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Productive and inductive constructions of graphs

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PRODUCTIVE AND INDUCTIVE CONSTRUCTIONS OF GRAPHS*

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0. Introduction

In [5], there is given a characterization of systems of antireflexive graphs in which any induced subgraph of a subdirectly irreducible (SI) graph is again SI. In the present paper we give a full characterization of hereditary subdirect irreducibility for graphs.

Importance of investigation this topic is following : having a list of SI C-graphs one can construct any C-graph using only products and restrictions to induced subgraphs. If a class C of graphs is hereditary subdirectly irreducible (HSI) then the subdirect dimension coincides with the product dimension (for the definitions see [4]).

1. Notations and known facts

1.1. For the description of HSI graphs we shall use some symbols from [5] and also introduce some new ones.

If A is an induced subgraph of B we shall write $A \hookrightarrow B$. For an arbitrary graph G denote V(G) its set of vertices and E(G) its set of edges. $L(G) \hookrightarrow G$ is a graph such that $V(L(G)) = \{v \in V(G) ; (v,v) \in E(G)\}$. $N(G) \hookrightarrow G$ is a graph with $V(N(G)) = V(G) - V(L(G))$. (Edges of L(G) are denoted as LL-edges, edges of N(G) as NN-edges, edges from L(G) to N(G) as LN-edges and edges from N(G) to L(G) as NL-edges.)

For any ordinal n denote $K_n = (n, \{(i,j) ; i,j \in n, i \neq j\})$ (i.e. the complete antireflexive graph with n vertices),

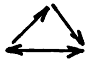
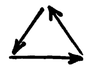
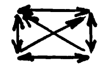
$K_n' = (n, \{(i,j) ; i,j \in n, i \neq j, (i,j) \neq (0,1)\})$,

$L_n = (n, \{(i,j) ; i,j \in n , i < j\})$,

$L_n^+ = (n, \{(i,j) ; i,j \in n , i < j\} \cup \{(1,0)\})$,

$L_n^- = (n, \{(i,j) ; i,j \in n , i > j\} \cup \{(0,1)\})$,

*) This paper is in final form and no version of it will be submitted for publication elsewhere.

$A_3 =$  $C_3 =$  $A_4 =$ 

$G_0 =$ $G_1 =$  $G_2 =$  $G_3 =$  $G_4 =$ 

$G_5 =$  $R_n = (n, n \times n)$ (the complete reflexive graph)

Further, put

$\underline{K} = \{K_n ; n \in \mathrm{Ord}\}$,
$\underline{K}' = \{K_n' ; n \in \mathrm{Ord}\}$,
$\underline{L}^+ = \{L_n^+ ; n \in \mathrm{Ord}\}$,
$\underline{L}^- = \{L_n^- ; n \in \mathrm{Ord}\}$,
$\underline{\mathrm{Set}} = \{(X,\emptyset) ; X$ is a set$\}$ (the class of sets = discrete graphs),
$\underline{T} = \{G ; |\{(x,y),(y,x)\} \cap E(G)| = 1$ for any $x \neq y \in V(G)\}$ (the class of all antireflexive tournaments)
$\underline{U} = \{(n,R) ; n \leq 6, |R| = \binom{n}{2} + [\frac{n}{2}], x \neq y \Rightarrow |\{(x,y),(y,x)\} \cap R| \geq 1$ and (n,R) contains neither K_3' nor A_3 as an induced subgraph$\}$,
$\underline{V} = \{(n,R) ; n \leq 4, x \neq y \Rightarrow |R \upharpoonright \{x,y\}| \geq 1, R \supseteq \{(0,1),(1,0),(2,3),(3,2)\} \cap n \times n$ and (n,R) does not contain K_3' as an induced subgraph$\}$,
$\underline{W} = \{A$; any induced subgraph of G with 3 vertices is either isomorphic to A_3, or to $L_3\}$,
$\underline{X}_0 = \{X ; V(X) = V \cup \{v\}, E(X) = E \cup \{(v,v)\}$ where $(V,E) \in \underline{X}\}$ for $\underline{X} \in \{\underline{K}, \underline{K}', \underline{L}^+, \underline{L}^-, \underline{T}, \underline{U}, \underline{V}, \underline{W}\}$,
$\underline{\mathrm{Sym}}_5 = \{A ; A$ is reflexive symmetric, $|V(A)| \leq 5\}$.

1.2. By a product of graphs we mean the categorical product (i.e. $\prod_{i \in I} (V_i, E_i) = (\prod_{i \in I} V_i, E)$ where $((x_i)_I, (y_i)_I) \in E$ iff $(x_i, y_i) \in E_i$ for any $i \in I$).

1.3. Let $\underline{C}$ be a class of graphs. Then $A \in \underline{C}$ (i.e. a $\underline{C}$-graph A) is said to be subdirectly irreducible if, whenever an isomorphic copy A' of A is contained as an induced subgraph in a product $\prod_{i \in I} B_i$ with $B_i \in \underline{C}$ and $p_j(A') = B_j$ for all the projections, there is a j such that the restriction of p_j to A' is an isomorphism onto B_j. (This formulation is due to A. Pultr - see [2].)

1.4. A class $\underline{C}$ of graphs is said to be hereditary with respect to subdirect irreducibility (HSI) if any induced subgraph of a SI $\underline{C}$-graph is again SI (see [5]).

1.5. If $V(A) = V(B)$ then the meet of graphs $A \wedge B$ denotes the graph $(V(A), E(A) \cap E(B))$. If $C = A \wedge B$, $C \neq A, B$ then C is subdirectly reducible in $\underline{C}$ (see [3]).

1.6. Let $\underline{D}$ be a family of graphs. Then $SP(\underline{D})$ denotes (similarly as in [1]) the class of all the graphs which can be embedded as induced subgraphs into products of graphs from $\underline{D}$.

1.7. Theorem (see[5]). Let $\underline{C}$ be a productive hereditary class of antireflexive graphs (i.e. a class closed to categorical products and to induced subgraphs). Then $\underline{C}$ is HSI iff either $\underline{C}$ = $\underline{\text{Set}}$ or $\underline{C} = SP(\underline{D})$ where $\underline{D}$ satisfies one of the following conditions :

(i) $\underline{D} \subseteq \underline{K} \cup \underline{K}'$

(ii) $\underline{D} \subseteq \underline{K} \cup \{K_3', A_4\}$

(iii) $\underline{D} \subseteq \underline{K} \cup \underline{L}^{+} \cup \underline{T}$

(iv) $\underline{D} \subseteq \underline{K} \cup \underline{L}^{-} \cup \underline{T}$

(v) $\underline{D} \subseteq \underline{K} \cup \underline{U}$

(vi) $\underline{D} \subseteq \underline{K} \cup \underline{V}$

(vii) $\underline{D} \subseteq \underline{K} \cup \underline{W}$

2. Hereditary subdirect irreducibility

Before giving the general characterization theorem for HSI in graphs we shall consider partial cases discussing possibilities for reflexive and antireflexive parts of graphs and for LL-,LN- and NL-edges.

Throughout this chapter, $\underline{C}$ denotes a productive hereditary class of graphs which is HSI.

2.1.Lemma. If any reflexive subgraph of a SI $\underline{C}$-graph A is complete then $|L(A)| \leq 2$.

Proof. Since $\underline{C}$ is HSI, any L(A) is SI whenever A is SI.Any reflexive complete graph is an induced subgraph of a power R_2^k for some k. Hence, $|L(A)| \leq 2$.

2.2.Lemma. If some LN-edge of a SI $\underline{C}$-graph A_0 is $\circlearrowleft\!\longleftrightarrow$ then no edge of a SI $\underline{C}$-graph A is $\longleftrightarrow$. Moreover, if some LL-edge of A_0 is $\circlearrowleft\!\longrightarrow\!\circlearrowright$ then no edge of A is $\longrightarrow$.

Proof. $\longleftrightarrow \; = \; \circlearrowleft\!\longleftrightarrow \wedge \longleftrightarrow\!\circlearrowright$, hence it is reducible and it cannot be an induced subgraph of a SI graph A. If $\circlearrowleft\!\longrightarrow\!\circlearrowright \in \underline{C}$ then $\longrightarrow \; = \; \longleftrightarrow \wedge \circlearrowleft\!\longrightarrow\!\circlearrowright$ is subdirectly reducible and hence it cannot be an induced subgraph of A.

2.3.Lemma. Let A_0,A be SI $\underline{C}$-graphs. If some LN-edge of A_0 is $\circlearrowleft\!\longrightarrow$ ($\circlearrowleft\!\longleftarrow$,resp.) then either any NN-edge of A is $\longleftrightarrow$ or any NN-edge of A is $\longrightarrow$.

Proof. $\longrightarrow \; = \; \circlearrowleft\!\longrightarrow \wedge \longleftrightarrow \; = \; \circlearrowleft\!\longleftarrow \wedge \longleftrightarrow$. Hence, graphs $\longleftrightarrow$ and $\longrightarrow$ cannot be both SI in $\underline{C}$.

2.4.Lemma. If $\circlearrowleft\!\longleftrightarrow$ and $\circlearrowleft\!\longrightarrow$ ($\longrightarrow\!\circlearrowright$,resp.) are both $\underline{C}$-graphs then $|N(A)| \leq 1$ for any SI $\underline{C}$-graph A.

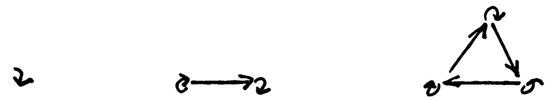
Proof. There is $\longleftrightarrow \; = \; \circlearrowleft\!\longleftrightarrow \wedge \longleftrightarrow\!\circlearrowright$, $\longleftarrow \; = \; \circlearrowleft\!\longleftrightarrow \wedge \longleftarrow\!\circlearrowright =$

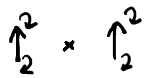
$= \longleftrightarrow \wedge \longleftarrow$, $\circ \quad \circ = \longrightarrow \wedge \longleftarrow = \longrightarrow \wedge \longleftarrow$, hence any antireflexive graph with at least 2 vertices is subdirectly reducible in $\underline{C}$ and $|N(A)| \leq 1$ for any SI $\underline{C}$-graph A.

2.5.Lemma. If $\longrightarrow$ and $\longrightarrow$ are both $\underline{C}$-graphs then any NN-edge of a SI $\underline{C}$-graph A is $\longleftrightarrow$ and no LN-edge of a SI $\underline{C}$-graph is $\circ$.

Proof. There is $\longrightarrow = \longrightarrow \wedge \longrightarrow$, $\circ \quad \circ = \longrightarrow \wedge \longleftarrow$.
Hence, any NN-edge of a SI $\underline{C}$-graph is $\longleftrightarrow$. Moreover, $\circ \quad \circ =$
$= \longrightarrow \wedge \longleftarrow$, hence no LN-edge of a SI $\underline{C}$-graph is $\circ \quad \circ$

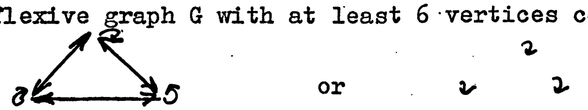
2.6.Lemma. If a reflexive tournament A is SI in $\underline{C}$ then A is one of the following graphs :



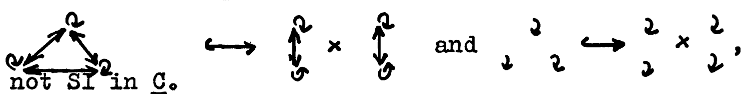
Proof.  is an induced subgraph of 
hence it is not SI. Any reflexive tournament with at least 4 vertices contains 
as an induced subgraph, hence it is not SI.

2.7.Lemma. If a symmetric reflexive graph A is SI in $\underline{C}$ then $|V(A)| \leq 5$.

Proof. Using Dirichlet principle, one can check that any symmetric reflexive graph G with at least 6 vertices contains either



as an induced subgraph. Since



G is not SI in $\underline{C}$.

2.8.Proposition. If a reflexive graph A is SI in $\underline{C}$ then $|V(A)| \leq 9$.

Proof. If A is symmetric then $|V(A)| \leq 5$ due to Lemma 2.7. If A is not symmetric then it contains $\longrightarrow$ as an induced subgraph. Hence, $\circ \quad \circ = \longrightarrow \wedge \longleftarrow$ is not SI in $\underline{C}$. Therefore, $\circ \quad \circ$ is not an induced subgraph of A. Using Ramsey theorem one can prove that any reflexive graph with at least 9 vertices which does not contain $\circ \quad \circ$ as an induced subgraph contains either

 or a tournament with 4 vertices as an induced subgraph. Using Lemmas 2.1 and 2.6 one proves that G is not SI.

2.9. Lemma. If a reflexive graph A is SI in $\underline{C}$ and contains G_5 as an induced subgraph, then $A = G_5$.

<u>Proof.</u> Using Proposition 4.6 from [3] one can prove that any reflexive graph is a $\underline{C}$-graph. Since any reflexive graph with at least 4 vertices is - due to [3] - subdirectly reducible and A is SI in $\underline{C}$, there is $A = G_5$.

<u>2.10. Lemma.</u> If G_5 is not a $\underline{C}$-graph and G_2 is a $\underline{C}$-graph then any reflexive SI $\underline{C}$-graph has the following property : For an $(x,y) \notin E(A)$ define $U(x,y)$ as the smallest subset of $V(A) \times V(A) - E(A)$ containing (x,y) and such that $(V(A), E(A) \cup U(x,y)) \in \underline{C}$. Then for any two (x_0,y_0), $(x_1,y_1) \in (V(A) \times V(A)) - E(A)$, $U(x_0,y_0) \cap U(x_1,y_1) \neq \emptyset$, and for every morphism $\varphi : A \longrightarrow B$ with $|V(B)| < |V(A)|$ there is an $(x,y) \in (V(A) \times V(A)) - E(A)$ with $(\varphi(x), \varphi(y)) \in E(B)$.

<u>Proof</u> follows directly from [3], Lemma 6.8.

<u>2.11. Proposition.</u> If a reflexive graph A is SI in $\underline{C}$ then it satisfies one of the following conditions :

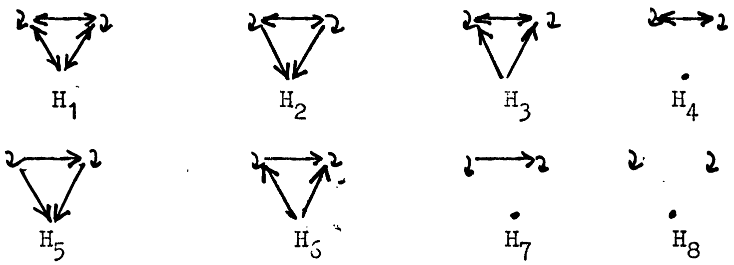
(i) $A \in \underline{Sym}_5$

(ii) $A \hookrightarrow G_5$

(iii) $|V(A)| \leq 9$ and A satisfies the conditions from Lemma 2.10.
(Denote this class of graphs by $\underline{Ref}_9$.)

<u>Proof</u> follows from 2.7-2.10.

<u>2.12. Lemma.</u> The following graphs are subdirectly reducible in $\underline{C}$ whenever they are $\underline{C}$-graphs:



<u>Proof.</u>
$H_1 \hookrightarrow G_3 \times R_2^2$, $H_2 \hookrightarrow \downarrow \times R_2^2$, $H_3 \hookrightarrow \uparrow \times R_2^2$, $H_4 \hookrightarrow \vdots \times R_2^2$,
$H_5 \hookrightarrow \updownarrow \times G_2^2$, $H_6 \hookrightarrow \uparrow \times G_2^2$, $H_7 \hookrightarrow \vdots \times G_2^2$, $H_8 \hookrightarrow \vdots \times \vdots$.

<u>2.13. Proposition.</u> If A is SI in $\underline{C}$, $L(A) \neq \emptyset$ is a complete graph and $(l,n),(n,l) \in E(A)$ whenever $l \in L(A)$, $n \in N(A)$ then either A is one of the following graphs : G_1, G_3, G_4 , or $|L(A)| = 1$ and N(A) is an antireflexive tournament. (Denote this class of graphs by $\underline{T}_1^{\circlearrowleft}$.)

<u>Proof.</u> According to 2.1, $|L(A)| \leq 2$. Hereditary subdirect irreducibility implies that N(A) satisfies conditions of Characterization Theorem 1.7. Consider two cases :

1. $|L(A)| = 1$. Consider possibilities for N(A) using 1.7.

(i) If $N(A) \in \underline{K} \cup \underline{K}'$ then by 2.2 there is $|N(A)| \leq 2$. For the case $|N(A)| = 0$ one obtains $A = G_1$, for the case $|N(A)| = 1$ there is $A = G_3$ and for the case $|N(A)| = 2$ there is $N(A) = K_2'$ which is an antireflexive tournament on 2 points.

(ii) If $N(A) = A_4$ then one obtains a contradiction using 2.2.

(iii) If $N(A) \in \underline{L}^+ \cup \underline{T}$ then $N(A)$ is an antireflexive tournament.

(iv) If $N(A) \in \underline{L}^- \cup \underline{T}$ then $N(A)$ is an antireflexive tournament as well.

(v) If $N(A) \in \underline{U}$ then one obtains $\longleftrightarrow$ as an induced subgraph which contradicts 2.2.

(vi) If $N(A) \in \underline{V}$ then one obtains a contradiction with 2.2, too.

(vii) If $N(A) \in \underline{W}$ then using 2.2 one obtains that $N(A)$ is an antireflexive tournament.

2. $|L(A)| = 2$.

If $N(A) = \emptyset$ then $A = G_4$.

If $N(A) \neq \emptyset$ then A contains H_1 as an induced subgraph which contradicts 2.12.

__2.14. Proposition.__ If A is SI in $\underline{C}$, $L(A) \neq \emptyset$ is a complete graph, $N(A) = \emptyset$ and $(l,n) \in E(A)$, $(n,l) \notin E(A)$ whenever $l \in L(A)$, $n \in N(A)$, then $|L(A)| = 1$ and either

(i) $N(A) = K_n$ for some n,

or

(ii) $N(A)$ is an antireflexive tournament.

(Denote the class of graphs satisfying (i) ((ii),resp.) by $\underline{K}_1^{\downarrow}$ ($\underline{T}_1^{\downarrow}$,resp.).

__Proof.__ Lemmas 2.1 and 2.12 imply that $|L(A)| = 1$. Using 1.7 and 2.3 one obtains that $N(A)$ is either antireflexive complete or an antireflexive tournament.

__2.15. Proposition.__ If A is SI in $\underline{C}$, $L(A) \neq \emptyset$ is complete, $N(A) \neq \emptyset$ and $(l,n) \notin E(A)$, $(n,l) \in E(A)$ whenever $l \in L(A)$, $n \in N(A)$, then $|L(A)| = 1$ and either $N(A) = K_n$ for some n or $N(A)$ is an antireflexive tournament.(Denote the corresponding class of graphs by $\underline{K}_1^{\uparrow} \cup \cup \underline{T}_1^{\uparrow}$.)

__Proof__ is similar to the proof of Proposition 2.14.

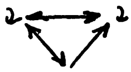
__2.16. Proposition.__ If $A_i (i=0,1)$ are SI in $\underline{C}$, $L(A_i) \neq \emptyset$ are complete, $N(A_0) \neq \emptyset$, $(l,n) \in E(A_i)$ whenever $l \in L(A_i)$, $n \in N(A_i)$ and if there exist $n_1, n_2 \in N(A_0)$, $l_1, l_2 \in L(A_0)$ such that $(n_1,l_1) \notin E(A_0)$, $(n_2,l_2) \notin E(A_0)$ then

$A_i \longleftrightarrow$ 

Proof. By 2.2 and 2.3, $\longleftrightarrow$ and $\longrightarrow$ (and also $\circ \quad \circ = \longrightarrow \wedge \longleftarrow$) are subdirectly reducible. Hence, $|N(A_i)| \leq 1$. By 2.1 and the assumptions of Proposition, $|L(A_i)| \leq 2$. Therefore,

$A_o =$  , $A_1 \hookrightarrow A_o$.

2.17. Proposition. If $A_i (i = 0,1)$ are SI in $\underline{C}$, $L(A_i) \neq \emptyset$ are complete, $N(A_o) \neq \emptyset$, $(n,l) \in E(A_i)$ whenever $l \in L(A_i)$, $n \in N(A_i)$ and if there exist $n_1, n_2 \in N(A_o)$, l_1, $l_2 \in L(A_o)$ such that $(l_1,n_1) \in E(A_o)$, $(l_2,n_2) \in E(A_o)$ then

$A_i \hookrightarrow$ 

Proof is similar to the proof of 2.16.

2.18. Proposition. If $A_i (i = 0,1)$ are SI in $\underline{C}$, $L(A_o) \neq \emptyset$ is either complete or a tournament, $N(A_o) \neq \emptyset$, $|\{(n,l),(l,n)\} \cap E(A_i)| = 1$ for any $(n,l) \in N(A_i) \times L(A_i)$ and if there exist $n_1, n_2 \in N(A_o)$, $l_1, l_2 \in L(A_o)$ such that $(l_1,n_1) \in E(A_o)$, $(n_2,l_2) \in E(A_o)$ then $N(A_i) = K_n$ for some n, $(l',n) \in E(A_i) \Leftrightarrow (n,l'') \in E(A_i)$ whenever $l' \neq l'' \in L(A_i)$. (Denote the class of graphs A_1 satisfying these conditions by $\underline{L}$.)

Proof. By 2.5, any NN-edge of A_i is $\longleftrightarrow$. Hence, $N(A_i) = K_n$ for some n. By 2.12, A_i does not contain neither H_2 nor H_3 as an induced subgraph.

2.19. Proposition. If A is SI in $\underline{C}$, $L(A) \neq \emptyset$, $N(A) \neq \emptyset$ and if there are no NL- and LN-edges in A then $|L(A)| = 1$.

Proof follows from 2.12 because A cannot contain H_4, H_7 and H_8 as induced subgraphs.

2.20. Proposition. If $A_i (i = 0,1)$ are SI in $\underline{C}$, $L(A_i) \neq \emptyset$ is complete, $N(A_i) \neq \emptyset$, $\emptyset^1 \neq L(A_o) \times N(A_o) \cap E(A_o)^1 \neq L(A_o) \times N(A_o)$, and for any $(l,n) \in L(A_i) \times N(A_i)$ there is $(l,n) \in E(A_i) \Leftrightarrow (n,l) \in E(A_i)$, then $|L(A_i)| \leq 2$ and $N(A_i)$ is an antireflexive tournament. Moreover, if $|L(A_i)| = 2$ then $L(A_i) = \{l',l''\}$ such that $(l',n) \notin E(A_i) \Leftrightarrow (l'', n) \in E(A_i)$ for any $n \in N(A_i)$. (Denote the class of graphs satisfying these conditions by $\underline{M}$.)

Proof follows from 1.7, 2.1, 2.2 and 2.12.

2.21. Proposition. If $A_i (i = 0,1)$ are SI in $\underline{C}$, $L(A_o) \neq \emptyset$, $N(A_o) \neq \emptyset$, $\emptyset \neq L(A_o) \times N(A_o) \cap E(A_o) \neq L(A_o) \times N(A_o)$ and $N(A_i) \times L(A_i) \cap E(A_i) = \emptyset$, then $N(A_i)$ is either an antireflexive tournament or an antireflexive complete graph. Moreover, if $l' \neq l'' \in L(A_i)$ then $(l',n) \in E(A_i) \Leftrightarrow (l'',n) \notin E(A_i)$ for any

$n \in N(A_i)$.(Denote the class of graphs satisfying these conditions by $\underline{N}$.)

Proof follows from 1.7, 2.3 and 2.12.

2.22.Proposition. If $A_i (i = 0,1)$ are SI in $\underline{C}$, $L(A_0) \neq \emptyset$, $N(A_0) \neq \emptyset \neq N(A_0) \times L(A_0) \cap E(A_0) \neq N(A_0) \times L(A_0)$ and $L(A_i) \times N(A_i) \cap E(A_i) = \emptyset$ then $N(A_i)$ is either an antireflexive tournament, or an antireflexive complete graph.Moreover, if $l' \neq l'' \in L(A)$ then $(n,l') \in E(A_i) \iff (n,l'') \notin E(A_i)$ for any $n \in N(A_i)$. (Denote the class of graphs satisfying these conditions by $\underline{P}$.)

Proof follows from 1.7, 2.3 and 2.12.

2.23.Lemma. If there are $(l_1,n_1),(l_2,n_2),(l_3,n_3) \in L(A) \times N(A)$ such that $(l_1,n_1),(n_1,l_1),(n_2,l_2),(l_3,n_3) \notin E(A)$, $(l_2,n_2),(n_3,l_3) \in E(A)$ then A is subdirectly reducible in $\underline{C}$.

Proof. Since $\circlearrowleft \quad \bullet$, $\circlearrowleft\!\longrightarrow$ and $\circlearrowleft\!\longleftarrow$ are induced subgraphs of A, $\circlearrowleft \quad \bullet = \circlearrowleft\!\longrightarrow \wedge \circlearrowleft\!\longleftarrow$ and $\underline{C}$ is HSI, A cannot be SI.

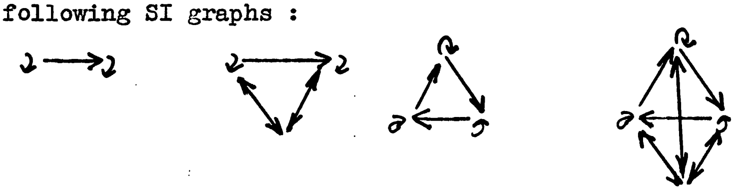
2.24.Lemma. If L(A) is complete and if there are $(l_1,n_1),(l_2,n_2),(l_3,n_3) \in L(A) \times N(A)$ such that $(l_1,n_1),(l_3,n_3),(n_3,l_3) \in E(A)$ and $(n_2,l_2) \notin E(A),(l_2,n_2) \in E(A)$ $((n_2,l_2) \in E(A),(l_2,n_2) \notin E(A)$ resp.) then A is subdirectly reducible in $\underline{C}$.

Proof. Suppose A be SI.Then 2.1 implies that $|L(A)| \leq 2$, 2.4 implies that $|N(A)| \leq 1$ and there are at most 2 LN- and 2 NL-edges which contradicts assumptions of Lemma.

2.25.Proposition. If A is a SI $\underline{C}$-graph which contains G_2 as an induced subgraph, $L(A) \times N(A) \subset E(A)$, $N(A) \times L(A) \subset E(A)$, then $|N(A)| \leq 1$. (Denote the class of graphs satisfying these conditions by $\underline{Q}$.)

Proof. Lemma 2.2 implies that no edge of A is neither $\longleftrightarrow$ nor $\longrightarrow$. Since $\circ \quad \bullet = \longrightarrow \wedge \longleftarrow$ there is $|N(A)| \leq 1$.

2.26.Corollary. For the case that L(A) is a tournament we obtain the following SI graphs :



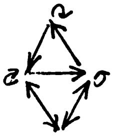
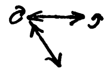
2.27.Lemma. If $G_2 \hookrightarrow A$, $N(A) \neq \emptyset$, $L(A) \times N(A) \subseteq E(A)$, $N(A) \times L(A) \cap E(A) = \emptyset$ $(L(A) \times N(A) \cap E(A) = \emptyset, N(A) \times L(A) \subseteq E(A)$ resp.), then A is subdirectly reducible in $\underline{C}$.

Proof follows from 2.12.

2.28.Lemma. If A contains G_2, G_3 and $\circlearrowleft\!\longrightarrow$ ($\longrightarrow\!\circlearrowleft$ resp.) as induced subgraphs then A is not SI in $\underline{C}$.

<u>Proof.</u> $= \wedge$, $= \wedge$, hence A is not SI in $\underline{C}$.

<u>2.29.Proposition.</u> If $A_i (i = 0,1)$ are SI in $\underline{C}$, $G_2 \hookrightarrow A_0$, $\emptyset \neq L(A_0) \times N(A_0) \cap E \neq L(A_0) \times N(A_0)$ and if for any $(l,n) \in L(A_i) \times N(A_i)$ there is $(l,n) \in E(A_i) \Leftrightarrow (n,l) \in E(A_i)$ then

$A_0 \hookrightarrow$  , $A_1 \hookrightarrow A_0$ or $A_1 \hookrightarrow$ 

<u>Proof</u> follows from 2.12 and 2.2.

<u>2.30.Lemma.</u> If a reflexive graph A is SI in $\underline{C}$ and is also SI in $\underline{C}$ then A is symmetric.

<u>Proof.</u> $= \wedge$, hence A cannot contain G_2 as an induced subgraph.

<u>2.31.Proposition.</u> If $A_i (i = 0,1)$ and are SI in $\underline{C}$, $L(A_0) \times N(A_0) \cap E(A_0) \neq \emptyset$ and $(n,l) \in E(A_i) \Leftrightarrow (l,n) \notin E(A_i)$ for any $(n,l) \in L(A_i) \times N(A_i)$ then $L(A_i) \times N(A_i) \subseteq E(A_i)$ and $N(A_i)$ is an antireflexive tournament. Moreover, $L(A_i) \hookrightarrow$ (Denote the class of graphs satisfying these conditions by $\underline{T}_2$.)

<u>Proof.</u> According to 2.30, $L(A_i)$ is symmetric. By 2.2, $N(A_i)$ is an antireflexive tournament. Since $= \wedge$ there is $L(A_i) \times N(A_i) \subseteq E(A_i)$. Lemma 2.12 implies that A_i cannot contain H_1 as an induced subgraph. Hence, any LL-edge of A_i is . Since

$\hookrightarrow \times$

there is $L(A_i) \hookrightarrow$.

<u>2.32.Proposition.</u> If A and are SI in $\underline{C}$, $L(A) \times N(A) \cap E(A) \neq \emptyset$ and $N(A) \times L(A) \cap E(A) = \emptyset$ ($L(A) \times N(A) \cap E(A) = \emptyset$ and $N(A) \times L(A) \cap E(A) \neq \emptyset$, resp.) then $L(A) \times N(A) \subseteq E(A)$ ($N(A) \times L(A) \subseteq E(A)$ resp.) and either $N(A) = K_n$ for some n or $N(A)$ is an antireflexive tournament and $L(A) \hookrightarrow$. (Denote the class of graphs satisfying these conditions by $\underline{K}_2^{\downarrow} \cup \underline{T}_2^{\downarrow}$ ($\underline{K}_2^{\uparrow} \cup \underline{T}_2^{\uparrow}$ resp.).)

<u>Proof.</u> By 1.7 and 2.3, $N(A)$ is either complete or a tournament. Since $= \wedge = \wedge$, there is $L(A) \times N(A) \subseteq E(A)$ ($N(A) \times L(A) \subseteq E(A)$ resp.). Lemma 2.12 implies that A cannot contain neither H_5 nor H_6 as an induced subgraph. Hence, any LL-edge of A is and $L(A) \hookrightarrow$.

<u>2.33.Proposition.</u> If $A_i (i = 0,1)$ are SI in $\underline{C}$, $\circlearrowleft \; \circlearrowleft \longleftrightarrow A_o$, $\circlearrowleft\!\longleftrightarrow \; \longleftrightarrow A_o$, $(L(A_o) \times N(A_o) \cup N(A_o) \times L(A_o)) \cap E(A_o) \neq E(A_o)$, $|\{(l,n),(n,l)\} \cap E(A_i)| \geq 1$ for any $(l,n) \in L(A_i) \times N(A_i)$, then $|N(A_i)| \leq 1$, $|L(A_i)| \leq 5$, $L(A_i)$ is symmetric and for any $n \in N(A_i)$ and $l', l'' \in L(A_i)$ such that $(l', l'') \in E(A_i)$ there does not hold both $(l',n) \in E(A_i) \Leftrightarrow (l'',n) \in E(A_i)$ and $(n,l') \in E(A_i) \Leftrightarrow$ $\Leftrightarrow (n,l'') \in E(A_i)$. (Denote the corresponding class of graphs by $\underline{R}$.)

<u>Proof.</u> By 2.30, $L(A_i)$ are symmetric. By 2.7, $|L(A_i)| \leq 5$. Lemma 2.4 implies that $|N(A_i)| \leq 1$. Lemma 2.12 - which implies that A_i cannot contain H_1, H_2, H_3 as induced subgraphs - finishes the proof.

<u>2.34.Proposition.</u> If A and $\circlearrowleft \; \circlearrowleft$ are SI in $\underline{C}$ and if for any $(l,n) \in L(A) \times N(A)$ there is $(l,n) \in E(A) \Leftrightarrow (n,l) \notin E(A)$ then $L(A)$ is symmetric, $|L(A)| \leq 5$, $N(A) = K_n$ for some n and for any $l', l'' \in L(A)$ such that $(l', l'') \in E(A)$ there is $(l',n) \in E(A) \Leftrightarrow$ $\Leftrightarrow (n,l'') \in E(A)$ for any $n \in N(A)$. (Denote the corresponding class of graphs by $\underline{S}$.)

<u>Proof.</u> By 2.30, $L(A)$ is symmetric. By 2.7, $|L(A)| \leq 5$. Lemma 2.5 implies that $N(A)$ is a tournament and $\circlearrowleft \; \circ$ is not an induced subgraph of A. By 2.12, for any $n \in N(A)$ and $l', l'' \in L(A)$ such that $(l', l'') \in E(A)$ there is $(l',n) \in E(A) \Leftrightarrow (n,l'') \in E(A)$.

<u>3. Characterization Theorem</u>

Now, we can prove the following :

<u>3.1. Theorem.</u> Let $\underline{C}$ be a productive hereditary class of graphs. Then $\underline{C}$ is HSI iff either $\underline{C} = \underline{Set}$ or $\underline{C} = SP(\underline{D})$ where $\underline{D}$ satisfies the following conditions :

(i) $\underline{D} \subseteq \underline{K} \cup \underline{K}' \cup \underline{K}_o \cup \underline{K}'_o$

(ii) $\underline{D} \subseteq \underline{K} \cup \{K'_3, A_4\} \cup \underline{K}_o \cup \{(K'_3)_o, (A_4)_o\}$

(iii) $\underline{D} \subseteq \underline{K} \cup \underline{L}^+ \cup \underline{T} \cup \underline{K}_o \cup \underline{L}^+_o \cup \underline{T}_o$

(iv) $\underline{D} \subseteq \underline{K} \cup \underline{L}^- \cup \underline{T} \cup \underline{K}_o \cup \underline{L}^-_o \cup \underline{T}_o$

(v) $\underline{D} \subseteq \underline{K} \cup \underline{U} \cup \underline{K}_o \cup \underline{U}_o$

(vi) $\underline{D} \subseteq \underline{K} \cup \underline{V} \cup \underline{K}_o \cup \underline{V}_o$

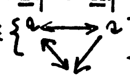
(vii) $\underline{D} \subseteq \underline{K} \cup \underline{W} \cup \underline{K}_o \cup \underline{W}_o$

(viii) $\underline{D} \subseteq \underline{Sym}_5 \cup \{G_5\} \cup \underline{Ref}_9$

(ix) $\underline{D} \subseteq \{G_1, G_3, G_4\} \cup \underline{T}_1^{\updownarrow}$

(x) $\underline{D} \subseteq \underline{T}_1^{\downarrow} \cup \underline{K}_1^{\downarrow}$

(xi) $\underline{D} \subseteq \underline{T}_1^{\uparrow} \cup \underline{K}_1^{\uparrow}$

(xii) $\underline{D} \subseteq \{$  $\}$

(xiii) $\underline{D} \le \{$  $\}$

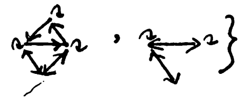
(xiv) $\underline{D} \le \underline{L}$

(xv) $\underline{D} \le \underline{M}$

(xvi) $\underline{D} \le \underline{N}$

(xvii) $\underline{D} \le \underline{P}$

(xviii) $\underline{D} \le \underline{Q}$

(xix) $\underline{D} \le \{$  $\}$

(xx) $\underline{D} \le \underline{T}_2^{\updownarrow}$

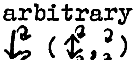
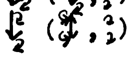
(xxi) $\underline{D} \le \underline{K}_2^{\downarrow} \cup \underline{T}_2^{\downarrow}$

(xxii) $\underline{D} \le \underline{K}_2^{\uparrow} \cup \underline{T}_2^{\uparrow}$

(xxiii) $\underline{D} \le \underline{R}$

(xxiv) $\underline{D} \le \underline{S}$

Proof. I. If $\underline{C}$ is HSI then we can consider the following cases :
a) All $\underline{C}$-graphs are antireflexive. Then by 1.7, $\underline{C}$ = SP($\underline{D}$) where $\underline{D}$ satisfies one of the conditions (i)-(vii).
b) All $\underline{C}$-graphs are reflexive. Then by 2.11, $\underline{C}$ = SP($\underline{D}$) where $\underline{D}$ satisfies (viii).
c) There exist $\underline{C}$-graphs with loops and also $\underline{C}$-graphs without loops. Then we can divide the proof of Theorem discussing possibilities for LL-, LN- and NL-edges of SI $\underline{C}$-graphs:

LL-edges	LN-and NL-edges	see	$\underline{C}$=SP($\underline{D}$) where D satisfies :
$\circlearrowleft\longleftrightarrow\circlearrowright$	$\circlearrowleft\longleftrightarrow$	2.13	(ix)
$\circlearrowleft\longleftrightarrow\circlearrowright$	$\circlearrowleft\longrightarrow$	2.14	(x)
$\circlearrowleft\longleftrightarrow\circlearrowright$	$\circlearrowleft\longleftarrow$	2.15	(xi)
$\circlearrowleft\longleftrightarrow\circlearrowright$	$\circlearrowleft\longleftrightarrow$ $\circlearrowleft\longrightarrow$	2.16	(xii)
$\circlearrowleft\longleftrightarrow\circlearrowright$	$\circlearrowleft\longleftrightarrow$ $\circlearrowleft\longleftarrow$	2.17	(xiii)
$\circlearrowleft\longleftrightarrow\circlearrowright$ or $\circlearrowleft\longrightarrow\circlearrowright$	$\circlearrowleft\longrightarrow$ $\circlearrowleft\longleftarrow$	2.18	(xiv)
arbitrary	$\circlearrowleft$ $\bullet$	2.19	(i)-(vii)
$\circlearrowleft\longleftrightarrow\circlearrowright$	$\circlearrowleft\longleftrightarrow$ $\circlearrowleft$ $\bullet$	2.20	(xv)
arbitrary	$\circlearrowleft\longrightarrow$ $\circlearrowleft$ $\bullet$	2.21	(xvi)
arbitrary	$\circlearrowleft\longleftarrow$ $\circlearrowleft$ $\bullet$	2.22	(xvii)
arbitrary	$\circlearrowleft\longleftrightarrow$ $\circlearrowleft\longrightarrow$ $\circlearrowleft\longleftarrow$	2.1	contradiction
arbitrary	$\circlearrowleft\longrightarrow$ $\circlearrowleft\longleftarrow$ $\circlearrowleft$ $\bullet$	2.23	contradiction
arbitrary	$\circlearrowleft\longleftrightarrow$ $\circlearrowleft\longrightarrow$ $\circlearrowleft$ $\bullet$	2.24	contradiction
arbitrary	$\circlearrowleft\longleftrightarrow$ $\circlearrowleft\longleftarrow$ $\circlearrowleft$ $\bullet$	2.24	contradiction
	$\circlearrowleft\longleftrightarrow$	2.25	(xviii)
	$\circlearrowleft\longrightarrow$	2.27	contradiction

$\downarrow^{2}_{5}(\updownarrow^{2}_{5}, {}^{2}_{5})$		${}^{2}\!\longleftarrow$	2.27	contradiction
$\downarrow^{2}_{2}(\updownarrow^{2}_{2}, {}^{2}_{2})$	${}^{2}\!\longleftrightarrow$	${}^{2}\!\longrightarrow$	2.28	contradiction
$\downarrow^{2}_{2}(\updownarrow^{2}_{2}, {}^{2}_{2})$	${}^{2}\!\longleftrightarrow$	${}^{2}\!\longleftarrow$	2.28	contradiction
$\downarrow^{2}_{2}(\updownarrow^{2}_{2}, {}^{2}_{2})$	${}^{2}\!\longleftrightarrow$	${}^{2}\ \cdot$	2.29	(xix)
${}^{2}_{2}(\updownarrow^{2}_{2}, \downarrow^{2}_{2})$	${}^{2}\!\longleftrightarrow$	$({}^{2}\ \cdot)$	2.31	(xx)
${}^{2}_{2}(\updownarrow^{2}_{2}, \downarrow^{2}_{2})$	${}^{2}\!\longrightarrow$	$({}^{2}\ \cdot)$	2.32	(xxi)
${}^{2}_{2}(\updownarrow^{2}_{2}, \downarrow^{2}_{2})$	${}^{2}\!\longleftarrow$	$({}^{2}\ \cdot)$	2.32	(xxii)
${}^{2}_{2}(\updownarrow^{2}_{2}, \downarrow^{2}_{2})$	${}^{2}\!\longleftrightarrow$	${}^{2}\!\longrightarrow$ $({}^{2}\!\longleftarrow)$	2.33	(xxiii)
${}^{2}_{2}(\updownarrow^{2}_{2}, \downarrow^{2}_{2})$	${}^{2}\!\longrightarrow$	${}^{2}\!\longleftarrow$	2.34	(xxiv)

II. One can check that systems satisfying one of the conditions (i)-(xxiv) are hereditary. Hence, systems SP($\underline{D}$) are HSI.

REFERENCES

[1] NEŠETŘIL J. - PULTR A. "On classes of relations and graphs determined by subobjects and factorobjects", Discr. Math. 22 (1978), 287-300.

[2] PULTR A. "On productive classes of graphs determined by prohibiting given subgraphs", Colloq. Math. Soc. J. Bolyai 18 (1976), 805-819.

[3] PULTR A. - VINÁREK J. "Productive classes and subdirect irreducibility, in particular for graphs", Discr. Math. 20 (1977), 159-176.

[4] VINÁREK J. "Remarks on dimensions of graphs", Abstracta Eighth Winter School on Abstract Analysis, Praha 1980, 180-193.

[5] VINÁREK J. "Hereditary subdirect irreducibility in graphs", Acta Univ. Carolinae - Math. et Phys. 25, 2 (1984), 3-10.

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