## Davide Carlo Demaria; Garbaccio Rosanna Bogin On shape groups and Cech homology groups of a compact space

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ON SHAPE GROUPS AND ČECH HOMOLOGY GROUPS OF A COMPACT SPACE

Davide Carlo Demaria - Rosanna Garbaccio Bogin

Given a pretopological space S=(X,P), we associate to any interior covering X of S a symmetrical pf-space  $S_{\chi}$  on the set X (see [2], [3]). Precisely, to obtain the pretopology of  $S_{\chi}$ , we take for each point x of X the principal filter St(x,X).

Then we associate to S the inverse system  $\hat{S}=(S_X, P_{XX'}, Cov(S))$ , where  $P_{XX'}:S_X, \rightarrow S_X$  is the identity in X and Cov(S) is the collection of all interior coverings of S.

For each dimension n, we associate to  $\hat{S}$  an inverse system of prehomotopy groups  $\Pi_n(S_\chi)$ , a) and an inverse system of singular homology groups  $H_n(S_\chi)$ . Taking the inverse limits  $\lim_n \Pi_n(S_\chi)$ , a) and  $\lim_n H_n(S_\chi)$ , we obtain the shape groups  $\check{\Pi}_n(S, a)$  and the Čech homology groups  $\check{H}_n(S)$  of the pretopological space S. In this way, if S is a topological space, instead to approximate it by means

In this way, if S is a topological space, instead to approximate it by means of polyhedra, we reduce the more the set of admissible functions into S, in such a way to obtain the set of continuous maps.

Here we prove that our shape groups and Čech homology groups of a connected compact topological space S are isomorphic to the classical ones.<sup>(1)</sup>

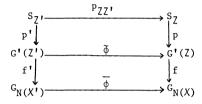
In [2] we proved that, if the covering  $X=\{X_i\}(i\in J)$  is finite, then  $S_X$  belongs to the same homotopy type of a finite symmetrical pf-space (i.e. an undirected graph) G'(X), that we obtain in the following way. The vertices  $v_{i_1...i_n}$  of G'(X) correspond to the maximal subsets  $\{i_1,...,i_n\}$  of J such that  $\bigcap_{r=1}^n X_i \neq \emptyset$ , and there is the edge  $v_{i_1...i_n}v_{j_1...j_m}$  iff  $\{i_1,...,i_n\} \cap \{j_1,...,j_m\}\neq \emptyset$ .

Here (§2, §3) we consider a suitable collection Cov'(S) of open coverings of S which is cofinal in Cov(S), and for any X€Cov'(S) we construct an open covering Z such that the nerve N(X) of X is isomorphic to the complex  $K_{G'(Z)}$  of the graph G'(Z). This is possible if the covering X is independent and non singular. In fact, if X is independent, we obtain Z such that the graph  $G_{N(X)}$  of the edges of N(X) is

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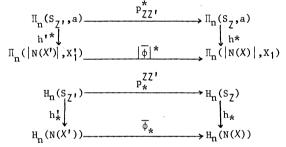
(.<sup>1</sup>) Any compact topological space is supposed to be Hausdorff. Moreover we consider only infinite spaces, since any finite connected compact space is a singleton. isomorphic to G'(Z). Moreover, if X is also non singular, the complex N(X) is complete and therefore isomorphic to the complex  $K_{G_N(X)}$ .

Afterwards, given  $X = \{X_i\} (i \in I)$  and  $X' = \{X_h\} (h \in H)$  in Cov'(S) such that  $X \leq X'$  and  $Z \leq Z'$  and a suitable function  $\phi : H \rightarrow J$  such that  $X_h \leq X_{\phi(h)}$  for each  $h \in H$ , we show that the following diagram over pretopological spaces:



where  $\overline{\phi}$  and  $\overline{\phi}$  are precontinuous maps induced by  $\phi$ , is such that  $\phi f'=f\widetilde{\phi}$  and  $\widetilde{\phi}p'\sim pp_{77'}$ .

Hence (§4) we obtain the following commutative diagrams:



where h'\*, h\*, h\*, h, are isomorphisms.

Since also the collection Cov''(S) of the coverings Z is cofinal in Cov(S), we obtain:

 $\begin{array}{ll} \underbrace{\lim}_{n} & \Pi_{n}(S_{Z}, a) & \underline{\sim} & \underbrace{\lim}_{n} & \Pi_{n}(|N(X)|, X_{1});\\ \underbrace{\lim}_{n} & H_{n}(S_{Z}) & \underline{\sim} & \underbrace{\lim}_{n} & H_{n}(N(X)). \end{array}$ Finally we give some examples.

1. On some finite open coverings of S.

Let  $X = \{X_1, \ldots, X_p\}$  be a covering of a nonempty set S. For any positive integer  $n \leq p$  and any n-tuple  $(i_1, \ldots, i_n)$  such that  $1 \leq i_1 < i_2 < \ldots < i_n \leq p$ , we put:

 $\begin{aligned} \mathbf{x}_{i_1 \dots i_n} &= \mathbf{x}_{i_1} \cap \dots \cap \mathbf{x}_{i_n}; \\ \mathbf{x}_{i_1 \dots \hat{\mathbf{1}}_r \dots i_n} &= \mathbf{x}_{i_1} \cap \dots \cap \mathbf{x}_{i_{r-1}} \cap \mathbf{x}_{i_{r+1}} \cap \dots \cap \mathbf{x}_{i_n}. \end{aligned}$ 

1.1 Definition The covering X is independent if:

 $\begin{array}{ccc} x_{i_1,\ldots i_n} \neq \emptyset \implies x_{i_1\ldots i_n} \not\subseteq & U\{x_j/j\notin \{i_1,\ldots,i_n\}\} & \text{for any n-tuple } (i_1,\ldots,i_n) \\ \text{with } 1 \leq n < p. \end{array}$ 

1.2 Definition Let N be an integer such that  $3\leq n \leq p$ .  $\{X_{i_1}, \dots, X_{i_n}\}$  is a singularity of X with degree n and indices  $i_1, \dots, i_n$ , if the following conditions

hold:

 $\begin{array}{l} x_{i_{1} \dots i_{n}} = \emptyset; \\ x_{i_{1} \dots \widehat{i_{r}} \dots i_{n}} \neq \emptyset \quad \text{for } r=1,2,\dots,n. \end{array}$ 

Then X is non singular, if there are no singularities of X.

1.3 Proposition Let S be a connected compact topological space. Any open covering  $X = \{x_1, \ldots, x_p\}$  of S has an independent open refinement  $Y = \{y_1, \ldots, y_p\}$ .

*Proof:* First we construct a finite set X of distinct points of S, taking a point  $x_{i_1...i_n}$  in each  $X_{i_1...i_n} \neq \emptyset$  for n=1,...,p. (This is possible since any nonempty open subset of S is infinite). Then we put:

 $Y_i = X_i - X(i) \text{ where } X(i) = \{x_{i_1,\ldots,i_n} \in X/i \notin \{i_1,\ldots,i_n\}\}.$ 

1.4 Remark.  $x_{i_1...i_n} \in Y_{j_1...j_m}$  iff  $\{j_1,...,j_m\} \subseteq \{i_1,...,i_n\}$ ; so Y is minimal. Moreover  $Y_{i_1...i_n} \neq \emptyset$  iff  $X_{i_1...i_n} \neq \emptyset$ . The point  $x_i$  will be called characteristical point of  $Y_i$ , since  $Y_i$  is the only element of Y containing  $x_i$ .

1.5 Proposition Let S be a connected compact topological space. Any independent open covering  $X = \{x_1, \ldots, x_p\}$  of S has an independent shrinking  $Y = \{Y_1, \ldots, Y_p\}$  such that  $Y_{i_1, \ldots, i_n} \neq \emptyset$  iff  $X_{i_1, \ldots, i_n} \neq \emptyset$  for any n-tuple of indices.

*Proof:* Construct a finite set X of distinct points of S, taking a point  $x_{i_1...i_n}$ in  $X_{i_1...i_n} - \bigcup \{x_j/j \notin \{i_1,...,i_n\}\}$  whenever  $X_{i_1...i_n} \neq \emptyset$ , for n=1,...,p. Then consider the closed subset:

 $Y^{(i)} = X(i) \cup (S - \bigcup_{j \neq i} X_j)$  where  $X(i) = \{x_{i_1, \dots, i_n} \in X/i \in \{i_1, \dots, i_n\}\}$ . Finally take an open subset  $Y_i$  of S such that:

 $Y^{(i)} \subseteq Y_i \subseteq \overline{Y_i} \subseteq X_i$ .

1.6 Remark.  $x_{i_1...i_n} \notin \overline{Y_j}$  for  $j \notin \{i_1,...,i_n\}$ , since  $\overline{Y_j} \subseteq X_j$ .

1.7 Lemma Let S be a connected compact topological space and  $X=\{x_1,\ldots,x_p\}$  an independent open covering of S. If  $\{x_1, x_2, \ldots, x_n\}$  is the singularity of X relative to  $(1,2,\ldots,n)$ , we can construct an independent open refinement  $X' = \{U', V', X_2', \ldots, X_n'\}$  of X such that:

- (i)  $U' \subseteq X_1$ ,  $V' \subseteq X_1$ ,  $X'_1 \subseteq X_1$  for i=2,...,p;
- (ii) {U', V', X'<sub>2</sub>,..., X'<sub>n</sub>}, {U', X'<sub>2</sub>,..., X'<sub>n</sub>} and {V', X'<sub>2</sub>,..., X'<sub>n</sub>} are not singularities;
- (iii) given m indices i<sub>1</sub>,i<sub>2</sub>,...,i<sub>m</sub> such that 1<i<sub>1</sub><i<sub>2</sub><...<i<sub>m</sub>r</sub>>n for some r, we have:
  - a) {U', V',  $X_{i_1}^{!}, \ldots, X_{i_m}^{!}$ } is not a singularity;
  - b) if  $\{U', X_{i_1}^!, \dots, X_{i_m}^!\}$  or  $\{V', X_{i_1}^!, \dots, X_{i_m}^!\}$  is a singularity of X', then  $\{X_1, X_{i_1}, \dots, X_{i_m}\}$  is a singularity of X.

*Proof:* Construct a shrinking  $V = \{Y_1, \dots, Y_p\}$  of X with the process from Proposition 1.5, and put:

$$\begin{split} & Y_{1} = Y_{1} \cap \dots \cap Y_{i-1} \cap Y_{i+1} \cap \dots \cap Y_{n} \quad \text{for } i=2,3,\dots,n; \\ & U = Y_{1} \cap (S - \bigcup_{i>2} \overline{Y_{1}}); \\ & V = Y_{1} \cap (S - \overline{Y_{2}}); \\ & Y' = \{U, V, Y_{2},\dots, Y_{n}\}. \end{split}$$

Clearly V' is an open covering of S and {U, V,  $Y_2, \ldots, Y_n$ }, {U,  $Y_2, \ldots, Y_n$ },  $\{V, Y_2, \ldots, Y_n\}$  are not singularities of Y'.

Now consider m indices  $i_1, i_2, \dots, i_m$  such that  $1 < i_1 < i_2 < \dots < i_m \le p$  and  $i_r > n$  for some r, and distinguish two cases.

I) If  $Y_1 \cap Y_{i_1 \dots i_m} = \emptyset$ , then {U, V,  $Y_{i_1}, \dots, Y_{i_m}$ } is not a singularity of Y'. Moreover, if {U,  $Y_{i_1}, \ldots, Y_{i_m}$ } or {V,  $Y_{i_1}, \ldots, Y_{i_m}$ } is a singularity of Y', then  $\{Y_1, Y_{i_1}, \ldots, Y_{i_m}\}$  is a singularity of Y, and hence  $\{X_1, X_{i_1}, \ldots, X_{i_m}\}$  is a singularity of X.

II) If  $Y_1 \cap Y_{i_1 \cdots i_m} \neq \emptyset$ , put I={2,3,...,n} and distinguish three possibilities 1)  $I - \{i_1, \dots, i_m\} = \{2\}.$ 

Since  $Y_1 \cap Y_{i_1 \dots i_m} \subseteq Y_2 \subseteq U \subseteq Y_1$ , we obtain  $Y_1 \cap Y_{i_1 \dots i_m} = U \cap Y_{i_1 \dots i_m} \neq \emptyset$ ; therefore {U,  $Y_{i_1}, \ldots, Y_{i_m}$ }" is not a singularity of Y'. Moreover  $V \cap Y_{i_1...i_m} \subseteq Y_2 \subseteq S - V$ ; hence both  $\{V, Y_{i_1}, ..., Y_{i_m}\}$  and  $\{U, V, Y_{i_1}, ..., Y_{i_m}\}$ 

2)  $I - \{i_1, \dots, i_m\} = \{j\}$  with j > 2.

are not singularities of Y'.

Both {U,  $Y_{i_1}, \ldots, Y_{i_m}$ } and {U, V,  $Y_{i_1}, \ldots, Y_{i_m}$ } are not singularities of Y', since  $U \cap Y_{i_1 \cdots i_m} \subseteq Y_{\hat{j}} \subseteq S - U.$ 

Moreover { $\{V, Y_{i_1}, \ldots, Y_{i_m}\}$  is not a singularity of Y', because  $V \cap Y_{i_1 \cdots i_m} =$ =  $Y_1 \cap Y_{i_1...i_m} \neq \emptyset$ . 3)  $I - \{i_1,...,i_m\} \supseteq \{h,k\}$  with h<k.

The point  $z=x_{1i_1...i_m}$ , we fixed to construct the shrinking y of X, is such that  $z \notin \overline{Y_h} \cup \overline{Y_k}$ . So  $z \notin \overline{Y_1}$  for i=2,3,...,n; hence  $z \in U \cap V \cap Y_{i_1,...i_m}$ . Therefore {U,  $Y_{i_1}, \ldots, Y_{i_m}$ }, {V,  $Y_{i_1}, \ldots, Y_{i_m}$ }, {U, V,  $Y_{i_1}, \ldots, Y_{i_m}$ } are not singularities of Y'.

Finally, construct an independent open refinement X'={U', V', X',..., X', of Y' applying Proposition 1.3.

1.8 Remark. To construct X' we replace the element  $X_1$  of X with two subsets U' and V' of X1, that we can associate again to the index 1. Instead each element of X with index greater than 1 is replaced with one subset with the same index. From each singularity of X containing  $X_1$  and different from  $\{X_1, X_2, \ldots, X_n\}$  we obtain at least one singularity of X' with the same indices, where  $X_1$  is replaced by one of the sets U', V'. So, if X has q singularities of index 1, then X' has at least q-1 and at most 2(q-1) singularities containing either U' or V', that we call again of index 1. Instead each singularity of X non containing X1 determines a singularity of X' with the same indices.

1.9 Proposition Let S be a connected compact topological space and  $X = \{X_1, \ldots, X_p\}$ an independent open covering with q singularities containing  $X_1$ . We can construct an independent open refinement  $\tilde{X} = \{\tilde{U}_{1,1}, \ldots, \tilde{U}_{1,h}, \tilde{X}_2, \ldots, \tilde{X}_p\}$  of X which has no singularities containing some U1.r.

*Proof:* Let  $s_1 = \{x_1, x_{i_2,1}, \dots, x_{i_{m_1},1}\}, s_2 = \{x_1, x_{i_2,2}, \dots, x_{i_{m_2},2}\}, \dots, s_{m_{m_2},m_{m_$  $s_q = \{x_1, x_{i_2,q}, \dots, x_{i_{m_q},q}\}$  be the singularities of  $\tilde{X}$  with index 1. Applying

 $x^{(1)} =$ Lemma 1.7, we eliminate s<sub>1</sub> and we obtain an independent open covering = { $U_1^{(1)}$ ,  $U_2^{(1)}$ ,  $X_2^{(1)}$ ,...,  $X_p^{(1)}$ }, which has at most 2(q-1) singularities of index 1, i.e. containing one of the subsets  $U_1^{(1)}$ ,  $U_2^{(1)}$  and generated from  $s_2, \ldots, s_q$ .

For the singularities generated from s2 we have two possibilities:

(i) only one of the collections  $\{U_1^{(1)}, X_{12,2}^{(1)}, \dots, X_{im_2,2}^{(1)}\}$  $\{U_2^{(1)}, X_{i_2,2}^{(1)}, \dots, X_{i_{m_2}}^{(1)}, 2\}$  is a singularity of  $X^{(1)}$ ; (ii) both of them are singularities of  $X^{(1)}$ .

Applying Lemma 1.7 once in case (i) and twice in case (ii), we obtain an

independent open covering  $X^{(2)}$  of form: { $U_1^{(2)}$ ,  $U_2^{(2)}$ ,  $U_3^{(2)}$ ,  $X_2^{(2)}$ ,...,  $X_p^{(2)}$ } in case (i);  $\{U_1^{(2)}, U_2^{(2)}, U_3^{(2)}, U_4^{(2)}, x_2^{(2)}, \dots, x_n^{(2)}\}$  in case (ii).

The covering  $X^{(2)}$  has at most 4(q-2) singularities with index 1, i.e. containing one of the sets  $U_r^{(2)}$  and generated from  $s_3, \ldots, s_q$ . The other singularities of  $X^{(2)}$  have the same indices of those of X.

Afterwards we eliminate successively the singularities generated from s3, from  $s_4,\ldots,$  from  $s_q$  applying an analogous process. So we obtain the independent open covering  $\tilde{X}$  we were looking for.

1.10 Theorem Let S be a connected compact topological space. Any finite open covering has a finite open refinement which is independent and non singular. *Proof:* Given an open covering  $A = \{A_1, \ldots, A_p\}$  of S, we take an independent open refinement  $X = \{x_1, \dots, x_p\}$  of A.

We denote by  $S_1, S_2, \ldots, S_{n-2}$  the sets of the singularities of X whose lowest index is 1,2,...,p-2 respectively.

 $\tilde{\chi}(1)$  = If  $S_1 \neq \emptyset$ , applying Proposition 1.9, we obtain a refinement

= { $\tilde{u}_{1,1}^{(1)}$ ,...,  $\tilde{u}_{1,h_1}^{(1)}$ ,  $\tilde{x}_2^{(1)}$ ,...,  $\tilde{x}_p^{(1)}$ } of X whose singularities are generated from  $S_2,...,S_{p-2}$ . Instead, if  $S_1=\emptyset$ , we take  $\tilde{X}^{(1)}=X$ .

Then, similarly, we construct a refinement  $\tilde{\chi}^{(2)} = \{\tilde{u}^{(2)}_{1,1}, \dots, \tilde{u}^{(2)}_{1,h_1}, \tilde{v}^{(2)}_{2,1}, \dots, \tilde{v}^{(2)}_{2,h_2}, \tilde{x}^{(2)}_{3}, \dots, \tilde{x}^{(2)}_{p}\}$ of  $\tilde{\chi}^{(1)}$  whose singularities are generated from  $S_3, \dots, S_{p-2}$ .

In this way, after p-2 steps, we obtain an open refinement of X which is non singular and independent.

2. Isomorphism between the pretopological spaces  $G_N(X)$  and G'(Z).

Let S be a connected compact space and  $X = \{X_1, \dots, X_p\}$  an independent open covering of S, such that  $X_i \neq \emptyset$  for i=1,2,...,p. Then let  $Y = \{Y_1, \ldots, Y_p\}$  be an independent shrinking of X (see Proposition 1.5).

For each positive integer  $n \le p$  and any n-tuple  $(i_1, \ldots, i_n)$  of indices of X such that  $i_1 < i_2 < \ldots < i_n$  and  $X_{i_1 \cdots i_n} \neq \emptyset$ , we put:

2.1 Lemma  $X_{i_1...i_n} = \bigcup \{A_{(j_1...j_m)} / \{j_1,...,j_m\} \supseteq \{i_1,...,i_n\} \}.$ 

2.2 Lemma Under the assumption  $B[\phi] = \phi$ , we have  $B[i_1 \dots i_n]^{\cap B}[j_1 \dots j_m]^{=B}[h_1 \dots h_s]$ , where  $\{h_1, \dots, h_s\} = \{i_1, \dots, i_n\} \cap \{j_1, \dots, j_m\}$ .

2.3 Definition We denote by  $A_{\chi}$  the collection of all subsets of S of form  $A_{(i_1...i_n)}$  and by  $B_{\chi}$  the collection of the  $B_{[i_1...i_n]}$  with maximal sets of indices.

2.4 Lemma Any  $A_{(i_1...i_n)} \in A_{\chi}$  is nonempty. Moreover  $A_{\chi}$  is an open covering of S and refines X.

2.5 Lemma  $B_{\chi}$  is an open covering of S.

2.6 Lemma Let  $X_j \in X$  and  $B_{[i_1 \dots i_n]} \in B_X$ . We have  $X_j \cap B_{[i_1 \dots i_n]} \neq \emptyset$  if and only if  $j \in \{i_1, \dots, i_n\}$ . Moreover, if  $j \in \{i_1, \dots, i_n\}$ , then  $X_j \subseteq St(B_{[i_1 \dots i_n]}, B_X)$  and  $B_{[i_1 \dots i_n]} \subseteq St(X_j, X)$ .

2.7 Definition For each  $i \in \{1, 2, ..., p\}$ , let  $Z_i = Y_i - \bigcup_{j \neq i} \overline{Y_j}$ . We put  $Z = B_X \vee \{Z_1, ..., Z_p\}$ .

2.8 Lemma  $Z_i \neq \emptyset$  for each  $i \in \{1, 2, ..., p\}$ . Moreover  $Z_i \cap Z_i = \emptyset$  whenever  $i \neq j$ .

Now let us consider the pf-space SZ and the graph G'(Z) that we obtain from Z (see [2], §6).

2.9 Theorem Given an open covering  $X=\{X_1,\ldots,X_p\}$  of S, let Z be the open covering of S associated to X with the foregoing process. Then the graph  $G_{N(X)}$  of the edges of the nerve N(X) of X is isomorphic to the graph G'(Z).

**Proof:** Each vertex of G'(Z) corresponds to a maximal collection of elements of Z with a nonempty intersection. Since in each of such collections we find exactly one element  $Z_i \in Z$ , the set of the vertices of G'(Z) is bijective to the collection  $\{Z_i\}(i=1,2,\ldots,p)$ , and we denote by  $w_i$  the vertex corresponding to  $Z_i$ . Clearly  $\{w_1, w_2, \ldots, w_p\}$  is bijective to the set  $\{X_1, X_2, \ldots, X_p\}$  of the vertices of N(X). Moreover, given two distinct indices i,j, in G'(Z) there is the edge  $w_i w_j$  iff there is some  $B_{[i_1 \cdots i_n]} \in Z$  such that  $\{i, j\} \subseteq \{i_1, \ldots, i_n\}$ , and hence iff  $X_i \cap X_i \neq \emptyset$ .

2.10 Corollary Under the same assumptions, if the covering X is non singular, then the nerve N(X) of X is isomorphic to the complex  $K_{G'(Z)}$  of the graph G'(Z). Proof: Since X is non singular, N(X) is a complete complex (see [1], §3).

# 3. Isomorphism between the inverse systems $(S_{\chi}, |p_{\chi\chi}|, Cov(S))$ and $(G_N(\chi), |\overline{\phi}_{\chi\chi}|, Cov(S))$ .

Let  $R=\{A_i\}(i\in J)$  and  $R'=\{A_h^{\dagger}\}(h\in H)$  be finite open coverings such that  $R\leq R'$ , and let  $\phi: H \rightarrow J$  be a function such that  $A_h^{\dagger} \subseteq A_{\phi(h)}$  for any  $h\in H$ .

3.1 Definition We denote by  $\overline{\phi}$  the function from  $G_{N(R')}$  to  $G_{N(R)}$  given by  $\overline{\phi}(A_h^{\prime}) = A_{\phi(h)}$  for any h $\epsilon_{H}$ .

3.2 Lemma  $\overline{\Phi}: G_{N}(R') \rightarrow G_{N}(R)$  is a precontinuous map. Moreover, if  $\phi': H \rightarrow J$  is another function such that  $A_{h}^{\mathsf{L}} \subseteq A_{\phi'}(h)$  for any h $\in \mathbb{H}$ , then  $\overline{\phi'}$  and  $\overline{\phi}$  are homotopic. *Proof:* Clearly  $\overline{\phi}$  is precontinuous, and the function  $H: G_{N}(R') \times I \rightarrow G_{N}(R)$  given by:

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$$H(A_{h}', t) = \begin{cases} A_{\phi(h)} & \text{if } t \in [0, 1/2] \\ A_{\phi'(h)} & \text{if } t \in [1/2, 1] \end{cases}$$
  
is a prehomotopy of  $\overline{\phi}$  to  $\overline{\phi'}$ .

3.3 Definition A function  $\tilde{\phi}:G'(R') \rightarrow G'(R)$  is called induced by  $\phi:\mathbb{H} \rightarrow J$ , if, for any vertex  $v'_{h_1...h_n}$  of G'(R'), we have  $\tilde{\phi}(v'_{h_1...h_n})=v_{i_1...i_m}$  with  $\{i_1,...,i_m\} \ge 2\phi(\{h_1,...,h_n\})$ .

3.4 Lemma Under the foregoing assumptions, we have:

(i) any function  $\tilde{\phi}: G'(R') \to G'(R)$  induced by  $\phi$  is precontinuous;

- (ii) any two functions  $\tilde{\phi}$  and  $\tilde{\phi}'$  from G'(R') to G'(R) induced by  $\phi$  are homotopic;
- (iii) if  $\psi: \mathbb{H} \to J$  is another function such that  $A_h^{\dagger} \subseteq A_{\psi(h)}$  for any h $\in \mathbb{H}$ , and if  $\tilde{\psi}: G^{\dagger}(\mathbb{R}') \to G^{\dagger}(\mathbb{R})$  is a function induced by  $\psi$ , then  $\tilde{\psi}$  and  $\tilde{\phi}$  are homotopic.

*Proof:* Since the pretopological spaces  $S_R$  and G'(R) belong to the same homotopy type, we find two precontinuous maps  $p:S_R \rightarrow G'(R)$  and  $q:G'(R) \rightarrow S_R$  such that  $qp \sim 1_{S_R}$  and  $pq \sim 1_{G'(R)}$  in the following way (see [2], §6).

For any vertex  $v_{i_1...i_n}$  of G'(R), we put  $q(v_{i_1...i_n}) = x_{i_1...i_n}$  where  $x_{i_1...i_n}$ belongs to  $A_{i_1...i_n} - U[A_j/j \in J - \{i_1,...,i_n\}\}$ .

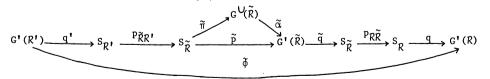
To define  $p:S_R \rightarrow G'(R)$ , we consider the graph  $G^{U}(R)$  (<sup>2</sup>), and we put  $p=\alpha\pi$  where  $\pi:S_R \rightarrow G^{U}(R)$  is the canonical projection and  $\alpha:G^{U}(R) \rightarrow G'(R)$  is a function such that  $\alpha(v_{i_1}...i_n)$  is a vertex  $v_{i_1}...i_m$  of G'(R) with  $\{i_1,...,i_m\} \supseteq \{i_1,...,i_n\}$ . Similarly we obtain  $p':S_{R'} \rightarrow G'(R')$  and  $q':G'(R') \rightarrow S_{R'}$ .

Now we construct a finite open covering  $\tilde{R} = \{\tilde{A}_i\}(i \in J)$  of S such that  $R \leq \tilde{R} \leq R'$ , putting:

$$\begin{split} &\tilde{A}_{i} = A_{i} - \{x_{h_{1}...h_{n}} = q'(v_{h_{1}...h_{n}}) / i \notin \phi(\{h_{1},...,h_{n}\})\}, \\ &\text{where } v_{h_{1}...h_{n}} \text{ denotes a vertex of } G'(R'). \\ &\text{Clearly } x_{h_{1}...h_{n}} \in \tilde{A}_{i} \text{ iff } i \in \phi(\{h_{1},...,h_{n}\}); \text{ moreover the point } x_{i_{1}...i_{m}} \in \tilde{A}_{i} \text{ iff } i \in \{i_{1},...,i_{m}\}. \end{split}$$

Afterwards we define  $\tilde{p}:S_{\tilde{R}} \to G'(\tilde{R})$  and  $\tilde{q}:G'(\tilde{R}) \to S_{\tilde{R}}$  like p and q respectively, and we consider the precontinuous maps  $p_{\tilde{R}R}$ ,  $S_{R}$ ,  $S_{\tilde{R}}$  and  $p_{R\tilde{R}}:S_{\tilde{R}} \to S_{R}$  given by the the identity in S.

Now we define a precontinuous map  $\tilde{\phi}: G'(\mathcal{R}') \rightarrow G'(\mathcal{R})$  in the following way:

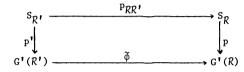


(<sup>2</sup>) The vertices of G<sup>U</sup>(R) are the classes of the equivalence relation  $\sigma$  in S, given by xoy iff  $I_x=I_y$ , where  $I_x=\{i\in J/x\in A_i\}$  and J is the set of the indices of R. We will write  $v_{i_1...i_n}$  to denote the equivalence class [x] such that  $I_x=\{i_1,...,i_n\}$ . We recall that in G<sup>U</sup>(R) there is the edge  $v_{i_1...i_n}v_{j_1...j_m}$  if and only if  $\{i_1,...,i_n\} \cap \{j_1,...,j_m\} \neq \emptyset$ . We easily see that  $\tilde{\phi}(\mathbf{v}_{h_1}^{t}...h_n)=\mathbf{v}_{i_1}...i_m$  with  $\{i_1,...,i_m\} \supseteq \phi(\{h_1,...,h_n\})$ , i.e.  $\tilde{\phi}$  is induced by  $\phi$ .

Then  $\tilde{\phi}$  is unique up to homotopies, since  $\tilde{\phi} \sim pp_{RR} q'$ , where  $p_{RR} : S_{R'} \to S_{R'}$  is the identity in S.

Finally, also  $\tilde{\psi}$  is homotopic to  $pp_{pp},q'$ ; and hence  $\tilde{\psi}$  and  $\tilde{\phi}$  are homotopic.

3.5 *Remark*. For any precontinuous map  $\tilde{\phi}: G'(R') \to G'(R)$  induced by  $\phi: H \to J$ , we obtain the following homotopy commutative diagram:



3.6 Definition Let Cov'(S) denote the collection of the finite independent non singular coverings of S, whose elements are nonempty open sets.

3.7 Proposition Cov'(S) is cofinal in Cov(S).

*Proof:* Observe that any  $A \in Cov(S)$  has a refinement R which is a finite open covering of S; then recall Theorem 1.10.

3.8 Definition Let Cov"(S) denote the collection of all finite open coverings Z associated to some  $X \in Cov'(S)$  (see §2).

3.9 Proposition Cov"(S) is cofinal in Cov(S).

*Proof:* Given RECov(S); take a finite open star-refinement R' of R and X'ECov(S) such that  $R' \leq X'$ . It is easy to see that any covering Z' associated to X' refines R.

3.10 Proposition Let  $X=\{X_i\}(i\in J) \in Cov'(S)$  and let  $Z\in Cov''(S)$  be associated to X. If we take  $X'=\{X_h'\}(h\in H)$  in Cov'(S) such that X' star-refines  $A_X$ , then any covering Z' associated to X' refines Z. Moreover, if  $\Lambda$  is the set of the indices of  $A_X$  and  $\chi: H \to \Lambda$  is any function such that  $St(X_h', X') \subseteq A_{\chi(h)}$  for each hell, then, taking  $\phi(h) \in \xi_{\chi}(h)$ , we can define a function  $\phi: H \to J$  such that:

- (1)  $X'_h \subseteq X_{\phi(h)}$  for any h $\in H$ ;
- (II) for any  $B[h_1...h_n] \in Z'$  there is  $B[i_1...i_m] \in Z$  such that  $B[h_1...h_n] \subseteq G[i_1...i_m]$  and  $\{i_1,...,i_m\} \supseteq \phi(\{h_1,...,h_n\});$
- (III) the function  $\tilde{\phi}_{ZZ'}:G'(Z') \rightarrow G'(Z)$ , that we obtain putting  $\tilde{\phi}_{ZZ'}(w_h')=w_{\phi(h)}$  for any hEH, is induced by  $\phi$ .

*Proof:* Ad (I). Observe that  $X'_h \subseteq A_{\chi(h)} \subseteq X_i$  for any  $i \in \chi(h)$ .

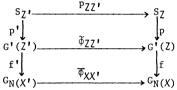
Ad (II).  $B_{[h_1...h_n]} \subseteq St(X_{h_r}, X') \subseteq A_{\chi(h_r)} \subseteq \cap \{X_j / j \in \chi(h_r)\}$  for r=1,2,...,n. Therefore  $B_{[h_1...h_n]} \subseteq \cap \{X_j / j \in \bigcup_{r=1}^n \chi(h_r)\}$ . Hence  $B_{[j = 1}^n \chi(h_r)]$  is nonempty; so there is  $B_{[i_1...i_m]} \in \mathbb{Z}$  such that  $\{i_1, ..., i_m\} \supseteq \bigcup_{r=1}^n \chi(h_r) \supseteq \phi(\{h_1, ..., h_n\})$  and  $B_{[i_1...i_m]} \supseteq A_{\chi(h_r)} \supseteq B_{[h_1...h_n]}$ .

Ad (III). Let  $w'_h$  be a vertex of G'(Z').  $w'_h$  corresponds to the nonempty intersection of  $Z'_h$  and of all  $B_{[h_1...h_n]} \in Z'$  such that  $h \in \{h_1, ..., h_n\}$ . By (II), for

each of `such B'[h<sub>1</sub>...h<sub>n</sub>] there is B[i<sub>1</sub>...i<sub>m</sub>]  $\epsilon$ Z such that {i<sub>1</sub>,...,i<sub>m</sub>}  $\geq \phi(\{h_1,...,h_n\}) \geq \phi(\{h_1,...,h_n\}) \geq \phi(\{h_1,...,h_n\}) \geq \phi(\{h_1,...,h_n\}) \geq \phi(\{h_1,...,h_n\}) \leq B[i_1...i_m]$ . Moreover each of such B'[h<sub>1</sub>...h<sub>n</sub>] contains Z'<sub>h</sub>. Hence the vertex w<sub>k</sub>= $\tilde{\phi}_{ZZ'}(w'_h)$  of G'(Z) must correspond to a maximal nonempty intersection of a collection of elements of Z containing all the B[i<sub>1</sub>...i<sub>m</sub>] we just mentioned. For example w<sub>k</sub> may correspond to the collection containing Z'<sub> $\phi(h)$ </sub> and all B[i<sub>1</sub>...i<sub>m</sub>]  $\epsilon$ Z such that  $\phi(h) \in \{i_1,...,i_m\}$ .

3.11 Remark. Similarly, let  $X \in Cov'(S)$  and let  $Z \in Cov''(S)$  be associated to X. If we take  $Z' \in Cov''(S)$  such that Z' star-refines  $A_X$ , then any  $X' \in Cov''(S)$ , to which we can associate Z', is a refinement of X. Moreover we obtain the statements analogous to the ones from Proposition 3.10.

3.12 *Proposition* Under the foregoing assumptions, we obtain the following homotopy commutative diagram:



where'p, p' are the precontinuous maps from Lemma 3.4, and f, f' are the isomorphisms from Theorem 2.9.

Proof:  $pp_{ZZ'} \sim \tilde{\phi}_{ZZ'}p'$  by Remark 3.5, and  $\overline{\phi}_{ZZ'}f' = f\tilde{\phi}_{ZZ'}$ .

3.13 Theorem The inverse systems  $(S_{\chi}, [p_{\chi\chi'}], Cov(S))$  and  $(G_N(\chi), [\overline{\phi}_{\chi\chi'}], Cov(S))$ , where  $[p_{\chi\chi'}]$  and  $[\overline{\phi}_{\chi\chi'}]$  are the homotopy classes represented by  $p_{\chi\chi'}$ , and  $\overline{\phi}_{\chi\chi'}$ , respectively, are isomorphic.

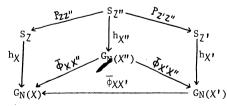
*Proof:* First we define a function  $\Phi$ :Cov'(S)  $\rightarrow$  Cov"(S), taking for each X $\in$ Cov'(S) an element Z= $\Phi(X)$  of Cov"(S) which is associated to X (see §2).

Then, for each X**C**Cov'(S), we consider the precontinuous map  $h_X:S_X \to G_{N(X)}$  given by:

$$S_{Z} \xrightarrow{p} G'(Z) \xrightarrow{f} G_{N}(X)$$

$$h_{X} \xrightarrow{f}$$

where p and f are the precontinuous maps before mentioned. Given  $X \leq X'$  in Cov'(S), take  $X'' \in Cov'(S)$  such that X'' star-refines both  $A_X$  and  $A_{X'}$ . Under this assumption, the following diagram is homotopy commutative:



Hence  $(h_{\chi}, \Phi)$  is a morphism from  $(S_{Z}, [P_{ZZ'}], Cov''(S))$  to  $(G_{N}(\chi), [\overline{\Phi}_{\chi\chi'}], Cov'(S))$ . With a similar process we define a morphism  $(k_{Z}, \Psi)$  from  $(G_{N}(\chi), [\overline{\Phi}_{\chi\chi'}], Cov'(S))$  to  $(S_{Z}, [P_{ZZ'}], Cov''(S))$ . Precisely we define  $\Psi$ :Cov''(S) + Cov'(S), taking for each

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 $Z \notin Cov''(S)$  an element  $X=\Psi(Z)$  of Cov'(S) such that Z is associated to X. Then we consider the precontinuous map  $k_Z:G_N(X) \rightarrow S_Z$  given by  $k_Z=qf^{-1}$ , where  $f:G'(Z) \rightarrow G_N(X)$  and  $q:G'(Z) \rightarrow S_7$  are the before mentioned functions.

Afterwards, each of the morphisms  $(h_{\chi}, \Phi)$  and  $(k_{\chi}, \Psi)$  is the inverse of the other. Finally recall Propositions 3.7 and 3.9.

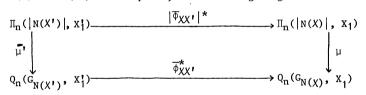
#### 4. Shape groups and Čech homology groups of a connected compact topological space S.

To calculate the shape groups  $\Pi_n(S,a)$  based at a point a(S, we have to fix, for each covering X, an open set X(X such that a(X.

Therefore we have to consider some pointed open coverings of the pointed space (S,a), such that there exists exactly one element of each covering X containing a. We denote such an element by X<sub>1</sub>, and we choose the characteristical point x<sub>1</sub> of X<sub>1</sub> taking x<sub>1</sub>=a. So a is a point of the element  $A_{(1)} \in A_X$ , and a belongs to the open set  $Z_1 \in Z$  and to each  $B_{[1i_2...i_m]} \in B_X$ . Then, "mutatis mutandis", we obtain that the inverse systems  $((S_X,a), [P_{XX'}], Cov(S))$  and  $((G_{N(X)},X_1), [\overline{\phi}_{XX'}], Cov(S))$  are isomorphic.

So, for each dimension n the inverse systems  $(\Pi_n(S_X,a), p_{XX}^*)$ , Cov(S)) and  $(Q_n(G_{N(X)},X_1), \overline{\varphi}_{XX}^*)$ , Cov(S)) are isomorphic.

Afterwards, if X, X'  $\epsilon$  Cov(S) and X  $\leq$  X', since X and X' are non singular and the complexes N(X) and N(X') are complete, the following diagram commutes:

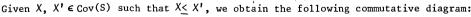


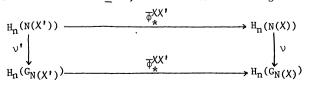
where  $\mu$  and  $\mu'$  are the isomorphisms given by the canonical projections from the polyhedron |N(X)| to the graph  $G_{N(X)}$  of the edges of N(X) and from |N(X')| to  $G_{N(X')}$  respectively (see [1], §3).

Hence the inverse systems  $(\Pi_n(S_{\chi,a}), p_{\chi\chi}^*, Cov(S))$  and  $(\Pi_n(|N(\chi)|, \chi_1), \overline{\phi}_{\chi\chi}, |*, Cov(S))$  are isomorphic. Therefore:

$$\underbrace{\lim}_{\mathbf{n}} (\Pi_{\mathbf{n}}(\mathbf{S}_{\mathbf{X}}, \mathbf{a}), \mathbf{p}_{\mathbf{X}\mathbf{X}}^{*}, \mathbf{Cov}(\mathbf{S})) \simeq \underbrace{\mathbb{I}}_{\mathbf{n}}(\mathbf{S}, \mathbf{a}) \simeq \underbrace{\lim}_{\mathbf{n}} (\Pi_{\mathbf{n}}(|\mathbf{N}(\mathbf{X})|, \mathbf{X}_{1}), |\overline{\phi}_{\mathbf{X}\mathbf{X}'}|^{*}, \mathbf{Cov}(\mathbf{S})).$$

In the case of Čech homology groups,  $\mathcal{K}$  for any XECov(S) and each dimension n, we consider the homology group  $H_n(N(X))$  of the simplicial complex N(X) and the singular homology group  $H_n(G_N(X))$  of the graph  $G_{N(X)}$  (see [5]).





where v and v' are the isomorphisms considered in [5], §5. Hence:

 $\underbrace{\lim}_{\mathsf{H}_{n}}(\mathsf{H}_{n}(\mathsf{S}_{X}), \mathsf{p}_{\star}^{XX'}, \operatorname{Cov}(\mathsf{S})) \stackrel{}{\scriptstyle{\sim}} \overset{}{\mathsf{H}}_{n}(\mathsf{S}) \stackrel{}{\scriptstyle{\sim}} \underbrace{\lim}_{\mathsf{H}_{n}}(\mathsf{H}_{n}(\mathsf{N}(X)), \overline{\phi}_{\star}^{XX'}, \operatorname{Cov}(\mathsf{S})).$ 

#### 5. Examples.

5.1 Let S be the polyhedron |K| of a finite simplicial complex K of dimension m. In this case we can calculate the groups  $\Pi_n(S,a)$  and  $H_n(S)$  more simply in the following way.

For any 
$$i \in \mathbb{N}$$
, we take the derived  $K^{(i)}$  of K, and we denote by  $V^{(i)}$  the vertex set  
of  $K^{(i)}$  and by  $\sigma_p^{(i)}$  a p-dimensional simplex whatever of  $K^{(i)}$ . Then we put:  
 $r_i = \frac{1}{m} \inf\{d(x_h^{(i)}, x_k^{(i)})\}$ , where  $x_h^{(i)}, x_k^{(i)} \in V^{(i)}$ ;  
 $R_i = \{V(\sigma_p^{(i)}, r_i)\}(\sigma_p^{(i)} \in K^{(i)}; 0 \le p \le m)$ , where  $V(\sigma_p^{(i)}, r_i) = \{y \in S / d(y, \sigma_p^{(i)}) < r_i\};$   
 $\Gamma = \{R_i\}(i \in \mathbb{N})$ .

It is easy to see that each  $R_i$  is an open covering of S, and that the graph G'( $R_i$ ) is the graph of the edges of the complex  $K^{(i)}$ .

The set  $\Gamma$  is cofinal in Cov(S); so we have:

$$\dot{\Pi}_{n}(S,a) = \underbrace{\lim}_{\underline{\ell}} (\Pi_{n}(S_{R_{i}}, a), p_{R_{i}R_{j}}^{*}, \Gamma);$$
  
$$\check{H}_{n}(S) = \lim (H_{n}(S_{p_{i}})), p_{i}^{\cdot R_{i}}j, \Gamma).$$

 $H_n(S) = \underbrace{\lim}_{m} (H_n(S_{R_i})), p_*^{1} J, \Gamma).$ Since, for i>0,  $\Pi_n(S_{R_i},a) \simeq \Pi_n(|K|,a), H_n(S_{R_i}) \simeq H_n(K)$ , and all functions  $p_{R_iR_j}^*$ and  $p_*^{R_iR_j}$  are isomorphisms, we obtain:

 $\check{\mathbb{H}}_{n}(S,a) = \mathbb{H}_{n}(S,a);$   $\check{\mathbb{H}}_{n}(S) = \mathbb{H}_{n}(K).$ 

5.2 Let (S,d) be a compact metric space.

For any  $\varepsilon>0$  we consider the symmetrical pf-space  $S_{\varepsilon}=(S,P_{\varepsilon})$  where  $P_{\varepsilon}=\{\overline{V(x,\varepsilon)}\}(x\in S)$ and  $V(x,\varepsilon)=\{y\in S / d(x,y)<\varepsilon\}$ . If  $\varepsilon'<\varepsilon$ , we consider the precontinuous map  $p_{\varepsilon\varepsilon}:S_{\varepsilon}:s_{\varepsilon}$  given by  $p_{\varepsilon\varepsilon}:(x)=x$  for any  $x\in S$ .

Then we easily see that, for each dimension n, we have:

$$\Pi_{n}(S,a) = \underbrace{\lim}_{t \to \infty} (\Pi_{n}(S_{\varepsilon},a), p_{\varepsilon\varepsilon}^{*}, E)$$

$$H_n(S) = \lim_{\epsilon \to \infty} (H_n(S_{\epsilon}), p_{\star}^{ce}, E),$$

where E is the directed set that we obtain taking the set  $R^+$  of all positive real numbers with the inverted order.

5.3 Let S be the Warsaw circle, i.e. the following subspace of  $\mathbb{R}^2$ . Given the points a=(0,1), b=(0,-2),  $c=(\frac{1}{2},-1)$ ,  $d=(\frac{1}{2},0)$ , we take the segments ab, bc, cd and all points  $(x,y)\in\mathbb{R}^2$  such that  $x\in]0,\frac{1}{2}]$  and  $y=\sin(\pi/2x)$ . Let  $\phi:[\frac{1}{2},1] \rightarrow ab$  U bc U cd be an homeomorphism such that  $\phi(1)=a$  and  $\phi(\frac{1}{2})=d$ , and let

f:]0,1]  $\rightarrow$  S be the continuous surjection given by:

Then for any  $\epsilon>0$  we consider the pretopological space  $S_\epsilon$  from 5.2 and the

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precontinuous loop  $\psi_{\varepsilon}: [0, 1] \rightarrow S_{\varepsilon}$  based at a, given by:

 $\psi_{\varepsilon}(\mathbf{x}) = \begin{cases} a & \text{if } 0 \leq \mathbf{x} \leq \lambda \\ \\ \Phi(\mathbf{x}) & \text{if } \lambda \leq \mathbf{x} \leq 1 \end{cases}$ 

where  $\lambda = 1/(4n+1)$  and n is the lowest positive integer such that  $1/(4n+1) < \varepsilon$ . The group  $\Pi_1(S,a)$  is isomorphic to (Z,+), and we observe that its generator can be associated to the sequence of the prehomotopy classes represented by the loops  $\psi_{\varepsilon}$  of  $S_{\varepsilon}$ .

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