ldots+x^m+a\rangle\}$$

for all $p \in \Pi$, $x \in X$ so that $F^o(p,x)$ is certianly convex compact and non-empty. Moreover, the correspondence F^o is easily seen to be closed and hence upper semicontinuous. Next, for $i=1,\ldots,m$ we define the continuous function $v^i : \Pi \times X \times X^i \to [0,\infty[$ by

$$v^i(p,x,y) := \psi^i(p,x)\mathrm{dist}((p,x,y), \Pi \times X \times X^i \setminus \mathrm{Graph}(\rho^i))$$

and then the correspondence $F^i : \Pi \times X \twoheadrightarrow X^i$ by

$$F^i(p,x) := \{y \in \varphi^i(p,x) : v^i(p,x,y) = \sup v^i(p,x,\varphi^i(p,x))\}$$

for all $p \in \Pi$, $x \in X$, $y \in X^i$. Again, it is clear that $F^i(p,x)$ is compact and non-empty, but not necessarily convex. Furthermore, since φ^i is upper semicontinuous everywhere and lower semicontinuous at every point (p,x) satisfying $\psi^i(p,x) > 0$, it is not hard to check that F^i is closed and hence upper semicontinuous on $\Pi \times X$. We finally introduce the correspondence $F : \Pi \times X \twoheadrightarrow \Pi \times X$ by

$$F(p,x) := F^0(p,x) \times \text{co}\, F^1(p,x) \times \ldots \times \text{co}\, F^m(p,x)$$

for all $p \in \Pi$ and $x \in X$. Then, by Kakutani's fixed-point theorem, there exists some $(p,x) \in \Pi \times X$ such that $(p,x) \in F(p,x)$. We claim that this pair has the desired properties: Condition (5) is fulfilled, since the sets $\varphi^i(p,x)$ are assumed to be convex. Moreover, condition (7) follows now from (3) and the definition of F^0. In order to prove (6), suppose that $\psi^i(p,x) > 0$ and that there exists a $y \in \varphi^i(p,x) \cap \rho^i(p,x)$. Then $v^i(p,x,y) > 0$ and therefore $v^i(p,x,z) > 0$ for all $z \in F^i(p,x)$. Thus $z \in \rho^i(p,x)$ for all $z \in F^i(p,x)$, which implies that

$$x^i \in \text{co}\, F^i(p,x) \subseteq \text{co}\, \rho^i(p,x).$$

This contradiction to assumption (4) completes the proof.

2. REMARK. If one takes Π to be the standard simplex $\mathbb{R}^n_1$, then condition (7) means exactly $x^1 + \ldots + x^m + a \geq 0$. Hence, for $m=1$ and $\rho^1 \equiv \emptyset$, $\psi^1 \equiv 0$ the preceding result yields a slight extension of the Gale-Nikaido-Debreu theorem referred to before. On the other hand, for $\Pi = \{0\}$ the theorem reduces to the result of Greenberg.

3. INTERPRETATION. In economic applications, Π stands for the set of all admissible price vectors for the commodity space $\mathbb{R}^n$ and is usually taken to be $= \mathbb{R}^n_1$, whereas $X^1,\ldots,X^m \subseteq \mathbb{R}^n$ have to be viewed as the choice sets for the finitely many agents (for instance traders or consumers) of a given economic system and $a \in \mathbb{R}^n$ is the initial endowment of this economy. Given a price vector $p \in \Pi$ and an allocation $x = (x^1,\ldots,x^m) \in X$, $\varphi^i(p,x)$ is the set of all those actions, which are admissible for the agent i under p and x, whereas $\rho^i(p,x)$ stands for the set of all actions of the agent i, which are strictly preferred to the respective component x^i of the allocation x under p and x. We emphasize that both sets may depend on the prices as well as on the choices of the other agents. A correspondence $\rho^i : \Pi \times X \twoheadrightarrow X^i$ having an open graph and being irreflexive in the sense of (4) will be called a *preference correspondence*; this generalizes the classical concept of strong preference relations.

Condition (3) is the usual Walras law in the general sense; it is decisive for the balance of supply and demand expressed by condition (7). A pair $(p,x) \in \Pi \times X$ satisfying (5), (6), (7) is sometimes called a *quasi-equilibrium* for the given abstract economy; it is said to be an *equilibrium* if it satisfies (5), (7) and instead of (6) the stronger condition $\varphi^i(p,x) \cap \rho^i(p,x) = \emptyset$ for $i=1,\dots,m$. The role of the critical functions ψ^i will soon become apparent; see also [4]. The interested reader will find plenty of further information on the classical background material in [1], [2], [6], and [7].

4. EXAMPLE. Suppose again that $\Pi, X^1,\dots,X^m \subseteq \mathbb{R}^n$ are convex compact and non-empty, let $a \in \mathbb{R}^n$, and consider continuous functions $\theta^1,\dots,\theta^m : \Pi \to \mathbb{R}$, which will be viewed as income functions for the agents of an abstract economy. Thus it is natural to assume that $\sup\langle p,X^i\rangle \geq \theta^i(p)$ holds for all $p \in \Pi$ and $i=1,\dots,m$. Then the correspondences $\varphi^i : \Pi \times X \Rightarrow X^i$ for $i=1,\dots,m$ given by

$$\varphi^i(p,x) := \varphi^i(p) := \{z \in X^i : \langle p,z\rangle \geq \theta^i(p)\} \quad \forall\, p \in \Pi,\ x \in X$$

are certainly upper semicontinuous and satisfy condition (1). Moreover it is not hard to see that (2) is fulfilled for the typical choice

$$\psi^i(p,x) := \psi^i(p) := \sup\langle p,X^i\rangle - \theta^i(p) \qquad \forall\, p \in \Pi,\ x \in X.$$

And it is clear that condition (3) holds, whenever

$$\theta^1(p) + \dots + \theta^m(p) + \langle p,a\rangle \geq 0 \qquad \forall\, p \in \Pi.$$

Finally if $f^i : X^i \to \mathbb{R}$ for $i=1,\dots,m$ denote continuous concave functions, which will be interpreted as individual utility functions, then the mappings $\rho^i : \Pi \times X \Rightarrow X^i$ for $i=1,\dots,m$ given by

$$\rho^i(p,x) := \rho^i(x^i) := \{z \in X^i : f^i(z) > f^i(x^i)\} \ \forall\, p \in \Pi,\ x \in X$$

are obviously preference correspondences. Hence, in the present situation, Theorem 1 guarantees the existence of a quasi-equilibrium. Note that every quasi-equilibrium of this abstract economy is necessarily an equilibrium, if one assumes the strict inequality $\sup\langle p,X^i\rangle > \theta^i(p)$ for all $p \in \Pi$ and $i=1,\dots,m$.

In some applications, one is mainly interested in strictly positive price vectors so that the choice $\Pi = \{p \in \mathbb{R}^n_1 : p_1,\dots,p_n > 0\}$ is more appropriate than $\Pi = \mathbb{R}^n_1$. The following variant of Theorem 1 is well-suited for this purpose, but it requires a somewhat technical additional assumption (12). This condition is automatically satisfied for $\Pi = \mathbb{R}^n_1$, but it is a non-trivial restriction for the general case; see [5] for a detailed discussion of this assumption.

5. THEOREM. *Let* $\Pi \subseteq \mathbb{R}^n$ *be convex and σ-compact such that* $\overline{\Pi} = \mathbb{R}_1$, *consider convex closed and non-empty subsets* $X^1,\dots,X^m$ *of* $\mathbb{R}^n$, *and define* $X := X^1 \times \dots \times X^m$. *Moreover, let* $a \in \mathbb{R}$ *and for* $i=1,\dots,m$ *consider an upper semicontinuous correspondence* $\varphi^i : \Pi \Rightarrow X^i$ *a correspondence* $\rho^i : \Pi \times X \Rightarrow X^i$ *with an open graph, and a continuous function* $\psi^i : \Pi \to [0,\infty[$ *such that for all* $p \in \Pi$ *and all* $x = (x^1,\dots,x^m) \in X$ *the following conditions are fulfilled:*

(8) $\varphi^i(p)$ *is convex compact and non-empty.*

(9) *If* $\psi^i(p) > 0$, *then* φ^i *is also lower semicontinuous at* p.

(10) $\langle p, y^1 + \dots + y^m + a\rangle \geq 0$ *whenever* $y^i \in \varphi^i(p)$ *for* $i=1,\dots,m$.

(11) $x^i \notin \mathrm{co}\, \rho^i(p,x)$.

(12) *If a sequence* $(p_k)_k$ *in* Π *converges to some* $p \in \mathbb{R}^n_1 \setminus \Pi$ *and if* $z_k \in \varphi^1(p_k) + \dots + \varphi^m(p_k) + a$ *for* $k \in \mathbb{N}$ *are arbitrarily chosen, then there exists some* $q \in \Pi$ *such that* $\liminf_{k\to\infty} \langle q, z_k\rangle < 0$.

Then there exists some $(p,x) \in \Pi \times X$ *such that:*

(13) $x^i \in \varphi^i(p)$ *for* $i=1,\dots,m$.

(14) $\psi^i(p) = 0$ *or* $\varphi^i(p) \cap \rho^i(p,x) = \emptyset$ *for* $i=1,\dots,m$.

(15) $x^1 + \dots + x^m + a \geq 0$.

Proof. We choose an increasing sequence of compact convex and non-empty subsets Π_k of $\mathbb{R}^n$ such that $\Pi_k \uparrow \Pi$ as $k \to \infty$. Now given an arbitrary $k \in \mathbb{N}$, we obtain by compactness and upper semicontinuity compact convex subsets $\tilde{X}^i_k$ of X^i such that

$$\varphi^i(p) \subseteq \tilde{X}^i_k \qquad \text{for } i=1,\dots,m \text{ and all } p \in \Pi_k.$$

Then Theorem 1 supplies us with a pair

$$p_k \in \Pi_k,\ x_k = (x^1_k,\dots,x^m_k) \in \tilde{X}^1_k \times \dots \times \tilde{X}^m_k$$

such that the following conditions are fulfilled:

$$x^i_k \in \varphi^i(p_k) \text{ for } i=1,\dots,m.$$

$$\psi^i(p_k) = 0 \text{ or } \varphi^i(p_k) \cap \rho^i(p_k,x_k) = \emptyset \text{ for } i=1,\dots,m.$$

$$\langle q, x^1_k + \dots + x^m_k + a\rangle \geq 0 \text{ for all } q \in \Pi_k.$$

After passing to a subsequence if necessary, we may assume that $(p_k)_k$ converges to some $p \in \mathbb{R}^n_1$. In view of the above properties, one easily deduces from condition (12) that actually $p \in \Pi$. Now, since $\varphi^1,\dots,\varphi^m$ are upper semicontinuous at p, we arrive at

$$x^i_k \in \varphi^i(p_k) \subseteq \varphi^i(p) + U \text{ for } i=1,\dots,m \text{ and almost all } k \in \mathbb{N},$$

where U denotes the open unit ball in $\mathbb{R}^n$. Hence, taking again a subsequence if necessary, we may suppose that $(x_k)_k$ converges to some limit $x = (x^1,\dots,x^m)$. We claim that the pair (p,x) has the desired properties. Obviously $x \in X$ and hence by upper semicontinuity $x^i \in \varphi^i(p)$ for $i=1,\dots,m$. And because of $\overline{\Pi} = \mathbb{R}^n_1$ it is not hard to see that $x^1 + \dots + x^m + a \geqq 0$. It remains to show that condition (14) is fulfilled. Suppose that $\psi^i(p) > 0$ and that there exists some $y \in \varphi^i(p) \cap \rho^i(p,x)$. Since φ^i is lower semicontinuous at p, there are $y_k \in \rho^i(p_k)$ for almost all $k \in \mathbb{N}$ such that $y_k \to y$ as $k \to \infty$. We also know that $y_k \in \rho^i(p_k,x_k)$ holds for almost all $k \in \mathbb{N}$, since the graph of ρ^i is open. On the other hand, we have by continuity $\psi^i(p_k) > 0$ for almost all $k \in \mathbb{N}$ and hence $\varphi^i(p_k) \cap \rho^i(p_k,x_k) = \emptyset$ for all these k according to the choice of (p_k,x_k). This contradiction completes the proof.

A GENERALIZED ARROW-DEBREU MODEL

The preceding theory leads to rather general existence theorems for equilibria in various types of bounded and unbounded economies. Here we shall concentrate on the typical case of a certain private ownership economy. Our main result will extend the classical Arrow-Debreu theorem [7, Thm. 16.1] and a more recent result due to Mas-Colell [6, Thm. 1] concerning only the case of pure exchange economies. In a similar way, one can attack the problem of strictly positive price vectors and the investigation of desired and productive goods.

Throughout this section, we assume the following *situation* of a generalized Arrow-Debreu model; see also [2] and [7] for more information on the classical aspects of this model:

$M = \{1,\dots,m\}$, $P,Q \subseteq M$ disjoint such that $P \cup Q = M$;

$X^i \subseteq \mathbb{R}^n$ convex closed and non-empty for all $i \in M$;

$X := X^1 \times \dots \times X^m$ and $\Pi = \mathbb{R}^n_1$;

$\rho^j : \Pi \times X \Rightarrow X^j$ preference correspondence for all $j \in Q$;
$a^j \in \mathbb{R}^n$ for all $j \in Q$ and $a: = \sum_{j\in Q} a^j$;
$\alpha_{ij} : \Pi \to [0,1]$ continuous for $i \in P$, $j \in Q$ such that

$$\sum_{j\in Q} \alpha_{ij} \equiv 1 \text{ for } i \in P.$$

In this situation, the numerical *income functions* are defined as follows:

$$\theta^i(p): = \sup\langle p, X^i\rangle \leq \infty \qquad \text{for } i \in P,\ p \in \Pi;$$

$$\theta^j(p): = -\langle p, a^j\rangle - \sum_{i\in P} \alpha_{ij}(p)\theta^i(p) \geq -\infty \qquad \text{for } j \in Q,\ p \in \Pi.$$

6. DEFINITION. A pair $(p,x) \in \Pi \times X$ is said to be an *equilibrium* for the Arrow-Debreu model, if the following conditions are fulfilled:

(16)	$x^1 + \ldots + x^m + a \geq 0$.	
(17)	$\langle p, x^i\rangle = \theta^i(p)$	for all $i \in P$.
(18)	$\langle p, x^j\rangle \geq \theta^j(p)$	for all $j \in Q$.
(19)	$\langle p, x^j\rangle = \theta^j(p)$	for all $j \in Q$ satisfying $\rho^j(p,x) \neq \emptyset$
(20)	$\langle p, y\rangle < \theta^j(p)$	for all $j \in Q$ and $y \in \rho^j(p,x)$.

If one considers in condition (20) only those $j \in Q$ which satisfy $\sup\langle p, X^j\rangle > \theta^j(p)$, the pair (p,x) is called a *quasi-equilibrium*.

We shall use the method of a-priori bounds to deal with the possibly unbounded choice sets X^i. To this end, let us introduce for every closed convex subset Y of $\mathbb{R}^n$ the closed convex cone

$$T(Y) := \{u \in \mathbb{R}^n : u + Y \subseteq Y\}.$$

Elementary convex analysis shows that this set coincides with the asymptotic cone of Y, i.e. we have

$$T(Y) = \{u \in \mathbb{R}^n : \exists\ y_k \in Y, t_k > 0 \text{ such that } t_k \to 0, t_k y_k \to u\}.$$

The following observation will be useful:

7. LEMMA. *Assume that* $T: = T(X^1) + \ldots + T(X^m)$ *satisfies*

$$T \cap (-T) = \{0\} = T \cap \mathbb{R}^n_+ .$$

Then $X_a: = \{x = (x^1,\ldots,x^m) \in X : x^1 + \ldots + x^m + a \geq 0\}$ *is compa*

Proof. Suppose that the assertion is false. Then, for all $k \in \mathbb{N}$ there exist

$$x_k = (x_k^1,\dots,x_k^m) \in X_a \text{ such that } c_k := \sum_{i=1}^m \|x_k^i\| \geq k.$$

After passing to suitable subsequences if necessary, we may assume that for each i=1,...,m the sequence $(c_k^{-1}x_k^i)_k$ converges to some limit $u^i \in \mathbb{R}^n$. By the remark preceding this lemma, we have $u^i \in T(X^i)$ for all i=1,...,m. And from $\|u^1\| + \dots + \|u^m\| = 1$ it follows that $u^i \neq 0$ for some i. On the other hand, from

$$\sum_{i=1}^m c_k^{-1}x_k^i \geq -c_k^{-1}a \qquad \text{for all } k \in \mathbb{N}$$

we conclude that $u^1 + \dots + u^m \geq 0$ and hence $u^1 + \dots + u^m = 0$, since by assumption $T \cap \mathbb{R}_+^n = \{0\}$. Now $u^i \in T \cap (-T) = \{0\}$ for each i=1,...,m. This obvious contradiction completes the proof of the lemma.

8. REMARK. The preceding result applies to a number of concrete situations. To give a typical example, let us assume that each of the sets X^j for $j \in Q$ is bounded above, that $0 \in X^i$ for all $i \in P$, and that the aggregate technology set $X_P := \sum_{i\in P} X^i$ satisfies:

$$X_P \cap \mathbb{R}_+^n \text{ is bounded and } X_P \cap (-X_P) = \{0\}.$$

Then ist is easily seen that the assumption of Lemma 7 is fulfilled so that X_a turns out to be compact. Conditions of this type on the sets X^i are well-known in equilibrium analysis.

We now state the main result of the present paper. Here, assertion (ii) is an immediate consequence of assertion (i), since the stronger condition given there implies $\sup\langle p,X^j\rangle > \theta^j(p)$ for all $j \in Q$ and $p \in \Pi$. However, for technical reasons we first prove (ii) and then deduce (i) by means of a perturbation argument.

9. THEOREM. *Assume that* $T \cap (-T) = \{0\} = T \cap \mathbb{R}_+^n$.

(i) *There exists a quasi-equilibrium for the Arrow-Debreu model if*

$$(X^j + a^j + \sum_{i\in P} \alpha_{ij}(p)X^i) \cap \mathbb{R}_+^n \neq \emptyset \qquad \forall\, j \in Q, p \in \Pi.$$

(ii) *There exists an equilibirum for the Arrow-Debreu model if*

$$(X^j + a^j + \sum_{i\in P} \alpha_{ij}(p)X^i) \cap \operatorname{Int}(\mathbb{R}_+^n) \neq \emptyset \qquad \forall\, j \in Q, p \in \Pi.$$

Proof of (ii). For all $i \in P$, $j \in Q$, $p \in \Pi$ we obtain by assumption suitable vectors $u^j(p) \in X^j$ and $u^{ij}(p) \in X^i$ such that

$$u^j(p) + a^j + \sum_{i\in P} \alpha_{ij}(p)u^{ij}(p) \in \mathrm{Int}(\mathbb{R}^n_+) \qquad \forall\ j \in Q, p \in \Pi.$$

According to the continuity of the functions α_{ij}, we may choose for every $\bar{p} \in \Pi$ an open neighborhood $U(\bar{p})$ of $\bar{p}$ in Π such that

$$u^j(\bar{p}) + a^j + \sum_{i\in P} \alpha_{ij}(p)u^{ij}(\bar{p}) \in \mathrm{Int}(\mathbb{R}^n_+) \qquad \forall\ j \in Q, p \in U(\bar{p}).$$

By compactness, we then obtain finitely many $\bar{p}_1,\dots,\bar{p}_r \in \Pi$ such that $\Pi = U(\bar{p}_1) \cup \dots \cup U(\bar{p}_r)$. Now, for each $i \in M$ let X^i_a denote the image of the product set X_a under the projection onto the i-the component. It follows from the assumptions of Theorem 9 and from Lemma 7 that each of the sets X^i_a for $i \in M$ is convex compact and non-empty. Hence we may find a convex and compact subset W of $\mathbb{R}^n$ such that

$$X^i_a \subseteq \mathrm{Int}(W) \qquad \text{for all } i \in M \text{ and}$$

$$u^j(\bar{p}_k),\ u^{ij}(\bar{p}_k) \in W \qquad \text{for all } i \in P,\ j \in Q,\ k=1,\dots,r.$$

Using this set, we now establish an appropriate situation to which Theorem 1 will apply. Let:

$$\tilde{X}^i := X^i \cap W \text{ for } i \in M \text{ and } \tilde{X} := \tilde{X}^1 \times \dots \times \tilde{X}^m;$$

$$\tilde{\theta}^i(p) := \sup\langle p,\tilde{X}^i\rangle \qquad \forall\ i \in P,\ p \in \Pi;$$

$$\tilde{\theta}^j(p) := -\langle p,a^j\rangle - \sum_{i\in P} \alpha_{ij}(p)\tilde{\theta}^i(p) \qquad \forall\ j \in Q,\ p \in \Pi;$$

$$\tilde{\varphi}^i(p,x) := \{z \in \tilde{X}^i : \langle p,z\rangle \geq \tilde{\theta}^i(p)\} \quad \forall\ i \in M,\ p \in \Pi,\ x \in \tilde{X};$$

$$\tilde{\psi}^i(p,x) := \sup\langle p,\tilde{X}^i\rangle - \tilde{\theta}^i(p) \qquad \forall\ i \in M,\ p \in \Pi,\ x \in \tilde{X}.$$

Note that the functions $\tilde{\theta}^i : \Pi \to \mathbb{R}$ and hence $\tilde{\psi}^i : \Pi \times \tilde{X} \to \mathbb{R}$ are continuous for all $i \in M$. Moreover, it is clear that $\tilde{\psi}^i \equiv 0$ for all $i \in P$, and it follows from our construction that $\tilde{\psi}^j(p,x) > 0$ for all $j \in Q$, $p \in \Pi$ and $x \in \tilde{X}$. The correspondences $\tilde{\varphi}^i : \Pi \times \tilde{X} \Rightarrow X^i$ are certainly upper semicontinuous and satisfy condition (1) for all $i \in M$, and it is not hard to see that $\tilde{\varphi}^j$ is also lower semicontinuous on $\Pi \times \tilde{X}$ whenever $j \in Q$, so that (2) is fulfilled. The Walras law (3) is an immediate consequence of the identity

$$\tilde{\theta}^1(p) + \dots + \tilde{\theta}^m(p) + \langle p,a\rangle = 0 \qquad \text{for all } p \in \Pi,$$

For $i \in P$, $j \in Q$, $p \in \Pi$ and $x \in \tilde{X}$, we finally introduce the sets

$$\tilde{\rho}^i(p,x) := \emptyset,$$

$$\tilde{\rho}^j(p,x) := W \cap \{ty + (1-t)x^j : 0 < t \leq 1,\ y \in \rho^j(p,x)\}.$$

An elementary calculation shows that $\tilde{\rho}^i : \Pi \times \tilde{X} \Rightarrow \tilde{X}^i$ is a preference correspondence for every $i \in M$. Hence, by Theorem 1, there exist a $p \in \Pi$ and an $x = (x^1,\dots,x^m) \in \tilde{X}$ such that the following conditions are fulfilled:

$$\langle p,x^i\rangle \geq \tilde{\theta}^i(p) \qquad \text{for all } i \in M$$

$$\langle p,y\rangle < \tilde{\theta}^j(p) \qquad \text{for all } j \in Q \text{ and } y \in \tilde{\rho}^j(p,x)$$

$$x^1 + \ldots + x^m + a \geq 0$$

We shall prove that the pair (p,x) is an equilibrium for the Arrow-Debreu model. We first claim that $\langle p,x^i\rangle = \theta^i(p)$ holds for every $i \in P$. Assuming this to be false, we have $\tilde{\theta}^i(p) = \langle p,x^i\rangle < \langle p,z\rangle$ for some suitable $z \in X^i$. Now $z_t := tz + (1-t)x^i$ satisfies $\tilde{\theta}^i(p) < \langle p,z_t\rangle$ for all $0 < t \leq 1$ as well as $z_t \in X_i \cap W$ for all sufficiently small $0 < t \leq 1$, since $x^i \in X^i_a \subseteq \mathrm{Int}(W)$. This obvious contradiction shows that

$$\tilde{\theta}^i(p) = \langle p,x^i\rangle = \theta^i(p) \qquad \text{for all } i \in P$$

and therefore $\tilde{\theta}^j(p) = \theta^j(p)$ for all $j \in Q$. Hence it is clear that the pair (p,x) satisfies the conditions (16), (17), (18). We next observe that $\rho^j(p,x) \neq \emptyset$ implies $\tilde{\rho}^j(p,x) \neq \emptyset$. Indeed, given a $y \in \rho^j(p,x)$, we have $y_t := ty + (1-t)x^j \in W$ for all sufficiently small $0 < t \leq 1$ because of $x^j \in X^j_a \subseteq \mathrm{Int}(W)$ and hence $y_t \in \tilde{\rho}^j(p,x)$ for all these t. Now, if $\tilde{\rho}^j(p,x)$ non-empty, then x^j certainly belongs to the closure of this set and hence $\langle p,x^j\rangle = \theta^j(p)$ as an immediate consequence of the properties stated above. Thus condition (19) is fulfilled as well. Let us finally assume that there exists some $y \in \rho^j(p,x)$ such that $\langle p,y\rangle \geq \theta^j(p)$. Then the same reasoning as before shows that $y_t := ty + (1-t)x^j$ satisfies $\langle p,y_t\rangle \geq \theta^j(p)$ for all $0 < t \leq 1$ as well as $y_t \in \tilde{\rho}^j(p,x)$ for all sufficiently small $0 < t \leq 1$, which is impossible by the choice of (p,x). Hence it follows that condition (20) is also satisfied. Thus (p,x) is the desired equilibrium.

Proof of (i). Let $e := (1,\ldots,1) \in \mathbb{R}^n$ and for $k \in \mathbb{N}$ define

$$a^j_k := a^j + \frac{1}{k}e \qquad \text{for } j \in Q \text{ and } a_k := \sum_{j\in Q} a^j_k.$$

Let $\theta^i_k = \theta^i$ for $i \in P$ and θ^j_k for $j \in Q$ denote the income functions corresponding to the initial endowments a^j_k for $k \in \mathbb{N}$. Then assertion (ii) applies to the perturbated situation so that for every $k \in \mathbb{N}$ there exists a pair $(p_k,x_k) \in \Pi \times X$ with the following properties:

$$x^1_k + \ldots + x^m_k + a_k \geq 0$$

$$\langle p_k, x_k^i\rangle = \theta^i(p_k) \qquad \text{for all } i \in P$$

$$\langle p_k, x_k^j\rangle \geq \theta_k^i(p_k) \qquad \text{for all } j \in Q$$

$$\langle p_k, x_k^j\rangle = \theta_k^j(p_k) \qquad \text{for all } j \in Q \text{ with } \rho^j(p_k, x_k) \neq \emptyset$$

$$\langle p_k, y\rangle < \theta_k^j(p_k) \qquad \text{for all } j \in Q \text{ and } y \in \rho^j(p_k, x_k).$$

Now observe that

$$x_k \in X_{a_k} \subseteq X_{a_1} \qquad \text{for all } k \in \mathbb{N},$$

where the latter set is known to be compact. Hence, taking subsequences if necessary, we may assume that $(p_k)_k$ and $(x_k)_k$ converge to limits $p \in \Pi$ and $x \in X$, respectively. One easily verifies that $\langle p, x^i\rangle = \theta^i(p)$ holds for all $i \in P$. This implies, in particular, the convergence $\theta_k^i(p_k) \to \theta^i(p)$ as $k \to \infty$ for all $i \in P$ and consequently for all $i \in M$, although the income functions need not be continuous on Π. Now, since the correspondences ρ^j have an open graph, one easily deduces from the properties stated above that (p,x) is a quasi-equilibrium for the Arrow-Debreu model.

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