ON PROPERTIES PRESERVED BY THE APPROXIMATE DOMINATION

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ABSTRACT. In this paper we identify properties which are preserved by Mardešić's notion of approximate domination. In particular, we show that internal versions of certain shape theoretic and certain approximate properties, various forms of the fixed point property, and also compactness, pseudocompactness, and strong paracompactness are examples of such properties.

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INTRODUCTION

In his approach to the problem of extending the notion of shape fibration to arbitrary topological spaces, Mardešić [19] introduced the following definition. A space X is approximately dominated by a class $\mathcal{D}$ of spaces, in notation $X \preccurlyeq \mathcal{D}$, provided for every normal (open) cover $\mathcal{U}$ of X there is a $Y \in \mathcal{D}$ and maps $u: X \longrightarrow Y$ and $d: Y \longrightarrow X$ such that $d \circ u$ is $\mathcal{U}$-close to the identity map 1_X on X, i. e., such that for every $x \in X$ there is a member of $\mathcal{U}$ which contains both x and $d \circ u(x)$. Except for few results in [19], [15], and [21] on spaces which are approximately dominated by the class of all polyhedra, there is

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no evidence in the literature of attempts to further investigate the new concept.

In the present paper we address ourselves to the identification of properties which are preserved by the approximate domination (which we call $\preccurlyeq$-invariant properties). More precisely, we are looking for properties Π which satisfy the following condition: If X is approximately dominated by a class of spaces each of which has the property Π, then X also has the property Π.

Our list of $\preccurlyeq$-invariant properties has more than two dozen entries. It can be roughly divided into six parts.

The first part form all 2_h-invariant and all 3_h-invariant properties from [3]. These are shape theoretic properties defined in terms of homotopy commutative diagrams involving nerves of either two or three normal covers. For example, the properties "to have deformation dimension $\leq$ n" [14], "to have the k-th Betti number with respect to Čech homology $\leq$ n", "to have trivial shape", tameness [3], smoothness [3], and movability [20] are of this type.

The second part consists of all 2_a-invariant and all 3_a-invariant properties from [3]. Those properties are approximate in nature because they are defined in terms of diagrams that commute up to a given normal cover. Again, they involve nerves of either two or three normal covers. The properties "to have covering dimension $\leq$ n", approximate movability [21], and the strong fixed point property [3] are in this group.

The third part comprise internal versions of properties from the first part. More precisely, the third part form properties which are invariant under the weak homotopy domination. This notion of domination is somewhere between the notion of homotopy domination and the notions of shape domination, 2_h-domination [3], and 3_h-domination [3]. The idea is to use genuine maps between spaces as in the ordinary homotopy theory and then project into nerves of normal covers to compare those maps as in the shape theory. The properties in this group are created by requering that certain maps which appear in the definitions of properties in the first part are maps into a space (this explains the use of the word "internal") rather than into a nerve of a normal cover of a space. The examples of such properties are (weak) internal tameness, (weak) internal smoothness, and internal movability [4], [21].

The fourth part form internal versions of properties from the second part. They can be also regarded as approximate versions of properties from the previous group. Hence, (weak) internal approximate tameness, (weak) internal approximate smoothness, and internal approximate movability [21], are examples of properties of this kind.

The fifth part includes the following three properties related to the fixed point property: approximate fixed point property, proximate fixed point property [18], and the internal strong fixed point property.

Finally, the last part consists of properties from the general topology related to the notion of compactness. For example, compactness, almost compactness [1], pseudocompactness, strong paracompactness, and τ-boundness.

The last three parts include properties which are preserved only by the approximate domination and are not preserved by other weaker notions of domination.

All of the listed properties are preserved by the m-convergence [12] on hyperspaces. Consequently, this paper establishes for the m-convergence results analogous to the ones proved about the q-convergence in [3]. On the other hand, since on metric spaces the m-convergence is equivalent to the convergence in the metric of continuity [5], our results extend to arbitrary topological spaces the results in [5] and [13] and also considerably enlarge the number of properties now known to be preserved by the convergence in the metric of continuity.

PRELIMINARIES

While the abstract and the introduction were directed to the widest possible audience, the rest of this paper is written for specialists. This part requires careful reading and consistent application notions, explanations, and abbreviations given in the present section.

We shall use standard logical symbols. The symbols $\forall$ and $\exists$ are used in front of variables that are put in paranthesis and are followed by a formula or some other sequence of symbols. The meaning of the variables and the unusual symbols are explained below or in the text. Each parenthesis containing the symbol $\exists$ should be followed mentally with words "such that". The expression within square brackets following a variable explains limitations on this variable and it should be preceded mentally with the word "with". An underlined expression between the parenthesis containing the symbol $\exists$ and the first colon is an abbreviation for the rest of the formula and is not a part of the formula. This abbreviation clearly lists all preceding variables.

IF NOT STATED OTHERWISE

X, Y, Z are topological spaces.

A, B are subsets of X.

$\mathcal{C}$, $\mathcal{D}$ are classes of topological spaces.

$\mathcal{Si}$ denotes both the class of all simplicial complexes and the category of simplicial complexes and simplicial maps.

Π is a topological property.

$\tilde{\Pi}$ is a class of all spaces which have the property Π.

$X \in \Pi \equiv X \in \tilde{\Pi}$.

$\tilde{X}$ denote all normal (open) covers of X [2].

$\mathcal{U}, \mathcal{V}, \mathcal{S}, \mathcal{M} \in \tilde{X}$ and $\mathcal{W}, \mathcal{Z}, \mathcal{T}, \mathcal{N} \in \tilde{Y}$.

$\mathcal{U}_A$ [<u>restriction</u> of $\mathcal{U}$ to A] $= \{U \cap A \mid U \in \mathcal{U}\}$.

$\mathcal{V} < \mathcal{U}$ [$\mathcal{V}$ <u>refines</u> $\mathcal{U}$] iff $(\forall V \in \mathcal{V})(\exists U \in \mathcal{U})\ V \subset U$.

A choice of a cover of X is always made so that it refines all covers that have previously been defined on X.

$2_{\mathcal{U}}A$ or $2A$ [<u>star</u> of A (w. r. t. $\mathcal{U}$)] $= \bigcup \{U \in \mathcal{U} \mid U \cap A \neq \emptyset\}$

$2_{\mathcal{U}}\mathcal{V}$ or $2\mathcal{V}$ [<u>star</u> of $\mathcal{V}$ (w. r. t. $\mathcal{U}$)] $= \{2V \mid V \in \mathcal{V}\}$.

We do not distinguish in our notation between a complex and its realization.

x_v is a barycentric coordinate of a point x in a complex with respect to a vertex v.

We do not distinguish in our notation between a normal cover and its nerve. Hence, $\mathcal{U}$ denotes also a simplicial complex whose vertices are members of $\mathcal{U}$ and $U_1, \ldots, U_n$ in $\mathcal{U}$ span a simplex iff $\bigcap_{i=1}^{n} U_i \neq \emptyset$.

Maps are continuous functions and between simplicial complexes all maps are simplicial.

$\mathcal{C}$-map is a map whose domain is a member of a class $\mathcal{C}$.

1_X denotes the identity map on X.

$i_{A,X}$ denotes the inclusion of A into X.

$\mathcal{V}_f$ is a cover $f^{-1}(\mathcal{V}) = \{f^{-1}(V) \mid V \in \mathcal{V}\}$ induced by a map $f:X \longrightarrow Y$.

$f_{\mathcal{V}}$ is a (simplicial) map $\mathcal{V}_f \longrightarrow \mathcal{V}$ defined by $f_{\mathcal{V}}(f^{-1}(V)) = V$ for every $V \in \mathcal{V}$.

We do not distinguish in our notation between a map and its homotopy class. We write $f \simeq g$ if maps $f, g:X \longrightarrow Y$ are homotopic and $f \overset{\mathcal{D}}{\simeq} g$ or $f \overset{\mathcal{D}}{=} g$ for homotopy classes if they are $\mathcal{D}$-homotopic (i. e., if $f \circ h \simeq g \circ h$ for every $\mathcal{D}$-map $h:Z \longrightarrow X$).

$\widetilde{p}:\mathcal{V} \longrightarrow \mathcal{U}$ ["p is a <u>projection</u> (of $\mathcal{V}$ into $\mathcal{U}$)"] iff $(\forall V \in \mathcal{V})\ V \subset p(V$

$\mathcal{U}\mathcal{V}$ denotes the unique homotopy class of projections of $\mathcal{V}$ into $\mathcal{U}$.

$\widetilde{p}:X \longrightarrow \mathcal{U}$ ["p is a <u>projection</u> (of X into $\mathcal{U}$)"] iff $(\forall U \in \mathcal{U},\ x \in U)$ $p(x)_U > 0$.

$\mathcal{U}$ denotes the unique homotopy class of projections of X into $\mathcal{U}$.

WARNING: Some symbols have several different usages. However, it will be always clear from the context which one is meant. For example, when we write $\mathcal{U} \circ d \circ u = \mathcal{U}$ it is clear that $\mathcal{U}$ must denote a morphism because the left hand side contains the symbols for composition. Hence, $\mathcal{U}$ can not denote a cover or a nerve of a cover. This implies that d and u actually denote homotopy classes of maps and not maps themselves.

WEAK HOMOTOPY DOMINATION

In this section we shall introduce a relation of weak homotopy domination. It is weaker than the relation of homotopy domination, but stronger than the homotopy relations 2_h and 3_h from [3].

(3.1) NOTATION. Write $X \overset{\mathcal{U}}{<} Y$ provided $(\exists u:X \longrightarrow Y,\ d:Y \longrightarrow X)$ $\mathcal{U}\circ d\circ u = \mathcal{U}$.

We shall say that X is <u>weakly</u> <u>homotopy</u> <u>dominated</u> by a class $\mathcal{D}$, in notation $X < \mathcal{D}$, provided $(\forall \mathcal{U})(\exists Y \in \mathcal{D})\ X \overset{\mathcal{U}}{<} Y$. When $\mathcal{D} = \{Y\}$, a class with a single element Y, we use $X < Y$ instead of $X < \{Y\}$. If for every $X \in \mathcal{C}$, $X < \mathcal{D}$, then the class $\mathcal{D}$ <u>weakly</u> <u>homotopy</u> <u>dominates</u> the class $\mathcal{C}$ and we write $\mathcal{C} < \mathcal{D}$.

(3.2) THEOREM. The relation of weak homotopy domination is reflexive and transitive.

PROOF. $(X < \mathcal{C},\ \mathcal{C} < \mathcal{D} \Rightarrow X < \mathcal{D})$. Let $\mathcal{U} \in \tilde{X}$. Since $X < \mathcal{C}$, there is a $Y \in \mathcal{C}$ and maps $a:X \longrightarrow Y$ and $b:Y \longrightarrow X$ such that $\mathcal{U}\circ b\circ a = \mathcal{U}$. Let $\mathcal{W} = \mathcal{U}_b \in \tilde{Y}$. Since $\mathcal{C} < \mathcal{D}$, there is a $Z \in \mathcal{D}$ and maps $m:Y \longrightarrow Z$ and $n:Z \longrightarrow Y$ such that $\mathcal{W}\circ n\circ m = \mathcal{W}$. Put $u = m\circ a:X \longrightarrow Z$ and $d = b\circ n:Z \longrightarrow X$. Then $\mathcal{U}\circ d\circ u = \mathcal{U}\circ b\circ n\circ m\circ a = b_{\mathcal{U}}\circ\mathcal{W}\circ n\circ m\circ a = b_{\mathcal{U}}\circ\mathcal{W}\circ a = \mathcal{U}\circ b\circ a = \mathcal{U}$. In other words, $X \overset{\mathcal{U}}{<} Z$ and, therefore, $X < \mathcal{D}$.

Recall [3] that we write $X\ 3_h^{\mathcal{U}}\ Y$ provided $(\exists \mathcal{W})$ $\underline{3_h^{\mathcal{U}}(\mathcal{W})}$: $(\forall \mathcal{Z})(\exists \mathcal{V})$ $\underline{3_h^{\mathcal{U}}(\mathcal{W}, \mathcal{Z}, \mathcal{V})}$: $(\forall \mathcal{S})(\exists \mathcal{T},\ e:\mathcal{W} \longrightarrow \mathcal{U},\ v:\mathcal{V} \longrightarrow \mathcal{Z},\ f:\mathcal{T} \longrightarrow \mathcal{S})$ $e\circ\mathcal{W}\mathcal{Z}\circ v = \mathcal{U}\mathcal{V}$ and $e\circ\mathcal{W}\mathcal{T} = \mathcal{U}\mathcal{S}\circ f$.

(3.3) THEOREM. $X \overset{\mathcal{U}}{<} Y \Rightarrow X\ 3_h^{\mathcal{U}}\ Y$.

PROOF. Since $X \overset{\mathcal{U}}{<} Y$, there are maps $u:X \longrightarrow Y$ and $d:Y \longrightarrow X$ such that $\mathcal{U}\circ d\circ u = \mathcal{U}$. Let $\mathcal{W} = \mathcal{U}_d \in \tilde{Y}$. We claim that $3_h^{\mathcal{U}}(\mathcal{W})$ is true.

Indeed, let $\mathcal{Z} \in \tilde{Y}$. Put $\mathcal{V} = \mathcal{Z}_u \in \tilde{X}$. Pick an $\mathcal{S} \in \tilde{X}$ with $\mathcal{S} < \mathcal{U}, \mathcal{V}$. Then $d_{\mathcal{U}}\circ\mathcal{W}\mathcal{Z}\circ u_{\mathcal{Z}}\circ\mathcal{V}\mathcal{S}\circ\mathcal{S} = d_{\mathcal{U}}\circ\mathcal{W}\mathcal{Z}\circ u_{\mathcal{Z}}\circ\mathcal{V} = d_{\mathcal{U}}\circ\mathcal{W}\mathcal{Z}\circ\mathcal{Z}\circ u = d_{\mathcal{U}}\circ\mathcal{W}\circ u = \mathcal{U}\circ d\circ u = \mathcal{U} =$

$\mathcal{US}\circ\mathcal{S}$. It follows from [20, p. 328], that there is an $\mathcal{M}\in\tilde{X}$ such that

(1) $d_{\mathcal{U}}\circ\mathcal{WZ}\circ u_{\mathcal{Z}}\circ\mathcal{VM} = \mathcal{UM}$.

Our claim will be proved provided we show that $3_h^{\mathcal{U}}(\mathcal{W}, \mathcal{Z}, \mathcal{M})$ is true.

In order to see this, let $\mathcal{A}\in\tilde{X}$. Put $\mathcal{T} = \mathcal{A}_d\in\tilde{Y}$. Choose a $\mathcal{B}\in\tilde{Y}$ which refines both $\mathcal{Z}$ and $\mathcal{T}$. Let $\tilde{p}:\mathcal{M}\longrightarrow\mathcal{V}$ and $\tilde{q}:\mathcal{B}\longrightarrow\mathcal{T}$. Put $e = d_{\mathcal{U}}: \mathcal{W}\longrightarrow\mathcal{U}$, $v = u_{\mathcal{Z}}\circ p:\mathcal{M}\longrightarrow\mathcal{Z}$, and $g = d_{\mathcal{A}}\circ q:\mathcal{B}\longrightarrow\mathcal{A}$. Observe that $\mathcal{UA}\circ g\circ\mathcal{B} = \mathcal{UA}\circ d_{\mathcal{A}}\circ\mathcal{TB}\circ\mathcal{B} = \mathcal{UA}\circ d_{\mathcal{A}}\circ\mathcal{T} = \mathcal{UA}\circ\mathcal{A}\circ d = \mathcal{U}\circ d = d_{\mathcal{U}}\circ\mathcal{W} = e\circ\mathcal{WB}\circ\mathcal{B}$. Once again, [20, p. 328] implies that there is an $\mathcal{F}\in\tilde{Y}$ such that

(2) $\mathcal{UA}\circ g\circ\mathcal{MF} = e\circ\mathcal{WF}$.

Let $\tilde{s}:\mathcal{F}\longrightarrow\mathcal{B}$. Put $f = g\circ s:\mathcal{F}\longrightarrow\mathcal{A}$. It remains to check $e\circ\mathcal{WZ}\circ v = \mathcal{UM}$ and $e\circ\mathcal{WF} = \mathcal{UA}\circ f$. But, the first of these equalities is simply another form of (1), while the second follows from (2).

The relations 3_h and 2_h in the statement of the following corollary were introduced in [3].

(3.4) COROLLARY. (a) $X < \mathcal{D} \Rightarrow X\ 3_h\ \mathcal{D}$.

(b) $X < \mathcal{D} \Rightarrow X\ 2_h\ \mathcal{D}$.

(c) $\mathcal{C} < \mathcal{D} \Rightarrow \mathcal{C}\ 3_h\ \mathcal{D}$.

(d) $\mathcal{C} < \mathcal{D} \Rightarrow \mathcal{C}\ 2_h\ \mathcal{D}$.

PROOF. The implication (a) follows from the above theorem, (b) is a consequence of (a) because $X\ 3_h\ \mathcal{D}$ implies $X\ 2_h\ \mathcal{D}$, while (c) and (d) follow from (a) and (b), respectively.

Recall that Y <u>homotopy</u> <u>dominates</u> X provided there are maps $u:X \longrightarrow Y$ and $d:Y\longrightarrow X$ such that $d\circ u \simeq 1_X$. Since the compositions of homotopic maps with a map are homotopic, if Y homotopy dominates X, then $X < Y$.

The following observation shows that in order to check whether X is weakly homotopy dominated by a class $\mathcal{D}$ instead of the Čech system of X we can use arbitrary ANR-expansion $\underline{p} = (p_i):X\longrightarrow\underline{X} = (X_i, p_{ij}, I)$ of X [20, p. 48]. Write $X <_{\underline{p}} \mathcal{D}$ provided for every $i\in I$ there is a $Y\in\mathcal{D}$ and maps $u:X\longrightarrow Y$ and $d:Y\longrightarrow X$ such that $p_i\circ d\circ u = p_i$.

(3.5) THEOREM. For every ANR-expansion $\underline{p}$ of X, the relations

$X < \mathcal{D}$ and $X <_{\underline{p}} \mathcal{D}$ are equivalent.

PROOF. Since the morphism $\check{p} = (\mathcal{U}):X \longrightarrow \check{X} = (\mathcal{U}, \mathcal{UV}, \tilde{X})$ is an ANR-expansion of X [20, p.328], it suffices to show that for ANR-expansions $\underline{p}$ and $\underline{q}$ of X, the relation $X <_{\underline{p}} \mathcal{D}$ implies $X <_{\underline{q}} \mathcal{D}$.

Suppose $\underline{q} = (q_j):X \longrightarrow \underline{Z} = (Z_j, q_{jk}, J)$ and $j \in J$. The condition (E1) for $\underline{p}$ applied to a morphism $q_j:X \longrightarrow Z_j$ implies that there is an $i \in I$ and a morphism $r:X_i \longrightarrow Z_j$ in the homotopy category $H\mathcal{T}op$ such that $q_j = r \bullet p_i$. Since $X <_{\underline{p}} \mathcal{D}$, there is a $Y \in \mathcal{D}$ and maps $u:X \longrightarrow Y$ and $d:Y \longrightarrow X$ with $p_i \bullet d \bullet u = p_i$. Then $q_j \bullet d \bullet u = r \bullet p_i \bullet d \bullet u = r \bullet p_i = q_j$.

In view of [20, Theorem 4, p. 50], the above theorem includes as a special case the following criterion for a metrizable space to be weakly homotopy dominated by a class $\mathcal{D}$. It is useful in verifying the examples below.

(3.6) THEOREM. A metrizable space X is weakly homotopy dominated by a class $\mathcal{D}$ provided for every neighborhood U of X in an ANR which contains X as a closed subset there is a $Y \in \mathcal{D}$ and maps $u:X \longrightarrow Y$ and $d:Y \longrightarrow X$ such that $d \bullet u$ is homotopic to the inclusion of X into U.

(3.7) EXAMPLES. (a) The reverse implications in the Corollary (3.4) are not true. Indeed, the Warsaw circle W and the circle S satisfy $W\ 2_h\ S$ and $W\ 3_h\ S$ because W and S are shape equivalent. However, W is not weakly homotopy dominated by S because every map of S into W is null-homotopic and there are neighborhoods of W in the plane inside of which W can not be contracted to a point.

(b) The closure A of the graph of the function $\sin(1/x)$, $0 < x \leq 1$, is weakly homotopy dominated by the one-point space P because A has trivial shape. But, A is not homotopy dominated by P because A is not contractible.

THE $<$-INVARIANT PROPERTIES

The goal in this section is to identify properties that are preserved by the weak homotopy domination. Most of those properties might be regarded as internal versions of corresponding 2_h-invariant and 3_h-invariant properties from [3].

A property Π is $<$-invariant provided $X < \widetilde{\Pi}$ implies $X \in \Pi$. In other words, Π is $<$-invariant provided X has the property Π whenever X is weakly homotopy dominated by a class of spaces having the property Π.

We shall say that X has a property $<\mathcal{C}$ provided $X < \mathcal{C}$. The theorem (3.2) implies the following.

(4.1) COROLLARY. The property $<\mathcal{C}$ is $<$-invariant.

On the other hand, the corollary (3.4) implies the following.

(4.2) COROLLARY. All 2_h-invariant and all 3_h-invariant properties are $<$-invariant.

Let $H\mathcal{T}op$ denote the homotopy category of topological spaces. Let $\mathcal{F}$ and $\mathcal{G}$ be collections of morphisms in $H\mathcal{T}op$. We shall say that X has a property $\overline{[\,]^{\mathcal{F}}}$ provided $\mathcal{U} \in \mathcal{F}$ for every $\mathcal{U} \in \widetilde{X}$. Similarly, X has a property $\overline{(\mathcal{C};\ \mathcal{F} \Rightarrow \mathcal{G})}$ provided $(\forall \mathcal{U})(\exists \mathcal{V})(\forall C \in \mathcal{C},\ f:C \longrightarrow X)\ \mathcal{V}\bullet f \in \mathcal{F} \Rightarrow \mathcal{U}\bullet f \in \mathcal{G}$.

A subcollection $\mathcal{F}$ of the collection $\mathrm{Mor}\mathcal{K}$ of all morphisms of a category $\mathcal{K}$ is a <u>left ideal in</u> $\mathcal{K}$ provided $u\bullet f \in \mathcal{F}$ whenever $f \in \mathcal{F}$, $u \in \mathrm{Mor}\mathcal{K}$, and $u\bullet f$ is defined. One similarly defines a <u>right ideal in</u> $\mathcal{K}$. If $\mathcal{F}$ is both a right and a left ideal in $\mathcal{K}$, then $\mathcal{F}$ is an <u>ideal in</u> $\mathcal{K}$.

For every collection $\mathcal{F}$ of morphisms in $H\mathcal{T}op$, $X \in \overline{(\mathcal{T}op;\ \mathrm{Mor}H\mathcal{T}op \Rightarrow \mathcal{F})}$ implies $X \in \overline{[\,]^{\mathcal{F}}}$. The converse is true when $\mathcal{F}$ is a right ideal in $H\mathcal{T}op$. Hence, the next theorem includes the following statement: If $\mathcal{F}$ is an ideal in $H\mathcal{T}op$, then the property $\overline{[\,]^{\mathcal{F}}}$ is $<$-invariant.

(4.3) THEOREM. If $\mathcal{F}$ and $\mathcal{G}$ are left ideals in $H\mathcal{T}op$, then the property $\overline{(\mathcal{C};\ \mathcal{F} \Rightarrow \mathcal{G})}$ is $<$-invariant.

PROOF. Let $\mathcal{D}$ be a class of spaces with the property $\overline{(\mathcal{C};\ \mathcal{F} \Rightarrow \mathcal{G})}$. Suppose $X < \mathcal{D}$. We must show that $X \in \overline{(\mathcal{C};\ \mathcal{F} \Rightarrow \mathcal{G})}$.

Let $\mathcal{U} \in \widetilde{X}$. Since $X < \mathcal{D}$, there is a $Y \in \mathcal{D}$ and maps $u:X \longrightarrow Y$ and $d:Y \longrightarrow X$ such that $\mathcal{U}\bullet d\bullet u = \mathcal{U}$. Let $\mathcal{W} = \mathcal{U}_d \in Y$. Since $Y \in \overline{(\mathcal{C};\ \mathcal{F} \Rightarrow \mathcal{G})}$, there is a $\mathcal{Z} \in \tilde{Y}$ such that for every $\mathcal{C}$-map $g:C \longrightarrow Y$, $\mathcal{Z}\bullet g \in \mathcal{F} \Rightarrow \mathcal{W}\bullet g \in \mathcal{G}$. Let $\mathcal{S} = \mathcal{Z}_u \in \tilde{X}$. Pick a $\mathcal{V} \in \tilde{X}$ with $\mathcal{V} < \mathcal{U}, \mathcal{S}$.

Suppose $f:C \longrightarrow X$ is a $\mathcal{C}$-map and $\mathcal{V}\bullet f \in \mathcal{F}$. Since $\mathcal{F}$ is a left ideal in $H\mathcal{T}op$, $u_{\mathcal{Z}}\bullet \mathcal{S}\mathcal{V}\bullet\mathcal{V}\bullet f = \mathcal{Z}\bullet u\bullet f \in \mathcal{F}$. The way in which $\mathcal{Z}$ was chosen implies that $\mathcal{W}\bullet u\bullet f \in \mathcal{G}$. Since $\mathcal{G}$ is also a left ideal in $H\mathcal{T}op$, we get $d_{\mathcal{U}}\bullet\mathcal{W}\bullet u\bullet f = \mathcal{U}\bullet d\bullet u\bullet f = \mathcal{U}\bullet f \in \mathcal{G}$.

Internal versions of many familiar shape properties are of the form $\boxed{\;}^{\mathcal{F}}$ for a suitable choice of the ideal $\mathcal{F}$. For example, taking for the ideal $\mathcal{F}$ homotopy classes of maps $f:X \longrightarrow Y$ which satisfy (1) f is $(\mathcal{C}, \mathcal{D})$-tame [10]; (2) f is $(\mathcal{C}, \mathcal{D})$-smooth [10]; and (3) $\mathrm{cat}_F f \leq n$ [7] will give us properties $\boxed{\;}^{\mathcal{F}}$ such that (1) X is weakly internally $(\mathcal{C}, \mathcal{D})$-tame; (2) X is weakly internally $(\mathcal{C}, \mathcal{D})$-smooth; and (3) $\mathrm{icat}_F X \leq n$ (the internal F-category of X is less than or equal to n), respectively. In order to help the reader dechipher above notions, we shall define the first property. A space X is <u>weakly internally</u> $(\mathcal{C}, \mathcal{D})$-<u>tame</u> provided $(\forall \mathcal{U}, C \in \mathcal{C}, f:C \longrightarrow X)(\exists D \in \mathcal{D}, a:C \longrightarrow D, b:D \longrightarrow \mathcal{U})\ \mathcal{U}\bullet f = b\bullet a$.

Let R and T be binary relations on morphisms of $H\mathcal{T}op$. A space X has a property $\overline{(\mathcal{C};\ R \Rightarrow T)}$ provided $(\forall \mathcal{U})(\exists \mathcal{V})(\forall C \in \mathcal{C},\ f, g:C \longrightarrow X)$ $\mathcal{V}\bullet f\ R\ \mathcal{V}\bullet g \Rightarrow \mathcal{U}\bullet f\ T\ \mathcal{U}\bullet g$.

Let R be a binary relation on the collection $\mathrm{Mor}\mathcal{K}$ of all morphisms in a category $\mathcal{K}$. Then R is 1-<u>stable</u> provided $u\bullet f\ R\ u\bullet g$ whenever f, g, $u \in \mathrm{Mor}\mathcal{K}$, f R g, and $u\bullet f$ and $u\bullet g$ are defined.

(4.4) THEOREM. If R and T are 1-stable binary relations on morphisms of $H\mathcal{T}op$, then the property $\overline{(\mathcal{C};\ R \Rightarrow T)}$ is $<$-invariant.

PROOF. Analogous to the proof of the theorem (4.3).

A property of mild internal $(\mathcal{C}, \mathcal{D})$-smoothness provides an example of the property to which the theorem (4.4) applies. Here, X is <u>mildly internally</u> $(\mathcal{C}, \mathcal{D})$-<u>smooth</u> provided $(\forall \mathcal{U})(\exists \mathcal{V})(\forall C \in \mathcal{C},\ f, g:C \longrightarrow X)$ $\mathcal{V}\bullet f \overset{\mathcal{D}}{=} \mathcal{V}\bullet g \Rightarrow \mathcal{U}\bullet f = \mathcal{U}\bullet g$.

Let $Q_{\mathcal{C}}$ be a q-system on $\mathcal{C}$ [3]. A space X is <u>internally</u> $Q_{\mathcal{C}}$-<u>movable</u>

provided $(\forall \mathcal{U})(\exists \mathcal{V})$ $\underline{\overline{Q}^{mo}_{C}(\mathcal{U}, \mathcal{V})}$: $(\forall C \in \mathcal{C}, f:C \longrightarrow \mathcal{V})(\exists (i, D_0, D, D_1) \in Q_C, F:D \longrightarrow \mathcal{U}, g:D_1 \longrightarrow X)$ $F|D_0 = \mathcal{U}\mathcal{V}\circ f\circ i$ and $F|D_1 = \mathcal{U}\circ g$.

Observe that every internally Q_C-movable space is Q_C-movable [3]. Also, a space X in internally $H_{\mathcal{S}i}$-movable [3] iff X is internally movable [21].

(4.5) THEOREM. The internal Q_C-movability is a $\angle$-invariant property.

PROOF. Let $\mathcal{D}$ be a class of internally Q_C-movable spaces. Suppose $X \angle \mathcal{D}$. We must show that X is internally Q_C-movable.

Let $\mathcal{U} \in \widetilde{X}$. Since $X \angle \mathcal{D}$, there is a $Y \in \mathcal{D}$ and maps $u:X \longrightarrow Y$ and $d:Y \longrightarrow X$ such that $\mathcal{U}\circ d\circ u = \mathcal{U}$. Since Y is internally Q_C-movable, there is a $\mathcal{Z} \in \widetilde{Y}$ such that $\overline{Q}^{mo}_{C}(\mathcal{W}, \mathcal{Z})$ holds. Let $\mathcal{V} = \mathcal{Z}_u \in \widetilde{X}$. Pick an $\mathcal{S} \in \widetilde{X}$ with $\mathcal{S} \angle \mathcal{U}, \mathcal{V}$. Observe that $d_{\mathcal{U}}\circ\mathcal{W}\mathcal{Z}\circ u_{\mathcal{Z}}\circ\mathcal{V}\mathcal{S}\circ\mathcal{S} = d_{\mathcal{U}}\circ\mathcal{W}\mathcal{Z}\circ u_{\mathcal{Z}}\circ\mathcal{V} = d_{\mathcal{U}}\circ\mathcal{W}\mathcal{Z}\circ\mathcal{Z}\circ u = d_{\mathcal{U}}\circ\mathcal{W}\circ u = \mathcal{U}\circ d\circ u = \mathcal{U} = \mathcal{U}\mathcal{S}\circ\mathcal{S}$. By [20, p. 328], there is an $\mathcal{M} \in \widetilde{X}$ such that $d_{\mathcal{U}}\circ\mathcal{W}\mathcal{Z}\circ u_{\mathcal{Z}}\circ\mathcal{V}\mathcal{M} = \mathcal{U}\mathcal{M}$. We claim that $\overline{Q}^{mo}_{C}(\mathcal{U}, \mathcal{M})$ is true.

Indeed, let a $\mathcal{C}$-map $f:C \longrightarrow \mathcal{M}$ be given. It follows from $\overline{Q}^{mo}_{C}(\mathcal{W}, \mathcal{Z})$ that there is an $(i, D_0, D, D_1) \in Q_C$, a $G: D \longrightarrow \mathcal{W}$, and an $h:D_1 \longrightarrow Y$ such that $G|D_0 = \mathcal{W}\mathcal{Z}\circ u_{\mathcal{Z}}\circ\mathcal{V}\mathcal{S}\circ f\circ i$ and $G|D_1 = \mathcal{W}\circ h$. Put $F = d_{\mathcal{U}}\circ G:D \longrightarrow \mathcal{U}$ and $g = d\circ h:D_1 \longrightarrow X$. Then $F|D_0 = d_{\mathcal{U}}\circ\mathcal{W}\mathcal{Z}\circ u_{\mathcal{Z}}\circ\mathcal{V}\mathcal{S}\circ f\circ i = \mathcal{U}\mathcal{S}\circ f\circ i$ and $F|D_1 = d_{\mathcal{U}}\circ G|D_1 = d_{\mathcal{U}}\circ\mathcal{W}\circ h = \mathcal{U}\circ d\circ h = \mathcal{U}\circ g$.

A space X is internally $(\mathcal{C}, \mathcal{D})$-tame provided $(\forall \mathcal{U})(\exists \mathcal{V})$ $\underline{i(\mathcal{C}, \mathcal{D})_{ta}(\mathcal{U}, \mathcal{V})}$: $(\forall C \in \mathcal{C}, f:C \longrightarrow \mathcal{V})(\exists D \in \mathcal{D}, a:C \longrightarrow D, b:D \longrightarrow X)$ $\mathcal{U}\circ b\circ a = \mathcal{U}\mathcal{V}\circ f$.

Every internally $(\mathcal{C}, \mathcal{D})$-tame space is both $(\mathcal{C}, \mathcal{D})$-tame [3] and eakly internally $(\mathcal{C}, \mathcal{D})$-tame.

(4.6) THEOREM. The internal $(\mathcal{C}, \mathcal{D})$-tameness is a $\angle$-invariant property.

PROOF. Let $\mathcal{E}$ be a class of internally $(\mathcal{C}, \mathcal{D})$-tame spaces. Suppose $X \angle \mathcal{E}$. We must show that X is internally $(\mathcal{C}, \mathcal{D})$-tame.

Let $\mathcal{U} \in \widetilde{X}$. Since $X \angle \mathcal{E}$, there is a $Y \in \mathcal{E}$ and maps $u:X \longrightarrow Y$ and $d:Y \longrightarrow X$ such that $\mathcal{U}\circ d\circ u = \mathcal{U}$. Let $\mathcal{W} = \mathcal{U}_d \in \widetilde{Y}$. Since Y is internally

$(\mathcal{C}, \mathcal{D})$-tame, there is a $\mathcal{Z} \in \widetilde{Y}$ such that $i(\mathcal{C}, \mathcal{D})_{ta}(\mathcal{W}, \mathcal{Z})$ holds. Let $\mathcal{V} = \mathcal{Z}_u \in \widetilde{X}$. Pick an $\mathcal{S} \in \widetilde{X}$ with $\mathcal{S} < \mathcal{U}, \mathcal{V}$. Observe that $d_{\mathcal{U}} \circ \mathcal{W}\mathcal{Z} \circ u_{\mathcal{Z}} \circ \mathcal{V}\mathcal{S} \circ \mathcal{S} = d_{\mathcal{U}} \circ \mathcal{W}\mathcal{Z} \circ u_{\mathcal{Z}} \circ \mathcal{V} = d_{\mathcal{U}} \circ \mathcal{W}\mathcal{Z} \circ \mathcal{Z} \circ u = d_{\mathcal{U}} \circ \mathcal{W} \circ u = \mathcal{U} \circ d \circ u = \mathcal{U} = \mathcal{U}\mathcal{S} \circ \mathcal{S}$. By [20, p. 328], there is an $\mathcal{M} \in \widetilde{X}$ such that $d_{\mathcal{U}} \circ \mathcal{W}\mathcal{Z} \circ u_{\mathcal{Z}} \circ \mathcal{V}\mathcal{M} = \mathcal{U}\mathcal{M}$. We claim that $i(\mathcal{C}, \mathcal{D})_{ta}(\mathcal{U}, \mathcal{M})$ is true.

Indeed, let a $\mathcal{C}$-map $f : C \longrightarrow \mathcal{M}$ be given. Let $\widetilde{p} : \mathcal{M} \longrightarrow \mathcal{V}$. Put $g = u_{\mathcal{Z}} \circ p \circ f : C \longrightarrow \mathcal{Z}$. By $i(\mathcal{C}, \mathcal{D})_{ta}(\mathcal{W}, \mathcal{Z})$, there is a $D \in \mathcal{D}$ and maps $a : C \longrightarrow D$ and $h : D \longrightarrow Y$ such that $\mathcal{W} \circ h \circ a = \mathcal{W}\mathcal{Z} \circ g$. Put $b = d \circ h : D \longrightarrow X$.

A space X is <u>internally</u> $(\mathcal{C}, \mathcal{D})$-<u>smooth</u> provided $(\forall \mathcal{U})(\exists \mathcal{V})$ <u>$i(\mathcal{C}, \mathcal{D})_{sm}(\mathcal{U}, \mathcal{V})$</u>: $(\forall C \in \mathcal{C},\ a, b : C \longrightarrow \mathcal{V}\ [a \overset{\mathcal{V}}{=} b])(\exists\, a', b' : C \longrightarrow X)$ $\mathcal{U} \circ a' = \mathcal{U}\mathcal{V} \circ a$, $\mathcal{U} \circ b' = \mathcal{U}\mathcal{V} \circ b$, and $\mathcal{U} \circ a' = \mathcal{U} \circ b'$.

Every internally $(\mathcal{C}, \mathcal{D})$-smooth space is both $(\mathcal{C}, \mathcal{D})$-smooth [3] and weakly internally $(\mathcal{C}, \mathcal{D})$-smooth.

(4.7) THEOREM. The internal $(\mathcal{C}, \mathcal{D})$-smoothness is a $<$-invariant property.

PROOF. Analogous to the proofs of the theorems (4.6) and (4.5).

APPROXIMATE DOMINATION

The notion of approximate domination has been introduced by Mardešić in [19]. It is a concept typical of the "approximate geometric topology". In this area we replace "strict" conditions with "approximate" conditions with an error as small as we please. The error is measured by a normal cover and depends on the following definitions of closeness for maps.

Let $\mathcal{U} \in \widetilde{X}$. For $x, y \in X$, write $x \overset{\mathcal{U}}{\sim} y$ provided there is a $U \in \mathcal{U}$ such that $x, y \in U$.

Maps $f, g : Z \longrightarrow X$ are $\mathcal{U}$-<u>close</u> (in notation, $f \overset{\mathcal{U}}{\sim} g$) provided $f(z) \overset{\mathcal{U}}{\sim} g(z)$ for every $z \in Z$. In the case when $Z \subset X$ and $f \overset{\mathcal{U}}{\sim} i_{Z,X}$, then we say that f is $\mathcal{U}$-<u>small</u> (in notation, $f < \mathcal{U}$).

Let $A, B \subset X$. Let $\mathcal{V} \in \widetilde{A}$ and $\mathcal{W} \in \widetilde{B}$. For $x \in \mathcal{V}$ and $y \in \mathcal{W}$, write

$x \overset{\mathcal{U}}{\sim} y$ provided given $V \in \mathcal{V}$ and $W \in \mathcal{W}$ with $x_V > 0$ and $y_W > 0$, then there is a $U \in \mathcal{U}$ such that $V \cup W \subset U$.

Maps $f:Z \longrightarrow \mathcal{V}$ and $g:Z \longrightarrow \mathcal{W}$ are $\mathcal{U}$-<u>close</u> (in notation, $f \overset{\mathcal{U}}{\sim} g$) provided $f(z) \overset{\mathcal{U}}{\sim} g(z)$ for every $z \in Z$.

In the case when $C \subset X$, $\mathcal{Z} \in \widetilde{C}$, and $Z = \mathcal{Z}$, the above definition is equivalent to the following. Maps $f:\mathcal{Z} \longrightarrow \mathcal{V}$ and $g:\mathcal{Z} \longrightarrow \mathcal{W}$ are $\mathcal{U}$-close iff $(\forall V \in \mathcal{Z})(\exists U \in \mathcal{U})\ f(V) \cup g(V) \subset U$. In this situation, f is $\mathcal{U}$-<u>small</u> (in notation, $f < \mathcal{U}$) provided $f \overset{\mathcal{U}}{\sim} 1_{\mathcal{Z}}$.

Let $A \subset X$. Let $\mathcal{V} \in \widetilde{A}$. For an $x \in X$ and a $y \in \mathcal{V}$, write $x \overset{\mathcal{U}}{\sim} y$ provided given a $V \in \mathcal{V}$ with $y_V > 0$, then there is a $U \in \mathcal{U}$ such that $x \in U$ and $V \subset U$.

Maps $f:Z \longrightarrow X$ and $g:Z \longrightarrow \mathcal{V}$ are $\mathcal{U}$-<u>close</u> (in notation, $f \overset{\mathcal{U}}{\sim} g$) provided $f(z) \overset{\mathcal{U}}{\sim} g(z)$ for every $z \in Z$.

(5.1) NOTATION. Write $X \preccurlyeq^{\mathcal{U}} Y$ provided $(\exists u:X \longrightarrow Y,\ d:Y \longrightarrow X)$ $d \circ u < \mathcal{U}$.

We shall say that X is <u>approximately dominated</u> by a class $\mathcal{D}$, in notation $X \preccurlyeq \mathcal{D}$, provided $(\forall \mathcal{U})(\exists Y \in \mathcal{D})\ X \preccurlyeq^{\mathcal{U}} Y$. When $\mathcal{D} = \{Y\}$, a class with a single element Y, we use $X \preccurlyeq Y$ instead of $X \preccurlyeq \{Y\}$. If for every $X \in \mathcal{C}$, $X \preccurlyeq \mathcal{D}$, then the class $\mathcal{D}$ <u>approximately dominates</u> the class $\mathcal{C}$ and we write $\mathcal{C} \preccurlyeq \mathcal{D}$ [19, Definition 1].

(5.2) THEOREM. The relation of approximate domination is reflexive and transitive.

PROOF. See [19, Remark 1].

Recall [3] that we write $X\ 3_a^{\mathcal{U}}\ Y$ provided $(\exists \mathcal{V}, \mathcal{W})$ $\underline{3_a^{\mathcal{U}}(\mathcal{V}, \mathcal{W})}$: $(\forall \mathcal{Z})$ $(\exists \mathcal{S})$ $\underline{3_a^{\mathcal{U}}(\mathcal{V}, \mathcal{W}, \mathcal{Z}, \mathcal{S})}$: $(\forall \mathcal{M})(\exists \mathcal{T},\ e:\mathcal{W} \longrightarrow \mathcal{V},\ v:\mathcal{S} \longrightarrow \mathcal{Z},\ f:\mathcal{T} \longrightarrow \mathcal{M},\ \widetilde{p}, \widetilde{r})$ $e \circ p \circ v < \mathcal{U}$ and $e \circ r \overset{\mathcal{U}}{\sim} f$.

(5.3) THEOREM. $X \preccurlyeq^{\mathcal{U}} Y \Rightarrow X\ 3_a^{2\mathcal{U}}\ Y$.

PROOF. Choose maps $u:X \longrightarrow Y$ and $d:Y \longrightarrow X$ such that $d \circ u < \mathcal{U}$. Let $\mathcal{W} = \mathcal{U}_d \in \widetilde{Y}$. We claim that $3_a^{2\mathcal{U}}(\mathcal{U}, \mathcal{W})$ is true.

In order to check this, let $\mathcal{Z} \in \widetilde{Y}$. Put $\mathcal{V} = \mathcal{Z}_u \in \widetilde{X}$. Pick an $\mathcal{S} \in \widetilde{X}$ with $\mathcal{S} < \mathcal{U}, \mathcal{V}$. Now, we claim that $3_a^{2\mathcal{U}}(\mathcal{U}, \mathcal{W}, \mathcal{Z}, \mathcal{S})$ is true.

Indeed, let $\mathcal{M} \in \widetilde{X}$. Put $\mathcal{T} = \mathcal{M}_d \in \widetilde{Y}$. Pick an $\mathcal{N} \in \widetilde{Y}$ with $\mathcal{N} < \mathcal{Z}, \mathcal{T}$. Let $\widetilde{p}:\mathcal{Z} \longrightarrow \mathcal{W}$, $\widetilde{r}:\mathcal{N} \longrightarrow \mathcal{W}$, $\widetilde{q}:\mathcal{S} \longrightarrow \mathcal{V}$, and $\widetilde{s}:\mathcal{N} \longrightarrow \mathcal{T}$. Define $e = d_{\mathcal{U}}:\mathcal{W} \longrightarrow \mathcal{U}$, $v = u_Z \circ q:\mathcal{S} \longrightarrow \mathcal{Z}$, and $f = d_{\mathcal{M}} \circ s:\mathcal{N} \longrightarrow \mathcal{M}$. It remains to see that $e \circ p \circ v < 2\mathcal{U}$ and $e \circ r \overset{2\mathcal{U}}{\sim} f$.

Let $S \in \mathcal{S}$. Pick a $Z \in \mathcal{Z}$ such that $q(S) = u^{-1}(Z)$. Hence, $v(S) = Z$. Next, we choose a $U \in \mathcal{U}$ with $p(Z) = d^{-1}(U)$. In other words, $e \circ p \circ v(S) = U$. Let $x \in S$. Then $x \in q(S)$ so that $u(x) \in Z$. Since $Z \subset p(Z) = d^{-1}(U)$, we get $d \circ u(x) \in U$. But, $d \circ u < \mathcal{U}$ implies that there is a $U_1 \in \mathcal{U}$ with x, $d \circ u(x) \in U_1$. It follows that $S \bigcup e \circ p \circ v(S) = S \cup U \subset 2_{\mathcal{U}} U$, i. e., that $e \circ p \circ v < 2\mathcal{U}$.

On the other hand, let $N \in \mathcal{N}$. Pick an $M \in \mathcal{M}$ and a $U \in \mathcal{U}$ such that $s(N) = d^{-1}(M)$ and $r(N) = d^{-1}(U)$. Then $f(N) = M$ and $e \circ r(N) = U$. But, $N \subset s(N) \cap r(N)$ so that $M \cap U \neq \emptyset$. This means that $f(N) \bigcup e \circ r(N) \subset 2_{\mathcal{U}} U$, i. e., that $e \circ r \overset{2\mathcal{U}}{\sim} f$.

The relations 3_a and 2_a in the statement of the following corollary were defined in [3].

(5.4) COROLLARY. (a) $X \leqslant \mathcal{D} \Rightarrow X\ 3_a\ \mathcal{D}$.

(b) $X \leqslant \mathcal{D} \Rightarrow X\ 2_a\ \mathcal{D}$.

(c) $\mathcal{C} \leqslant \mathcal{D} \Rightarrow \mathcal{C}\ 3_a\ \mathcal{D}$.

(d) $\mathcal{C} \leqslant \mathcal{D} \Rightarrow \mathcal{C}\ 2_a\ \mathcal{D}$.

PROOF. The implication (a) follows from the above theorem, (b) is a consequence of (a) because $X\ 3_a\ \mathcal{D}$ implies $X\ 2_a\ \mathcal{D}$, while (c) and (d) follow from (a) and (b), respectively.

Recall [6] that Y r-<u>dominates</u> X provided there are maps $u:X \longrightarrow Y$ and $d:Y \longrightarrow X$ such that $d \circ u = 1_X$. Clearly, if Y r-dominates X, then $X \leqslant Y$.

The following observation shows that in order to check whether X is approximately dominated by a class $\mathcal{D}$ instead of the Čech system of X we can use arbitrary resolutions $\underline{p} = (p_i):X \longrightarrow (X_i, p_{ij}, I)$ of X [19]. Write $X \leqslant_{\underline{p}} \mathcal{D}$ provided $(\forall i \in I, \mathcal{U} \in \widetilde{X}_i)(\exists Y \in \mathcal{D}, u:X \longrightarrow Y, d:Y \longrightarrow X)\ p_i \circ d \circ u \overset{\mathcal{U}}{\sim} p_i$.

(5.5) THEOREM. For every resolution $\underline{p}$ of the space X, the relations $X \preccurlyeq \mathcal{D}$ and $X \preccurlyeq_{\underline{p}} \mathcal{D}$ are equivalent.

PROOF. Suppose $X \preccurlyeq \mathcal{D}$. Let an $i \in I$ and a $\mathcal{U} \in \widetilde{X}_i$ be given. Let $\mathcal{V} = \mathcal{U}_{p_i} \in \widetilde{X}$. Select a $Y \in \mathcal{D}$ and maps $u:X \longrightarrow Y$ and $d:Y \longrightarrow X$ with $d \circ u < \mathcal{V}$. Then $p_i \circ d \circ u \overset{\mathcal{U}}{\sim} p_i$.

Conversely, suppose $X \preccurlyeq_{\underline{p}} \mathcal{D}$. Let $\mathcal{U} \in \widetilde{X}$. Since every resolution satisfies the condition (B1) (see the proof of the Theorem 6 in [19]), there exists an $i \in I$ and a normal cover $\mathcal{V}$ of X_i such that $p_i^{-1}(\mathcal{V})$ refines $\mathcal{U}$. Now, use $X \preccurlyeq_{\underline{p}} \mathcal{D}$ to get a $Y \in \mathcal{D}$ and maps $u:X \longrightarrow Y$ and $d:Y \longrightarrow X$ with $p_i \circ d \circ u \overset{\mathcal{V}}{\sim} p_i$. Clearly, $d \circ u < \mathcal{U}$.

(5.6) THEOREM. $X \preccurlyeq \mathcal{D} \Rightarrow X < \mathcal{D}$.

PROOF. Let $\mathcal{U} \in \widetilde{X}$. Since $\mathcal{U}$ is an ANR, there is a normal cover $\mathcal{A}$ of $\mathcal{U}$ such that $\mathcal{A}$-close maps into $\mathcal{U}$ are homotopic [17]. Let $\widetilde{p}:X \longrightarrow \mathcal{U}$. Put $\mathcal{V} = \mathcal{A}_p \in \widetilde{X}$. Since $X \preccurlyeq \mathcal{D}$, there is a $Y \in \mathcal{D}$ and maps $u:X \longrightarrow Y$ and $d:Y \longrightarrow X$ such that $d \circ u < \mathcal{V}$. Then $p \circ d \circ u \overset{\mathcal{A}}{\sim} p$ so that $p \circ d \circ u \simeq p$. Hence, $\mathcal{U} \circ d \circ u = \mathcal{U}$.

(5.7) EXAMPLES. (a). The reverse implications in the corollary (5.4) are not true. Indeed, the circle S and the circle with the spiral C satisfy $C \ni_a S$ but C is not approximately dominated by S.

(b). The closure A of the graph of the function $\sin(1/x)$, $0 < x \leq 1$, is approximately dominated by the arc I but A is not r-dominated by I.

(c) Also, A is weakly homotopy dominated by the one-point space P but A is not approximately dominated by P.

The next result shows that the limit of an m-convergent net of subsets is approximately dominated by the members of the net.

Recall the definition of the m-convergence on the hyperspace $\mathfrak{a}Z$ of all non-empty subsets of a space Z [12]. For $X, Y \in \mathfrak{a}Z$ and an $\mathcal{E} \in \widetilde{Z}$, write $X \underset{m}{\overset{\mathcal{E}}{\sim}} Y$ provided $(\exists f:X \longrightarrow Y,\ g:Y \longrightarrow X)$ $f < \mathcal{E}$ and $g < \mathcal{E}$. We shall say that a net $\{X_i\}_{i \in I}$ in $\mathfrak{a}Z$ m-<u>converges</u> to an $X \in \mathfrak{a}Z$ (in notation, $X_i \underset{m}{\longrightarrow} X$) provided $(\forall \mathcal{E} \in \widetilde{Z})(\exists i_0 \in I)(\forall i \geq i_0)\ X_i \underset{m}{\overset{\mathcal{E}}{\sim}} X$.

(5.8) THEOREM. Let X be a P-embedded [2] subset of a topological space Z. Let $\{X_i\}_{i\in I}$ be a net in $\tilde{a}Z$. If $X_i \xrightarrow[m]{} X$, then $X \lll \{X_i\}_{i\in I}$.

PROOF. Let $\mathcal{U} \in \tilde{X}$. Choose a $\mathcal{V} \in \tilde{X}$ with $2\mathcal{V} < \mathcal{U}$. Since X is P-embedded in Z, there is an $\mathcal{E} \in \tilde{Z}$ with $\mathcal{E}_X < \mathcal{V}$. Pick an $i \in I$ such that $X \overset{\mathcal{E}}{\underset{m}{\frown}} X_i$. Clearly, $X \lll^{\mathcal{U}} X_i$.

(5.9) COROLLARY. Let $\{X_i\}_{i\in I}$ be a net of P-embedded subsets of a topological space Z. Let $X \in \tilde{a}Z$. If $X_i \xrightarrow[m]{} X$, then $X \lll \{X_i\}_{i\in I}$.

PROOF. By [11] and [12], X is P-embedded in X.

THE $\lll$-INVARIANT PROPERTIES

We now turn to the identification of properties that are preserved by the approximate domination.

A property Π is $\lll$-<u>invariant</u> provided $X \lll \tilde{\Pi}$ implies $X \in \Pi$. It is m-<u>invariant</u> if for every net $\{X_i\}_{i\in I}$ of subsets of a topological space Z and a P-embedded subset X of Z with $X_i \xrightarrow[m]{} X$, $X_i \in \Pi$ for every $i \in I$ implies $X \in \Pi$.

The corollaries (5.4), (5.6), and (5.8) imply the following statements, respectively.

(6.1) COROLLARY. All 2_a-invariant and all 3_a-invariant properties [3] are $\lll$-invariant.

(6.2) COROLLARY. All $<$-invariant properties are $\lll$-invariant.

(6.3) COROLLARY. All $\lll$-invariant properties are m-invariant.

We shall say that X has a property $\lll C$ provided $X \lll C$. The theorem (5.2) implies the following.

(6.4) COROLLARY. The property $\lll C$ is $\lll$-invariant.

Let $\mathcal{F} \subset \mathrm{Mor}\,\mathcal{T}op$. We shall say that X has a property $s\mathcal{F}$ provided $(\forall \mathcal{U})(\exists \mathcal{V}, f{:}X \longrightarrow \mathcal{V})\ f < \mathcal{U}$ and $f \in \mathcal{F}$. A space X has a property $\bar{s}\mathcal{F}$ provided $(\forall \mathcal{U})(\exists f{:}X \longrightarrow X)\ f < \mathcal{U}$ and $f \in \mathcal{F}$. Observe that $X \in \bar{s}\mathcal{F}$ implies $X \in s\mathcal{F}$ whenever $\mathcal{F}$ is a left ideal in $\mathcal{T}op$.

(6.5) THEOREM. If $\mathcal{F}$ is an ideal in $\mathcal{T}op$, then both $s\mathcal{F}$ and $\bar{s}\mathcal{F}$ are

are $\leqslant$-invariant properties.

PROOF. We shall prove that $s^{\mathcal{F}}$ is $\leqslant$-invariant. An analogous proof for the property $\bar{s}^{\mathcal{F}}$ is left to the reader.

Let $\mathcal{D}$ be a class of spaces having the property $s^{\mathcal{F}}$. Suppose $X \leqslant \mathcal{D}$. We must show that $X \in s^{\mathcal{F}}$.

Let $\mathcal{U} \in \widetilde{X}$. Select a $\mathcal{V} \in \widetilde{X}$ with $2\mathcal{V} < \mathcal{U}$. Since $X \leqslant \mathcal{D}$, there is a $Y \in \mathcal{D}$ and maps $u:X \longrightarrow Y$ and $d:Y \longrightarrow X$ with $d\circ u < \mathcal{V}$. Since $Y \in s^{\mathcal{F}}$ and $\mathcal{W} = \mathcal{V}_d \in \widetilde{Y}$, there is a $\mathcal{Z} \in \widetilde{Y}$ and a map $g:Y \longrightarrow \mathcal{Z}$ such that $g < \mathcal{W}$ and $g \in \mathcal{F}$. Let $\tilde{p}:\mathcal{Z} \longrightarrow \mathcal{W}$. Put $f = d_{\mathcal{V}}\circ p\circ g\circ u:X \longrightarrow \mathcal{V}$. One can easily check that $f \in \mathcal{F}$ and $f < \mathcal{U}$.

A space X is <u>internally</u> $\mathcal{C}$-e-<u>movable</u> provided $(\forall\mathcal{U})(\exists\mathcal{V})$ $\underline{iC^{e}_{mo}(\mathcal{U}, \mathcal{V})}$: $(\forall C \in \mathcal{C},\ f:C \longrightarrow \mathcal{V})(\exists g:C \longrightarrow X)\ f \overset{\mathcal{U}}{\sim} g$.

Observe that every internally $\mathcal{C}$-e-movable space is $\mathcal{C}$-e-movable [3]. Also, if $\mathcal{C}$ is a class of k-spaces, then a compactum X is internally $\mathcal{C}$-e-movable iff X is $\mathcal{C}$-e-movable [8].

(6.6) THEOREM. The internal $\mathcal{C}$-e-movability is a $\leqslant$-invariant property.

Let $\mathcal{D}$ be a class of internally $\mathcal{C}$-e-movable spaces. Suppose $X \leqslant \mathcal{D}$. We must show that X is internally $\mathcal{C}$-e-movable.

Let $\mathcal{U} \in \widetilde{X}$. Select a $\mathcal{V} \in \widetilde{X}$, a $Y \in \mathcal{D}$, and maps $u:X \longrightarrow Y$ and $d:Y \longrightarrow X$ with $2\mathcal{V} < \mathcal{U}$ and $d\circ u < \mathcal{V}$. Let $\mathcal{W} = \mathcal{V}_d \in \widetilde{Y}$. Since Y is internally $\mathcal{C}$-e-movable, there is a $\mathcal{Z} \in \widetilde{Y}$ such that $iC^{e}_{mo}(\mathcal{W}, \mathcal{Z})$ holds. Let an $\mathcal{S} \in \widetilde{X}$ refines both $\mathcal{V}$ and $\mathcal{Z}_u$. We claim that $iC^{e}_{mo}(\mathcal{U}, \mathcal{S})$ is true.

Indeed, consider a $\mathcal{C}$-map $f:C \longrightarrow \mathcal{S}$. Let $\tilde{p}:\mathcal{S} \longrightarrow \mathcal{Z}_u$. Let $h = u_{\mathcal{Z}}\circ p\circ f:C \longrightarrow \mathcal{Z}$. The condition $iC^{e}_{mo}(\mathcal{W}, \mathcal{Z})$ implies that there is a map $k:C \longrightarrow Y$ with $h \overset{\mathcal{W}}{\sim} k$. Put $g = d\circ k:C \longrightarrow X$. It remains to check that $f \overset{\mathcal{U}}{\sim} g$.

Let $x \in C$. Consider an $S \in \mathcal{S}$ with $f(x)_S > 0$. Let $Z = u(p(S)) \in \mathcal{Z}$. Then $h(x)_Z > 0$. The relation $h \overset{\mathcal{W}}{\sim} k$ implies that there is a $W \in \mathcal{W}$ such that $h(x) \in W$ and $Z \subset W$. Let $V = d(W)$. Clearly, $g(x) \in V$ and $d(Z) \subset V$. Let $y \in S$. Then $u(y) \in Z$ and $d\circ u(y) \in V$. Since $d\circ u < \mathcal{V}$, there is a $V_1 \in \mathcal{V}$ such that $d\circ u(y),\ y \in V_1$. In other words, $S \subset 2_{\mathcal{V}}V$. Pick a $U \in \mathcal{U}$

with $2_V V \subset U$. Then $g(x) \in U$ and $S \subset U$.

A space X is <u>weakly internally</u> $(\mathcal{C}, \mathcal{D})$-e-<u>tame</u> provided $(\forall \mathcal{U}, C \in \mathcal{C}, f:C \longrightarrow X)(\exists D \in \mathcal{D}, a:C \longrightarrow D, b:D \longrightarrow X)\ f \overset{\mathcal{U}}{\sim} b\circ a$. It is <u>internally</u> $(\mathcal{C}, \mathcal{D})$-e-<u>tame</u> whenever $(\forall \mathcal{U})(\exists \mathcal{V})\ \underline{i(\mathcal{C}, \mathcal{D})^e_{ta}(\mathcal{U}, \mathcal{V})}$: $(\forall C \in \mathcal{C}, f:C \longrightarrow \mathcal{V})(\exists D \in \mathcal{D}, a:C \longrightarrow D, b:D \longrightarrow X)\ f \overset{\mathcal{U}}{\sim} b\circ a$.

Observe that every internally $(\mathcal{C}, \mathcal{D})$-e-tame space is weakly internally $(\mathcal{C}, \mathcal{D})$-e-tame. The converse is not true in general. Indeed, the Warsaw circle W is weakly internally $(\{S^1\}, \{I\})$-e-tame, but it is not internally $(\{S^1\}, \{I\})$-e-tame. A compactum X is internally $(\mathcal{C}, \mathcal{D})$-e-tame iff it satisfies the definition (3.1) in [9]. Also, a (weakly) internally $(\mathcal{C}, \mathcal{D})$-e-tame space is (weakly) internally $(\mathcal{C}, \mathcal{D})$-tame.

(6.7) THEOREM. Both the weak internal $(\mathcal{C}, \mathcal{D})$-e-tameness and the internal $(\mathcal{C}, \mathcal{D})$-e-tameness are $\lll$-invariant properties.

PROOF. We shall prove that the internal $(\mathcal{C}, \mathcal{D})$-e-tameness is a $\lll$-invariant property. An analogous proof for the weak internal $(\mathcal{C}, \mathcal{D})$-e-tameness is left to the reader.

Let $\mathcal{E}$ be a class of internally $(\mathcal{C}, \mathcal{D})$-e-tame spaces. Suppose $X \lll \mathcal{E}$. We must show that X is internally $(\mathcal{C}, \mathcal{D})$-e-tame.

Let $\mathcal{U} \in \tilde{X}$. Pick a $\mathcal{V} \in \tilde{X}$ with $2\mathcal{V} < \mathcal{U}$. Since $X \lll \mathcal{E}$, there is a $Y \in \mathcal{E}$ and maps $u:X \longrightarrow Y$ and $d:Y \longrightarrow X$ such that $d\circ u < \mathcal{V}$. Let $\mathcal{W} = \mathcal{V}_d \in \tilde{Y}$. Since Y is internally $(\mathcal{C}, \mathcal{D})$-e-tame, there is a $\mathcal{Z} \in \tilde{Y}$ such that $i(\mathcal{C}, \mathcal{D})^e_{ta}(\mathcal{W}, \mathcal{Z})$ holds. Let $\mathcal{S} \in \tilde{X}$ refines both $\mathcal{V}$ and $\mathcal{Z}_u$. We claim that $i(\mathcal{C}, \mathcal{D})^e_{ta}(\mathcal{U}, \mathcal{S})$ is true.

Indeed, let $C \in \mathcal{C}$ and $f:C \longrightarrow \mathcal{S}$. Let $\hat{p}:\mathcal{S} \longrightarrow \mathcal{Z}_u$. Put $g = u_{\mathcal{Z}}\circ p\circ f: C \longrightarrow \mathcal{Z}$. By $i(\mathcal{C}, \mathcal{D})^e_{ta}(\mathcal{W}, \mathcal{Z})$, there is a $D \in \mathcal{D}$ and maps $a:C \longrightarrow D$ and $c:D \longrightarrow Y$ such that $g \overset{\mathcal{W}}{\sim} c\circ a$. Put $b = d\circ c:D \longrightarrow X$. Then $f \overset{\mathcal{U}}{\sim} b\circ a$.

A space X is <u>internally</u> $(\mathcal{C}, \mathcal{D})$-e-<u>smooth</u> provided $(\forall \mathcal{U})(\exists \mathcal{V})\ \underline{i(\mathcal{C}, \mathcal{D})^e_{sm}(\mathcal{U}, \mathcal{V})}$: $(\forall C \in \mathcal{C}, a, b:C \longrightarrow \mathcal{V}\ [a \overset{\mathcal{V}}{\simeq} b])(\exists a', b':C \longrightarrow X)\ a \overset{\mathcal{U}}{\sim} a'$, $b \overset{\mathcal{U}}{\sim} b'$, and $a' \simeq b'$. It is <u>weakly internally</u> $(\mathcal{C}, \mathcal{D})$-e-<u>smooth</u> whenever $(\forall \mathcal{U}, C \in \mathcal{C}, a, b:C \longrightarrow X\ [a \overset{\mathcal{V}}{\simeq} b])(\exists a', b':C \longrightarrow X)\ a \overset{\mathcal{U}}{\sim} a'$, $b \overset{\mathcal{U}}{\sim} b'$, and $a' \simeq b'$.

Observe that every internally $(\mathcal{C}, \mathcal{D})$-e-smooth space is weakly internally $(\mathcal{C}, \mathcal{D})$-e-smooth. The converse is not true in general. For example, the Warsaw circle is weakly internally $(\{S^1\}, \{1\})$-e-smooth but it is not internally $(\{S^1\}, \{1\})$-e-smooth. A (weakly) internally $(\mathcal{C}, \mathcal{D})$-e-smooth space is (weakly) internally $(\mathcal{C}, \mathcal{D})$-smooth.

(6.8) THEOREM. Both the weak internal $(\mathcal{C}, \mathcal{D})$-e-smoothness and the internal $(\mathcal{C}, \mathcal{D})$-e-smoothness are $\preccurlyeq$-invariant properties.

PROOF. The proof is similar to the proof of the theorem (6.7).

The next three properties are related to the fixed point property.

Let $\mathcal{F}$ be a collection of morphisms in the category $\mathcal{T}op$ of topological spaces and continuous functions. We shall say that a space X has the <u>approximate fixed point property with respect to</u> $\mathcal{F}$ (in notation, $X \in \mathrm{afpp}(\mathcal{F})$) provided $(\forall \mathcal{U},\ f:X \longrightarrow X\ [f \in \mathcal{F}])(\exists x \in X)\ x \overset{\mathcal{U}}{\sim} f(x)$.

Recall that X has the <u>fixed point property with respect to</u> $\mathcal{F}$ (in notation, $X \in \mathrm{fpp}(\mathcal{F})$) if for every map $f:X \longrightarrow X$ in $\mathcal{F}$ there is a point $x \in X$ with $x = f(x)$. The $\mathrm{fpp}(\mathrm{Mor}\mathcal{T}op)$ is the familiar fixed point property. Clearly, $X \in \mathrm{fpp}(\mathcal{F})$ implies $X \in \mathrm{afpp}(\mathcal{F})$.

(6.9) PROPOSITION. (Bestvina). Let X be a paracompact T_2 space. Then $X \in \mathrm{afpp}(\mathcal{F})$ implies $X \in \mathrm{fpp}(\mathcal{F})$.

PROOF. Suppose $X \notin \mathrm{fpp}(\mathcal{F})$. Then there is a map $f:X \longrightarrow X$ in $\mathcal{F}$ such that $x \neq f(x)$ for every $x \in X$. Since X is T_2, for every $x \in X$ there is an open neighborhood U_x of x and an open neighborhood V_x of f(x) with $U_x \cap V_x = \emptyset$. Since f is continuous, for every $x \in X$, there is an open neighborhood W_x of x such that $W_x \subset U_x$ and $f(W_x) \subset V_x$. Since X is paracompact, the open cover $\mathcal{U} = \{W_x\}_{x \in X}$ is a normal cover of X [2]. Clearly, for every $U \in \mathcal{U}$ and every $x \in X$, $x \in U$ implies $f(x) \notin U$. Hence, $X \notin \mathrm{afpp}(\mathcal{F})$.

(6.10) THEOREM. If $\mathcal{F}$ is an ideal in $\mathcal{T}op$, then the property $\mathrm{afpp}(\mathcal{F})$ is $\preccurlyeq$-invariant.

PROOF. Let $\mathcal{D}$ be a class of spaces having the property $\mathrm{afpp}(\mathcal{F})$.

Suppose $X \preccurlyeq \mathcal{D}$. We must show that $X \in \mathrm{afpp}(\mathcal{F})$.

Let an $f:X \longrightarrow X$ in $\mathcal{F}$ and a $\mathcal{U} \in \widetilde{X}$ be given. Pick a $\mathcal{V} \in \widetilde{X}$ with $2\mathcal{V} < \mathcal{U}$. Since $X \preccurlyeq \mathcal{D}$, there is a $Y \in \mathcal{D}$ and maps $u:X \longrightarrow Y$ and $d:Y \longrightarrow X$ such that $d \circ u < \mathcal{V}$. Let $g = u \circ f \circ d:Y \longrightarrow Y$. Observe that $g \in \mathcal{F}$. Since $Y \in \mathrm{afpp}(\mathcal{F})$, there is a $y \in Y$ such that $y \overset{\mathcal{W}}{\sim} g(y)$. Let $x = d(y)$. It is easy to check that $x \overset{\mathcal{U}}{\sim} f(x)$. Hence, $X \in \mathrm{afpp}(\mathcal{F})$.

The following definitions provide an extension from metric spaces to arbitrary topological spaces of proximate notions in [18].

Let $f:X \longrightarrow Y$ be a function between topological spaces and let $\mathcal{V}$ be an open cover of Y. The function f is $\mathcal{V}$-<u>continuous</u> <u>at</u> <u>a</u> <u>point</u> $x \in X$ provided there is a neighborhood U of x in X such that $f(U) \subset 2f(x)$. If f is $\mathcal{V}$-continuous at every point $x \in X$, then we say that f is $\mathcal{V}$-<u>continuous</u>.

Another, more restrictive, notion relies on normal covers of X. The function f is $\mathcal{V}$-<u>$\tilde{\text{c}}$ontinuous</u> provided there is a $\mathcal{U} \in \widetilde{X}$ such that $f(2x) \subset 2f(x)$ for every $x \in X$.

These two different concepts of proximate continuity lead to two different versions of the proximate fixed point property [18].

Let $\mathcal{F}$ be a collection of morphisms in the category *Sets* of sets and functions. Write $X \in \widetilde{\mathrm{p}}\mathrm{fpp}(\mathcal{F})$ provided $(\forall \mathcal{U})(\exists \mathcal{V})$ $\underline{\widetilde{\mathrm{p}}\mathrm{fpp}_{\mathcal{F}}(\mathcal{U}, \mathcal{V})}$: $(\forall f:X \longrightarrow X\ [f \in \mathcal{F},\ \mathcal{V}\text{-}\tilde{\text{c}}\text{ontinuous}])(\exists x \in X)\ x \overset{\mathcal{U}}{\sim} f(x)$. If an entirely analogous condition for $\mathcal{V}$-continuous functions in $\mathcal{F}$ holds, then we write $X \in \mathrm{pfpp}(\mathcal{F})$. Clearly, $X \in \mathrm{pfpp}(\mathcal{F})$ implies $X \in \widetilde{\mathrm{p}}\mathrm{fpp}(\mathcal{F})$. The converse is true for paracompact spaces. A compactum X has the proximate fixed point property [18] iff $X \in \mathrm{pfpp}(\mathrm{Mor}\,\mathcal{S}ets)$. Also, since a continuous function $f:X \longrightarrow X$ is $\mathcal{V}$-continuous for every open cover $\mathcal{V}$ of X, if $\mathcal{F}$ is a collection of morphisms in *Top*, then $X \in \mathrm{pfpp}(\mathcal{F})$ implies $X \in \mathrm{afpp}(\mathcal{F})$.

(6.11) THEOREM. If $\mathcal{F}$ is an ideal in *Sets*, then the properties $\mathrm{pfpp}(\mathcal{F})$ and $\widetilde{\mathrm{p}}\mathrm{fpp}(\mathcal{F})$ are $\preccurlyeq$-invariant.

PROOF. Let $\mathcal{D}$ be a class of spaces with the property $\mathrm{pfpp}(\mathcal{F})$. Suppose $X \leqslant \mathcal{D}$. We must show that $X \in \mathrm{pfpp}(\mathcal{F})$.

Let $\mathcal{U} \in \widetilde{X}$. Choose a $\mathcal{V} \in \widetilde{X}$ with $2\mathcal{V} < \mathcal{U}$. Since $X \leqslant \mathcal{D}$, there is a $Y \in \mathcal{D}$ and maps $u:X \longrightarrow Y$ and $d:Y \longrightarrow X$ such that $d \circ u < \mathcal{V}$. Let $\mathcal{W} = \mathcal{V}_d \in \widetilde{Y}$. Since $Y \in \mathrm{pfpp}(\mathcal{F})$, there is a $\mathcal{Z} \in \widetilde{Y}$ such that $\mathrm{pfpp}_{\mathcal{F}}(\mathcal{W}, \mathcal{Z})$ holds.Let $\mathcal{S} = \mathcal{Z}_u \in \widetilde{X}$. We claim that $\mathrm{pfpp}_{\mathcal{F}}(\mathcal{U}, \mathcal{S})$ is true.

Indeed, let $f:X \longrightarrow X$ be an $\mathcal{S}$-continuous function in $\mathcal{F}$. Put $g = u \circ f \circ d:Y \longrightarrow Y$. Observe that $g \in \mathcal{F}$. We shall first prove that g is $\mathcal{Z}$-continuous.

Let $z \in Y$. It suffices to show that g is $\mathcal{Z}$-continuous at the point z. Since f is $\mathcal{S}$-continuous, there is a neighborhood N of d(z) in X such that $f(N) \subset 2_{\mathcal{S}} f(d(z))$. In other words, $u \circ f(N) \subset 2_{\mathcal{Z}} g(z)$. Since d is continuous, there is a neighborhood M of z in Y with $d(M) \subset N$. Hence, $g(M) = u \circ f \circ d(M) \subset u \circ f(N) \subset 2_{\mathcal{Z}} g(z)$.

It follows now from the $\mathrm{pfpp}_{\mathcal{F}}(\mathcal{W}, \mathcal{Z})$ that there is a point $y \in Y$ and a $V \in \mathcal{V}$ such that $y, g(y) \in d^{-1}(V)$. Let $x = d(y) \in X$. Pick a $U \in \mathcal{U}$ with $2V \subset U$. Clearly, $x, d \circ g(y) \in V$. Since $d \circ g(y) = d \circ u(f(x))$ and $d \circ u < \mathcal{V}$, there is a $V_1 \in \mathcal{V}$ such that $f(x), d \circ u(f(x)) \in V_1$. Hence, $x, f(x) \in U$.

The above proof applies also to the property $\widetilde{\mathrm{p}}\mathrm{fpp}(\mathcal{F})$ provided we show that g is $\mathcal{Z}$-$\widetilde{\mathrm{c}}$ontinuous when f is $\mathcal{S}$-$\widetilde{\mathrm{c}}$ontinuous.

In order to prove this, first use the fact that f is $\mathcal{S}$-$\widetilde{\mathrm{c}}$ontinuous to get an $\mathcal{N} \in \widetilde{X}$ such that $f(2_{\mathcal{N}} d(z)) \subset 2_{\mathcal{S}} f(d(z)) = u^{-1}(2_{\mathcal{Z}} u \circ f \circ d(z)) = u^{-1}(2_{\mathcal{Z}} g(z))$ and, therefore, $u \circ f(2_{\mathcal{N}} d(z)) \subset 2_{\mathcal{Z}} g(z)$ for every $z \in Y$. Put $\mathcal{M} = \mathcal{N}_d \in \widetilde{Y}$. Since $2_{\mathcal{M}} z = d^{-1}(2_{\mathcal{N}} d(z))$, we get $g(2_{\mathcal{M}} z) = u \circ f \circ d(2_{\mathcal{M}} z) = u \circ f(2_{\mathcal{N}} d(z)) \subset 2_{\mathcal{Z}} g(z)$, for every $z \in Y$.

Let $\mathcal{F}$ be a collection of morphisms in $\mathcal{Top}$. Write $X \in \overline{\mathrm{s}}\mathrm{fpp}(\mathcal{F})$ provided $(\forall \mathcal{U})(\exists \mathcal{V})$ $\underline{\overline{\mathrm{Fix}}_{\mathcal{F}}(\mathcal{U}, \mathcal{V})}$: $(\forall f:X \longrightarrow \mathcal{V}\ [f \in \mathcal{F}])(\exists x \in X)\ x \overset{\mathcal{U}}{\sim} f(x)$. Recall [3] that we write $X \in \mathrm{sfpp}(\mathcal{F})$ provided $(\forall \mathcal{U})(\exists \mathcal{V})$ $\underline{\mathrm{Fix}_{\mathcal{F}}(\mathcal{U}, \mathcal{V})}$: $(\forall S,\ f:S \longrightarrow \mathcal{V}\ [f \in \mathcal{F}])(\exists s \in S)\ s \overset{\mathcal{U}}{\sim} f(s)$.

(6.12) PROPOSITION. If $\mathcal{F}$ is a right ideal in $\mathcal{Top}$, then $X \in \overline{\mathrm{s}}\mathrm{fpp}(\mathcal{F})$ implies $X \in \mathrm{sfpp}(\mathcal{F})$.

Let $\mathcal{U} \in \tilde{X}$. Pick $\mathcal{V}$, $\mathcal{S} \in \tilde{X}$ such that $2\mathcal{V} < \mathcal{U}$ and $\overline{\mathrm{Fix}}_{\mathcal{F}}(\mathcal{V}, \mathcal{S})$ holds. We claim that $\overline{\mathrm{Fix}}_{\mathcal{F}}(\mathcal{U}, \mathcal{S})$ is true.

Indeed, let an $\mathcal{M} \in \tilde{X}$ and an $f:\mathcal{M} \longrightarrow \mathcal{S}$ in $\mathcal{F}$ be given. Let $\tilde{p}:X \longrightarrow \mathcal{M}$. The composition $g = f \circ p:X \longrightarrow \mathcal{S}$ is in $\mathcal{F}$. Hence, there is an $x \in X$ with $x \overset{\mathcal{V}}{\sim} g(x)$. Pick an $M \in \mathcal{M}$ such that $x \in M$. One can easily check that $M \overset{\mathcal{U}}{\sim} f(M)$.

(6.13) PROPOSITION. If $\mathcal{F}$ is a left ideal in $\mathcal{T}op$, then $X \in \overline{\mathrm{s}}\mathrm{fpp}(\mathcal{F})$ implies $X \in \mathrm{afpp}(\mathcal{F})$.

PROOF. Let an $f:X \longrightarrow X$ in $\mathcal{F}$ and a $\mathcal{U} \in \tilde{X}$ be given. Since $X \in \overline{\mathrm{s}}\mathrm{fpp}(\mathcal{F})$, there is a $\mathcal{V} \in \tilde{X}$ such that $\overline{\mathrm{Fix}}_{\mathcal{F}}(\mathcal{U}, \mathcal{V})$ holds. Let $\tilde{p}:X \longrightarrow \mathcal{V}$. Put $g = p \circ f:X \longrightarrow \mathcal{V}$. Since $g \in \mathcal{F}$, by $\overline{\mathrm{Fix}}_{\mathcal{F}}(\mathcal{U}, \mathcal{V})$, there is an $x \in X$ with $x \overset{\mathcal{U}}{\sim} g(x)$. Clearly, $x \overset{\mathcal{U}}{\sim} f(x)$.

(6.14) THEOREM. If $\mathcal{F}$ is an ideal in $\mathcal{T}op$, then the property $\overline{\mathrm{s}}\mathrm{fpp}(\mathcal{F})$ is $\preccurlyeq$-invariant.

PROOF. Let $\mathcal{D}$ be a class of spaces with the property $\overline{\mathrm{s}}\mathrm{fpp}(\mathcal{F})$. Suppose $X \preccurlyeq \mathcal{D}$. We must show that $X \in \overline{\mathrm{s}}\mathrm{fpp}(\mathcal{F})$.

Let $\mathcal{U} \in \tilde{X}$. Pick a $\mathcal{V} \in \tilde{X}$ with $2\mathcal{V} < \mathcal{U}$. Since $X \preccurlyeq \mathcal{D}$, there is a $Y \in \mathcal{D}$ and maps $u:X \longrightarrow Y$ and $d:Y \longrightarrow X$ such that $d \circ u < \mathcal{V}$. Let $\mathcal{W} = \mathcal{V}_d \in \tilde{Y}$. Since $Y \in \overline{\mathrm{s}}\mathrm{fpp}(\mathcal{F})$, there is a $\mathcal{Z} \in \tilde{Y}$ such that $\overline{\mathrm{Fix}}_{\mathcal{F}}(\mathcal{W}, \mathcal{Z})$ holds. Let $\mathcal{S} = \mathcal{Z}_u \in \tilde{X}$. Let $\mathcal{M} \in \tilde{X}$ refines both $\mathcal{V}$ and $\mathcal{S}$. We claim that $\overline{\mathrm{Fix}}_{\mathcal{F}}(\mathcal{U}, \mathcal{M})$ is true.

Indeed, let an $f:X \longrightarrow \mathcal{M}$ in $\mathcal{F}$ be given. Let $\tilde{p}:\mathcal{M} \longrightarrow \mathcal{S}$. Put $g = u_{\mathcal{Z}} \circ p \circ f \circ d:Y \longrightarrow \mathcal{Z}$. Observe that $g \in \mathcal{F}$. By $\overline{\mathrm{Fix}}_{\mathcal{F}}(\mathcal{W}, \mathcal{Z})$, there is a $y \in Y$ with $y \overset{\mathcal{W}}{\sim} g(y)$. Let $x = d(y) \in X$. It is easy to check that $x \overset{\mathcal{U}}{\sim} f(x)$.

We shall now identify several $\preccurlyeq$-invariant properties from the general topology.

For a class of topological spaces $\mathcal{D}$, let $i\mathcal{D}$ denote a class of all spaces Y such that there is an $X \in \mathcal{D}$ and a map $f:X \longrightarrow Y$ of X onto Y.

(6.15) LEMMA. If $X \preccurlyeq \mathcal{D}$, then there is a net $\{A_i\}_{i \in I}$ in $(\tilde{a}X) \cap (i\mathcal{D})$ such that $A_i \xrightarrow[m]{} X$.

PROOF. Let $I = \tilde{X}$. For an $i = \mathcal{U} \in I$, choose a $Y \in \mathcal{D}$ and maps $u:X$

$\longrightarrow Y$ and $d:Y \longrightarrow X$ such that $d\circ u < \mathcal{U}$. Put $A_i = d(Y) \in (\tilde{a}X) \cap (i\mathcal{D})$. Let $f = d\circ u : X \longrightarrow A_i$ and let g denote the inclusion of A_i into X. Clearly, $f < \mathcal{U}$ and $g < \mathcal{U}$. Hence, $A_i \overset{\mathcal{U}}{\underset{m}{\sim}} X$ and $A_i \underset{m}{\longrightarrow} X$.

A topological property Π is m-<u>stable</u> provided $X \in \Pi$ and $Y \in i\{X\}$ imply $Y \in \Pi$.

(6.16) THEOREM. Every property Π which is both m-invariant and -stable is also $\preccurlyeq$-invariant.

PROOF. Let $\mathcal{D}$ be a class of spaces with the property Π. Suppose $X \preccurlyeq \mathcal{D}$. We must show that $X \in \Pi$.

By the above lemma, there is a net $\{A_i\}_{i \in I}$ in $(\tilde{a}X) \cap (i\mathcal{D})$ such that $A_i \underset{m}{\longrightarrow} X$. Since Π is m-stable, $A_i \in \Pi$ for every $i \in I$. Finally, since Π is also m-invariant, we get $X \in \Pi$.

(6.17) COROLLARY. The following properties are $\preccurlyeq$-invariant: compactness, almost compactness [1], countable compactness, Lindelöf, and "to have $\leqq$ n components".

PROOF. The listed properties are clearly m-stable and by the results in [11] and [12] they are also m-invariant.

Recall that a space X is <u>pseudocompact</u> provided every real-valued (continuous) map $f:X \longrightarrow R$ is bounded.

(6.18) THEOREM. Pseudocompactness is a $\preccurlyeq$-invariant property.

PROOF. Let $\mathcal{D}$ be a class of pseudocompact spaces. Suppose $X \preccurlyeq \mathcal{D}$. We must show that X is pseudocompact.

Let $f:X \longrightarrow R$ be a continuous map. Let $\mathcal{V}$ denote a cover of the real line R with open segments $(n, n+2)$ $(n \in Z)$. Let $\mathcal{U} = f^{-1}(\mathcal{V}) = \mathcal{V}_f \in \tilde{X}$. Since $X \preccurlyeq \mathcal{D}$, there is a $Y \in \mathcal{D}$ and maps $u:X \longrightarrow Y$ and $d:Y \longrightarrow X$ such that $d\circ u < \mathcal{U}$. Put $g = f\circ d : Y \longrightarrow R$. Since Y is pseudocompact, the function g is bounded, i. e., there is a $k \in N$ such that $g(Y) \subset [-k, k]$. Clearly, $g\circ u(X) = f\circ d\circ u(X) \subset [-k, k]$. But, $f\circ d\circ u$ and f are $\mathcal{V}$-close, so that $f(X) \subset [-k-2, k+2]$. In other words, f is bounded. Hence, X is pseudocompact.

Recall that a space X is called <u>strongly paracompact</u> [16, p. 404] if X is a Hausdorff space and every open cover of X has a star-finite open refinement.

(6.19) THEOREM. Let $\mathcal{D}$ be a class of strongly paracompact spaces. Let X be a paracompact Hausdorff space. If $X \leqslant \mathcal{D}$, then X is also strongly paracompact.

PROOF. Let $\mathcal{U}$ be an open cover of X. Since X is paracompact, $\mathcal{U} \in \widetilde{X}$. Pick a $\mathcal{V} \in \widetilde{X}$ with $2\mathcal{V} < \mathcal{U}$. Since $X \leqslant \mathcal{D}$, there is a $Y \in \mathcal{D}$ and maps $u: X \longrightarrow Y$ and $d: Y \longrightarrow X$ such that $d \circ u < \mathcal{V}$. Let $\mathcal{W} = \mathcal{V}_d \in \widetilde{Y}$. Since Y is strongly paracompact, there is a star-finite open refinement $\mathcal{Z}$ of $\mathcal{W}$. Put $\mathcal{S} = \mathcal{Z}_u$. It is easy to check that $\mathcal{S}$ is a star-finite refinement of $\mathcal{U}$.

Let τ be a cardinal number. Recall that a space X is τ-<u>bounded</u> provided a closure of every subset of X of cardinality $\leq \tau$ is a compact space.

(6.20) THEOREM. Let $\mathcal{D}$ be a class of τ-bounded spaces. Let X be a paracompact Hausdorff space. If $X \leqslant \mathcal{D}$, then X is also τ-bounded.

PROOF. Let A be a subset of X with $|A| \leq \tau$. For each $\mathcal{U} \in \widetilde{X}$ construct a compact set $A_{\mathcal{U}}$ in X as follows.

Pick a $\mathcal{V} \in \widetilde{X}$ with $2\mathcal{V} < \mathcal{U}$. Since $X \leqslant \mathcal{D}$, there is a $Y \in \mathcal{D}$ and maps $u: X \longrightarrow Y$ and $d: Y \longrightarrow X$ such that $d \circ u < \mathcal{V}$. Observe that $|u(A)| \leq \tau$. Since Y is τ-bounded, $\overline{u(A)}$ is a compact subset of Y. Put $A_{\mathcal{U}} = d(\overline{u(A)})$. It is easy to check that $A_{\mathcal{U}} \overset{\mathcal{U}}{\sim} \overline{A}$. Hence, the net $\{A_{\mathcal{U}}\}$ p-converges [11] to $\overline{A}$. Since X is paracompact Hausdorff space, $\overline{A}$ is P-embedded in X. It follows from [11], that $\overline{A}$ is compact. Hence, X is τ-bounded.

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