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NATURAL OPERATIONS WITH SECOND ORDER JETS

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Adopting the categorical point of view, we observe that several differential geometric operations can be interpreted as natural transformations of the corresponding functors. To preserve the usual geometrical terminology, we also say that such a natural transformation is a natural operation on the geometric objects in question. In the present paper we use such a systematic approach to determine all natural operations with the second order holonomic, semi-holonomic and non-holonomic jets, [1]. Our starting point have been two geometric operations with the semi-holonomic 2-jets: the canonical involution defined by Pradines, [5], and the difference tensor introduced by the first author, [3]. Interpreting the construction of jets as a functor on the product category $\mathcal{M}_n \times \mathcal{M}$, we first deduce analytically that all natural transformations of the semi-holonomic second order jet functor into itself are generated by those two operations in a simple way. Our analytical procedure yields that the only natural operations on the classical (i.e. holonomic) 2-jets are the identity and the so-called contraction, which transforms every 2-jet into the 2-jet of the related constant map. Then we determine all natural transformations of the non-holonomic second order jet functor into itself. An interesting consequence of the latter result is that there is no natural holonomization of the non-holonomic 2-jets except contraction. - All manifolds and maps are assumed to be infinitely differentiable.

1. Given an arbitrary fibred manifold $Y \to X$, its first prolongation J^1Y means the space of all 1-jets of the local sections of Y. The first prolongation of the fibred manifold $J^1Y \to X$ is called the second non-holonomic prolongation $\tilde{J}^2Y = J^1(J^1Y)$, [1]. Let J^2Y be the second (holonomic) prolongation of Y, i.e. the space of all 2-jets

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of the local sections of Y. We have a canonical inclusion $J^2Y \subset \tilde{J}^2Y$, $j^2_x s \mapsto j^1_x(j^1 s)$ for every local section s of Y and every $x \in X$. Let $\beta : J^1Y \to Y$ and $\beta_1 : \tilde{J}^2Y \to J^1Y$ be the target jet projections. The map $\beta : J^1Y \to Y$ is extended into $J^1\beta : \tilde{J}^2Y \to J^1Y$, $J^1\beta(j^1_x s) = j^1_x(\beta \circ s)$. This is another projection of $\tilde{J}^2Y$ into J^1Y. The second semiholonomic prolongation $\bar{J}^2Y$ is the subset of all $A \in \tilde{J}^2Y$ satisfying $\beta_1(A) = J^1\beta(A)$. Obviously, it holds $J^2Y \subset \bar{J}^2Y$.

Having two manifolds M, N, the space $J^2(M,N)$ of all 2-jets of M into N coincides with the second holonomic prolongation of the product fibred manifold $M \times N \to M$. The second non-holonomic or semi-holonomic prolongation of $M \times N \to M$ is said to be the space $\tilde{J}^2(M,N)$ of all non-holonomic 2-jets of M into N or the space $\bar{J}^2(M,N)$ of all semi-holonomic 2-jets of M into N, respectively. In the first order, the identification $J^1_x(M,N)_y \approx \mathrm{Hom}(T_xM, T_yN)$ of the set of all 1-jets of M into N with source x and target y with the set of all linear maps of the tangent space T_xM into T_yN is well known. In the second order, if we consider the iterated tangent bundles TTM and TTN, every $A \in \tilde{J}^2_x(M,N)_y$ can be interpreted as a map $\mu A : TT_xM \to TT_yN$ as follows. We have $A = j^1_x s$, where s is a local section of $J^1(M,N) \to M$. If we interpret every s(u) as a map $S_u : T_uM \to TN$, we obtain a local map S of TM into TN. Then we define μA as the restriction of the tangent map TS to TT_xM. If $B \in \tilde{J}^2_y(N,P)_z$ is another non-holonomic 2-jet, the composition $B \circ A \in \tilde{J}^2_x(M,P)_z$ can be defined by

$$(1) \qquad \mu(B \circ A) = (\mu B) \circ (\mu A)$$

The correctness of this formula follows from the coordinate expressions below.

If x^i or y^p are some local coordinates on M or N, then the induced coordinates on $J^1(M,N)$ will be denoted by $a^p_i = \partial y^p/\partial x^i$. Every local section of $J^1(M,N) \to M$ is given by some functions $y^p(x)$, $a^p_i(x)$, so that the induced coordinates on $\tilde{J}^2(M,N)$ are $a^p_{oi} = \partial y^p/\partial x^i$, $a^p_{ij} = \partial a^p_i/\partial x^j$. One finds easily that a jet $A = (x^i, y^p, a^p_i, a^p_{oi}, a^p_{ij})$ is semi-holonomic, if and only if $a^p_i = a^p_{oi}$, and a holonomic jet is further characterized by $a^p_{ij} = a^p_{ji}$.

Let ξ^i be the additional coordinates on TM corresponding to x^i and dx^i, $d\xi^i$ be the induced coordinates on TTM. A local section $y^p(x)$, $a^p_i(x)$ of $J^1(M,N)$ determines a local map $y^p = y^p(x)$, $\eta^p = a^p_i(x)\xi^i$ of TM into TN. Evaluating the tangent map, we find that the coordinate expression of $\mu A : TT_xM \to TT_yN$ is

(2) $\quad \eta^p = a^p_i \xi^i,\ dy^p = a^p_{oi} dx^i,\ d\eta^p = a^p_{ij} \xi^i dx^j + a^p_i d\xi^i$

If we consider μB

(3) $\quad \zeta^s = b^s_p \eta^p,\ dz^s = b^s_{op} dy^p,\ d\zeta^s = b^s_{pq} \eta^p dy^q + b^s_p d\eta^p$

and compose (2) with (3), we verify that (1) is a correct formula. In the same time, we obtain the coordinates c^s_i, c^s_{oi}, c^s_{ij} of $B \circ A$ in the form

(4) $\quad c^s_i = b^s_p a^p_i,\ c^s_{oi} = b^s_{op} a^p_{oi},\ c^s_{ij} = b^s_{pq} a^p_i a^q_{oj} + b^s_p a^p_{ij}$

In particular, this proves that our definition of the composition of the non-holonomic 2-jets coincides with that one introduced by Ehresmann, [1]. Clearly, the composition of two semi-holonomic or holonomic jets is semi-holonomic or holonomic, respectively.

2. Pradines, [5], has introduced an involutory map $i : \bar{J}^2 \to \bar{J}^2$ as follows. Consider the canonical involution of the second tangent bundle $i_x : TT_xM \to TT_xM$ or $i_y : TT_yN \to TT_yN$. By [2], for every $A \in \bar{J}^2_x(M,N)_y$ there exists a unique $iA \in \bar{J}^2_x(M,N)_y$ satisfying $\mu(iA) = i_y \circ \mu A \circ i_x$. The coordinate effect of i consists in the exchange of both subscripts of a^p_{ij}. Using the well-known fact that $J^1Y \to Y$ is an affine bundle for every fibred manifold $Y \to X$, we find easily that $\bar{J}^2(M,N) \to J^1(M,N)$ is an affine bundle, the associated vector bundle of which is $TN \otimes T^*M \otimes T^*M$. Hence for every $t \in R$ we can intrinsically define

(5) $\quad tA + (1-t)iA \in \bar{J}^2_x(M,N)_y$

On the other hand, two points A and iA of the same affine space determine a vector $\overrightarrow{(iA)A}$ of the associated vector space. Its coordinate expression is $a^p_{ij} - a^p_{ji}$, so that it belongs into $T_yN \otimes \otimes \Lambda^2 T^*M \subset T_yN \otimes T^*M \otimes T^*M$. Hence for every $k \in R$ we obtain

(6) $\quad k\overrightarrow{(iA)A} \in T_yN \otimes \Lambda^2 T^*_xM$

For $k = 1/2$, we get the difference tensor of A introduced by the first author, [3]. Further, the kernel of the projection $\beta_1 : \bar{J}^2_x(M,N)_y \to J^1_x(M,N)_y$ is also identified with $T_yN \otimes T^*M \otimes T^*M$.

Hence (6) can be interpreted as a map of $\bar{J}^2_x(M,N)_y$ into itself. Thus, (5) and (6) are two one-parameter families of natural operations with semi-holonomic 2-jets.

3. Let $\mathcal{M}$ be the category of all manifolds and all smooth maps and $\mathcal{M}_n$ be the category of n-dimensional manifolds and their local diffeomorphisms. We can consider $\tilde{J}^2$ as a functor on the product category $\mathcal{M}_n \times \mathcal{M}$ transforming every pair of manifolds (M,N) into $\tilde{J}^2(M,N)$ and every local diffeomorphism $f : M_1 \to M_2$ and every smooth map $g : N_1 \to N_2$ into $\tilde{J}^2(f,g) : \tilde{J}^2(M_1,N_1) \to \tilde{J}^2(M_2,N_2)$ defined by

$$(7) \qquad \tilde{J}^2(f,g)(A) = (j^2_y g) \circ A \circ j^2_{f(x)}(f^{-1})$$

for every $A \in \tilde{J}^2_x(M,N)_y$, where the inverse diffeomorphism f^{-1} is taken locally. The same formula defines $\bar{J}^2$ or J^2 as a functor on $\mathcal{M}_n \times \mathcal{M}$. If we consider a natural transformation t of any two of these functors, we always assume that every $t_{M,N}$ is a projectable map over the identity of $M \times N$.

<u>Proposition</u> 1. *All natural tranformations $\bar{J}^2 \to \bar{J}^2$ form the two one-parameters families* (5) *and* (6).

Proof. Consider first the subcategory $\mathcal{M}_n \times \mathcal{M}_m \subset \mathcal{M}_n \times \mathcal{M}$. Let L^2_n or L^2_m be the group of all invertible 2-jets on R^n or R^m with source and target 0 and $\bar{L}^2_{n,m} = \bar{J}^2_o(R^n,R^m)_o$. Similarly to Palais and Terng, [4], (7) induces the following left action of the group $L^2_n \times L^2_m$ on $\bar{L}^2_{n,m}$

$$(8) \qquad \begin{aligned} \bar{c}^p_i &= b^p_q c^q_j a^j_i \\ \bar{c}^p_{ij} &= b^p_{qr} c^q_k c^r_\ell a^k_i a^\ell_j + b^p_q c^q_{k\ell} a^k_i a^\ell_j + b^p_q c^q_k a^k_{ij}, \end{aligned}$$

where (b^p_q, b^p_{qr}) are the coordinates of an element $b \in L^2_m$ and, to simplify the evaluations, (a^i_j, a^i_{jk}) are the coordinates ot the inverse element of an $a \in L^2_n$. In particular, a^i_{jk} and b^p_{qr} are symmetric in both subscripts, while c^p_{ij} are arbitrary quantities. By naturality, the restriction of every natural transformation $\bar{J}^2 \to \bar{J}^2$ to $\bar{L}^2_{n,m}$ is an $L^2_n \times L^2_m$-equivariant map of $\bar{L}^2_{n,m}$ into itself. By definition, such a map $f = (f^p_i, f^p_{ij}) : \bar{L}^2_{n,m} \to \bar{L}^2_{n,m}$ satisfies

$$(9) \qquad b^p_q f^q_j(c_1,c_2) a^j_i = f^p_i(b^p_q c^q_j a^j_i,\ b^p_{qr} c^q_k c^r_\ell a^k_i a^\ell_j + b^p_q c^q_{k\ell} a^k_i a^\ell_j + b^p_q c^q_k a^k_{ij}),$$

$$(10)\qquad b^p_{qr}f^q_k(c_1,c_2)f^r_\ell(c_1,c_2)a^k_i a^\ell_j + b^p_q f^q_{k\ell}(c_1,c_2)a^k_i a^\ell_j +$$

$$+ b^p_q f^q_k(c_1,c_2)a^k_{ij} = f^p_{ij}(b^p_q c^q_j a^j_i,\ b^p_{qr}c^q_k c^r_\ell a^k_i a^\ell_j +$$

$$+ b^p_q c^q_{k\ell}a^k_i a^\ell_j + b^p_q c^q_k a^k_{ij})$$

where we use $c_1 = c^p_i$, $c_2 = c^p_{ij}$ as abbreviation. To solve (9) and (10), we shall discuss certain subgroups in $L^2_n \times L^2_m$. We have injections $L^1_n \to L^2_n$, $(a^i_j) \mapsto (a^i_j, 0)$ and $L^1_m \to L^2_m$, $(b^p_q) \to (b^p_q, 0)$. Consider first the homotheties of L^1_n, i.e. put $a^i_j = k\delta^i_j$, $a^i_{jk} = 0$, $b^p_q = \delta^p_q$, $b^p_{qr} = 0$. Then (9) yields

$$(11)\qquad kf^p_i(c_1,c_2) = f^p_i(kc_1,k^2c_2)$$

Hence f^p_i are globally defined smooth functions homogeneous of degree 1 in c_1 and of degree 1/2 in c_2, so that f^p_i must be linear in c_1 and independent on c_2, see e.g. [2]. Using the same substitution in (10), we find

$$(12)\qquad k^2 f^p_{ij}(c_1,c_2) = f^p_{ij}(kc_1,k^2c_2)$$

If we consider the homotheties of L^1_m, i.e. if we put $a^i_j = \delta^i_j$, $a^i_{jk} = 0$, $b^p_q = k\delta^p_q$, $b^p_{qr} = 0$, we obtain

$$(13)\qquad kf^p_{ij}(c_1,c_2) = f^p_{ij}(kc_1,kc_2)$$

By homogeneity, (12) and (13) imply that f^p_{ij} must be independent on c_1 and linear in c_2.

Consider now the subgroup $L^1_n \times L^1_m \subset L^2_n \times L^2_m$. Then (9) implies that f^p_i represent a linear $L^1_n \times L^1_m$-equivariant map of $R^n \otimes R^{m*}$ into itself. It is easy to find that every such a map is of the form

$$(14)\qquad \bar{c}^p_i = kc^p_i,\qquad k \in R.$$

Analogously, f^p_{ij} represent a linear $L^1_n \times L^1_m$-equivariant map of $R^n \otimes R^{m*} \otimes R^{m*}$ into itself. A simple result by Vadovičová is, [6], that every such a map is of the form

$$(15)\qquad \bar{c}^p_{ij} = ac^p_{ij} + bc^p_{ji},\qquad a,b \in R.$$

Consider further the kernels of the jet projections $L^2_n \to L^1_n$, and $L^2_m \to L^1_m$, which are characterized by $a^i_j = \delta^i_j$, $b^p_q = \delta^p_q$. Then (10) yields

$$(16) \qquad k^2 b^p_{qr} c^q_i c^r_j + k c^p_k a^k_{ij} = (a + b) b^p_{qr} c^q_i c^r_j + (a + b) c^p_k a^k_{ij}$$

This implies $k = a + b$, $k^2 = a + b$, so that $k^2 = k$ and we have two possibilities $k = 1$ or $k = 0$. In first case, we obtain

$$(17) \qquad \bar{c}^p_i = c^p_i, \qquad \bar{c}^p_{ij} = t c^p_{ij} + (1 - t) c^p_{ji}$$

which is the coordinate expression of (5). In the second case, we get

$$(18) \qquad \bar{c}^p_i = 0, \qquad \bar{c}^p_{ij} = k(c^p_{ij} - c^p_{ji})$$

which is the coordinate expression of (6).

For the whole category $\mathcal{M}_n \times \mathcal{M}$ we have one of the transformations (5) and (6) on each subcategory $\mathcal{M}_n \times \mathcal{M}_m$ for every integer m. Taking into account the canonical injection $i : R^m \to R^{m+k}$, $x \mapsto (x,0)$, our natural tranformations must commute with $\bar{J}^2(\mathrm{id},i) : \bar{J}^2(R^n,R^m) \to \bar{J}^2(R^n,R^{m+k})$ for all k. This implies directly that we have the same transformation from (5) or (6) for all m. This completes the proof of Proposition 1.

We remark that we have also determined all natural transformations of the holonomic second order functor J^2 into itself. The subspace $J^2(M,N) \subset \bar{J}^2(M,N)$ being characterized by $c^p_{ij} = c^p_{ji}$, we can add this symmetry condition at the very end of the previous consideration. Then (17) is reduced to the identity and (18) represents the so-called contraction transforming every 2-jet of M into N with source x and target y into the 2-jet at x of the constant map of M into y. This proves

Corollary 1. *The only natural transformations $J^2 \to J^2$ are the identity and the contraction.*

4. Quite analogously, we can determine all natural transformations of the non-holonomic second order jet functor $\tilde{J}^2$ into itself. Consider $\tilde{L}^2_{n,m} = \tilde{J}^2_0(R^n,R^m)_0$ with coordinates c^p_i, c^p_{oi}, c^p_{ij}. By (4), (7) induces the following left action of $L^2_n \times L^2_m$ on $\tilde{L}^2_{n,m}$

$$\bar{c}^p_i = b^p_q c^q_j a^j_i, \qquad \bar{c}^p_{oi} = b^p_q c^q_{oj} a^j_i,$$
(19)
$$\bar{c}^p_{ij} = b^p_{qr} c^q_k c^r_{o\ell} a^k_i a^\ell_j + b^p_q c^q_{k\ell} a^k_i a^\ell_j + b^p_q c^q_k a^k_{ij}$$

By analogous evaluations as in the proof of Proposition 1, which we omit here, we deduce

Proposition 2. *All natural transformations $\tilde{J}^2 \to \tilde{J}^2$ form the following two families*

(20) $$\bar{c}^p_i = kc^p_i,\ \bar{c}^p_{oi} = c^p_{oi},\ \bar{c}^p_{ij} = kc^p_{ij}, \qquad k \in R,$$

(21) $$\bar{c}^p_i = 0,\ \bar{c}^p_{oi} = ac^p_i + bc^p_{oi},\ \bar{c}^p_{ij} = 0,\ a,b \in R.$$

If we want to determine all natural transformations $\tilde{J}^2 \to \bar{J}^2$, we have to require $\bar{c}^p_i = \bar{c}^p_{oi}$ identically. This is impossible in (20) and implies a = b = 0 in (21). This rededuces the following result by Vadovičová, [6].

Corollary 2. *The only natural transformation $\tilde{J}^2 \to \bar{J}^2$ is the contraction.*

It is easy to construct geometrically the natural transformations (20) and (21). Since $\beta : J^1(M,N) \to M \times N$ is a vector bundle, $J^1\beta : \tilde{J}^2(M,N) \to J^1(M,N)$ is also a vector bundle. Clearly, (20) expresses the multiplication by a scalar on this vector bundle. Further, the zero section of $J^1\beta : \tilde{J}^2(M,N) \to J^1(M,N)$ is an injection $i : J^1(M,N) \to \tilde{J}^2(M,N)$. We have two geometric projections β_1, $J^1\beta : \tilde{J}^2(M,N) \to J^1(M,N)$ and (21) transforms every $A \in \tilde{J}^2(M,N)$ into the injection $i(a\beta_1(A) + bJ^1\beta(A))$ of a linear combination of the vectors $\beta_1 A$ and $J^1\beta(A)$ with prescribed coeficients $a,b \in R$.

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