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CONVEXITY IN COMBINATORIAL STRUCTURES (*)

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SUMMARY. The recent (since 1968) combinatorial developments of abstract convexity are surveyed. The combinatorial properties of convex sets (Helly property, Eckhoff's partition problem ...) are considered in the general setting of finitary closure systems ("Convexity Spaces" or "Alignements"). In ordered sets, in tree-like structures and in combinatorial structures inspired by Geometry (e.g. "oriented matroids") there are natural definitions of convex sets : an axiomatic common background is the theory of "convex geometries" (or anti-exchange convexities") of Edelman and Jamison or dually the theory of "shelling structures" (or "APS-greedoids")of Korte and Lovász. Convexity in graphs recently appeared of independent interest (contraction into complete graphs, universal properties of geodesic convexity ...)

I - INTRODUCTION

Properly speaking, convexity is not a mathematical theory, but rather a notional domain where five basic concepts operate : *betweenness* ($\to$ medians, convex dependance), *algebraicness* ($\to$ convex hull operator, dimension), *separation* ($\to$ hemispaces, copoints), *connectedness* and *optimization* ($\to$ extremal points, face-lattices, duality). Thus, convexity is present in almost all constituted combinatorial theories: Finite Geometry (namely oriented matroids) of course, but also Graphs, Set Systems, Ordered Sets, Extremal Set Theory, Enumeration Problems, Designs and Combinatorial Optimization.

The first aspect of the interplay between Combinatorics and Convexity is the use of combinatorial methods when studying ordinary convexity (i.e. standard convexity in Euclidean spaces $\mathbb{R}^d$) : combinatorial properties of families of convex sets, facial structure of polytopes.

(*) This paper is in final form and no version of it will be submitted for publication elsewhere.

The second aspect is the use of convexity concepts in combinatorial research. The title of the present article refers to a definite part of the second approach : we restrict our attention to those combinatorial structures where the notions relative to convexity are introduced by *intrinsic* means, or to put it differently, where the definition of the convex sets only depends on the combinatorial structure itself. Nevertheless, since most convex approaches of combinatorial structures are motivated and inspired by abstract generalization of Euclidean space situations, we briefly recall in Section 2 the major *combinatorial themes* of convexity theory in real vector spaces. But convexities that arise in an extrinsic way in Combinatorics are not considered in this survey. For instance the reader who is interested in polyhedral combinatorics (discrete optimization, linear programming, packing and covering problems) is refered to the classical literature of this important domain : Rockefellar [1970], Stoer and Witzgall [1970], Lawler [1976], Hammer et al. [1979], Schrijver [1979], Grötschel et al. [1981], Lovász [1984] Yemelichev et al. [1984], Karmarkar [1985].

The general context, a sort of "*Prototheory of Convexity*" is introduced in Section 3. The interest of an independent development of an abstract and axiomatic convexity is underlined by recent results that concern *combinatorial parameters* attached to convexity spaces (Section 4) and by Eckhoff's partition problem. A typically interesting direction is the theory of *Convex Geometries* (or anti-exchange convexities) where every convex set is the convex hull of its extreme points : this theory, described in Section 5, can be viewed as a part of the Theory of Greedoids (a relaxation of Matroids) In Section 6, we deal with *convex sets in graphs*. Graphical convexities appear as a very important example of finite convexity spaces with possible applications both in Graph Theory (e.g. Hadwiger's conjecture) and in Abstract Convexity (universalness of geodesic convexity). The last three sections present more specialized works on combinatorial structures where convexity appears in a very natural way : *oriented matroids* (Section 7) with facial structure and separation problems, *ordered sets* (Section 8) with betweeness or medians, *tree-like structures* (Section 9) with coherent convexities.

Throughout the paper, strict inclusion is denoted by $\subset$ and set difference by $\setminus$. Square brackets indicate the year of a reference issue. Open problems are numbered separately. A notation of the form (3.7) refer to the seventh paragraph of Section 3.

2. FROM EUCLIDEAN CONVEXITY ...

(2.1) Combinatorial properties of ordinary convex sets were progressively discovered when mathematicians tries to find geometrical proofs of properties of convex sets. So, in a parallel direction with geometric approaches by Brünn [1887], Bonnesen [1929], Alexandroff [1950], the combinatorial works follow one another : Carathéodory [1907], Radon [1921], Helly [1923], Kakutani [1937]. Further ideas and references may be found in usual books on Convexity and Combinatorial Geometry : Bonnesen and Fenchel [1934], Yaglom and Boltianski [1956, 1961], Hadwiger and Debrunner [1964] (wonderful !), Valentine [1964] Boltianski and Soltan [1978]. Interesting survey papers are : Dantzer et al. [1963], Eckhoff [1979,1986].

(2.2) The second major theme in Combinatorial Convexity is the following problem, attributed to Baker.

PROBLEM 1. (Baker) Characterize the nerves of finite families of convex sets in $\mathbb{R}^d$.

The nerve of a finite family $(C_i)_{i \in I}$ of convex sets in $\mathbb{R}^d$ is the abstract simplicial complex (*) whose vertex set is I and simplices (= faces) are those subsets $J \subseteq I$ such that $\bigcap (C_i ; i \in J) \neq \emptyset$. An important stage towards a solution of Baker's problem was recently reached : Kalai [1984], using technics of exterior algebra, characterized the f-vectors (**) associated to the nerves of families of convex sets — thus solving a conjecture due to Eckhoff —

A special unsolved part of Baker's problem deserves mention :

PROBLEM 2 (Wegner) Characterize the intersection graphs of families of convex sets in $\mathbb{R}^2$.

The characterization problem for lattices of Euclidean convex sets is closely related to Baker's problem : see Bennett [1974] and its references.

(2.3) At last, combinatorial Convexity includes the systematic examination of face lattices of convex bodies, espacilly of polytopes : McMullen [1971] solved the Upper Bound Conjecture (Motzkin) when

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(*) An *(abstract) simplicial complex* is a collection $\mathcal{K}$ of subsets — the *faces* of $\mathcal{K}$ — such that the properties $F \in \mathcal{K}$ and $F' \subseteq F$ imply $F' \in \mathcal{K}$. In Berge's terminology [1973,1986] a simplicial complex is a *hereditary hypergraph*.

(**) The *f-vector (face-vector)* of a cell complex $\mathcal{K}$ is $(f_0, f_1, \ldots, f_k, \ldots)$ where f_k is the number of k-dimensionnal cells of $\mathcal{K}$. In case of simplicial complexes, f_k is the number of $k+1$-elements simplices.

determining the maximum number of k-dimensional faces of real d-dimensional polytope with a given number of vertices. Stanley [1975] generalizes the result to shellable complexes. More, the characterizations of f-vectors of *simplicial* (*) polytopes was obtained by Stanley [1980] (necessity of McMullen's conditions) and Billera and Lee [1981] (sufficiency). Stanley intensively uses homological and commutative algebra.

(2.4) Evident similarities exist between Eckhoff's conditions for nerves (2.2) and McMullen's conditions for f-simplicial boundary complexes of polytopes (2.3). Attempts to find a common general setting (**) to both properties can also help for a solution to a nice conjecture due to Chvátal [1974a,b].

PROBLEM 3 (Chvátal) Let $\mathcal{K}$ *be a simplicial complex with at least two faces. Let* Δ *denote the maximum number of* $\mathcal{K}$*-faces that have a common vertex; Is it true that every family of* $\Delta+1$ *faces contains two disjoint faces ?*

Berge (see [1986]) conjectures a stronger property : *the line-graph* (***) *of the faces has chromatic number* Δ .

3. ... TO ABSTRACT CONVEXITY.

(3.1) How to do geometry with convexity ? This was the original motivation of pioneers of axiomatic convexity. The scheme was : *convexity* → *convex hull* → *segment* → *line* → *dimension* → *geometry*. The best achievements in this domain were obtained by Busemann [1955] , Prenowitz [1969], Bryant, Webster [1972,73,77], Cantwell [1974,76], Cantwell and Kay [1978], Prenowitz and Jantosciak [1979]. An axiomatic characterization of $\mathbb{R}^d$ by the means of axioms that only involve abstract convexities, "linearization problem" is possible : see Doignon [1976] and Whitfield and Yong [1981], where further references can be found.

(3.2) Motzkin [1951] was seemingly the first to advice an independent

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(*) i.e. in which every face is a simplex.

(**) I share Eckhoff's [1985] point of view : an homological interpretation of the f-vectors (or more precisely of the associated "h-vectors") of a wider class of simplicial complexes is highly desirable.

(***) The *line-graph* $L(\mathcal{F})$ (or *intersection graph*) of a family of sets $\mathcal{F}=(F_i)_{i\in I}$ has vertex set I : the vertices $i,j \in I$ are joined in $L(\mathcal{F})$ when $F_i \cap F_j \neq \emptyset$.

development of an axiomatic convexity in a very general abstract setting, namely in the theory of *algebraic closure systems* Cohn [1965], Birkhoff [1967], Graetzer [1968]. As a matter of fact, several approaches of what is now called *Abstract Convexity* pre-existed without any reference to Convexity : Moore [1910], Schmidt [1952,1953] Tarski [1930], Hammer [1963a]. The purely abstract point of view is now sustained by convergent problems and results. Recent general references on Abstract Convexity are Jamison [1972,1982] and Soltan [1984].

(3.3) CONVEXITY SPACES There are essentially two ways of defining a convex set : by intersection of "large basic convex sets" (for instance half-spaces in vector spaces), or by the property of being closed with respect to a certain family of finitary operators (for instance n-any operators of the form

$$x_1,\dots,x_n \in \mathbb{R}^d \to \sum_{i=1}^{n} \lambda_i x_i$$

where the λ_i's are non negative and sum to 1). This remark leads to the following definition : a set X , endowed with a set $\mathcal{C}$ of X-subsets forms a *convexity space* $(X, \mathcal{C})$ if the following axioms are satisfied :

(C_1) $\emptyset \in \mathcal{C}$, $X \in \mathcal{C}$

(C_2) $\mathcal{C}$ *is preserved under intersections*

(C_3) $\mathcal{C}$ *is preserved by nested union.*

Elements of X are *points*, members of $\mathcal{C}$ are *convex sets* or *convexes*. Set families that satisfy (C_1) and (C_2) are known as *Moore families* or *closure systems*. Families satisfying (C_3) are known as *inductive systems*. An alternative terminology for convexity spaces is *algebraic closure systems* (cf(1.3)), *alignements* [Jamison 1972], *geometries* [Wille 1970], *domain finite convexity structures* [Hammer 1963; Sierksma 1976].

(3.4) BASIS When infinite convexity spaces are considered, Choice Axiom is assumed. By Zorn's lemma, Axiom (C_3) implies the existence, for all $x \in X$, of a maximal convex set that does not contain x . Such a convex set is called a *copoint (relative to x)*. The set X and copoints form a *basis* $\mathcal{B} \subseteq \mathcal{C}$, i.e. every convex set is an intersection of members of $\mathcal{B}$. As easily seen, this set constitues the unique inclusion-minimal basis of $\mathcal{C}$. (cf. Soltan [1984])

(3.5) CONVEX HULL, OPERATORS. Let $\mathcal{C}$ be a Moore family (resp. a convexity) on a set X . The *closure* (resp. the *convex hull*) of a X-subset A , denoted by $\langle A\rangle_{\mathcal{C}}$ — or simply by $\langle A\rangle$ when no confusion can arise — is the intersection of all $\mathcal{C}$-members containing A .

Axiom (C_3) expresses the algebraic nature of the convex hull operator : for Moore families, this axiom is equivalent (Schmidt[1952], see Cohn[1981] p.95) (*), compare Hammer [1963a]) to the following property :

(CF) *If* $x \in \langle A \rangle$ *then* $x \in \langle F \rangle$ *for some finite* A*-subset* F .

Thus, every convexity $\mathcal{C}$ can be defined by a family of *algebraic operators "generating rules"* $\omega_i : X^n \to X$. A set $A \subseteq X$ is convex if and only if it is closed for all operators. This point of view allows recursive proofs.

(3.6) INTERVAL CONVEXITIES. A convexity is *n-ary* (or *n-generated*) if it can be defined by a family of n_i -ary operators with $n_i \leq n$ for all i . 2 -generated convexities are known as *Interval convexities*. Most usual convexities are interval convexities : Euclidean convexity in $\mathbb{R}^d$, geodesic convexity in metric spaces (graphs, Reimann manifolds ...). Although, the oriented matroid convexities (see section 7) are not interval convexities.

Intersection of convexities (see (3.8')) preserves n-arity. Hence *there exists a smallest n-ary convexity that contains a given convexity* $\mathcal{C}$ *on a set* X (Burris,[1968]) It may be defined as

$$\mathcal{C}^{(n)} = \{C \subseteq X \ ; \ A \subseteq C \ , \quad |A| \leq n \Rightarrow \langle A \rangle_{\mathcal{C}} \subseteq C\}$$

Burris [1972] has shown that every convexity space is isomorphic to an induced subspace (see (3.8)) for a definition) of an interval convexity space.

(3.7) METRIC CONVEXITY. In every metric space (X,d) one may consider an operator $D(x,y) = \{z \in X \ ; \ d(x,y) = d(x,z)+d(z,y)\}$. Any interval convexity which can be defined by a single operator of this form is called a metric convexity. To give an intrinsic characterization of metric convexities should not be difficult. See Busemann [1955] .

(3.8) MORPHISMS. Morphisms for convexity spaces are defined as for topological spaces : a *convex morphism* from a convexity space $(X,\mathcal{C})$ in another $(Y,\mathcal{D})$ is a mapping $\varphi : X \to Y$ such that $\varphi^{-1}(D) \in \mathcal{C}$ for every $D \in \mathcal{D}$. Endowed with these morphisms, the class of convexity spaces forms a category (or an "espèce de structure" in Bourbaki 's terminology [1966]). Initial and final objects in this category (~ "Structures dérivées") are easily defined. For instance the *product convexity space* of a family $(X_i, \mathcal{C}_i)$ $(i \in I)$ of convexity spaces is the convexity space $(X, \mathcal{C})$ where $X = \prod_i (X_i ; i \in I)$ and $\mathcal{C} = \{\prod(C_i ; i \in I) \ ; \ C_i \in \mathcal{C}_i\}$. The *intersection convexity space* is

(*)Cohn's definition for inductive system is uncorrect; replace by ours in the proof.

$(\bigcap(X_i; i \in I), \bigcap(\mathcal{C}_i; i \in I))$. The *convexity subspace* $(Y,\mathcal{C}_Y)$ of a convexity space $(X,\mathcal{C})$ *induced by* a subset $Y \subseteq X$ is defined by : $\mathcal{C}_Y = \{C \cap Y \; ; \; C \in \mathcal{C}\}$. Other derived convexities are considered below : (3.9) and (3.10) and in Degreef [1979,82], Sierksma [1981], Soltan [1984].

(3.9) MINORS. As in Matroid Theory (see Welsh [1976]), a very natural notion of minor space exists for convexity spaces (see Jamison [1982], compare Korte, Lovász [1984b]). Let $S = (X, \mathcal{C})$ be a convexity space and let $Y \subseteq X$. The *contracted space of* S *by* A , denoted by S/A is the convexity space $(X \setminus A \; , \; \mathcal{C}/A)$ with :

$$\mathcal{C}/A = \{C \setminus A \; ; \; A \subseteq C \text{ and } C \in \mathcal{C}.\}$$

The subspace $S \setminus A$ is the convexity subspace of S induced by $X \setminus A$ We have $S \setminus A = (X \setminus A \; , \; \mathcal{C} \setminus A)$ with :

$$\mathcal{C} \setminus A = \{C \setminus A \; ; \; C \in \mathcal{C}\} .$$

When A,B are disjoint subsets, we have $(S/A) \setminus B = (S \setminus B)/A$. A *minor* of S is every convexity space which is isomorphic to some induced subspace of a contracted space of S . If the flats of a matroids are viewed as convexes of a finite convexity structure, the notions "matroidal minor" and "convex minor" coincide. With this first point of view, matroids are exactly the finite convexity spaces with no minor isomorphic to the space $\mathbf{Q}_1^{(2)} = (\{1,2\},\{\{\emptyset\},\{1\},\{1,2\}\})$. Convex geometries (section 5) are another interesting class which is stable under taking minors.

(3.10) VARIETES. If we order by inclusion the set of all convexities on a given set X we obtain a complete lattice (Jamison,[1982]), in which the upper bound of a family of convexities is called the *joint* of the family. Most interesting classes of convexity spaces are preserve under the formation of joints. Jamison proposer to call *variety* a class **V** of convexity spaces that possesses the following properties :

(V_1) *Convexity spaces isomorphic to a number of* **V** *are in* **V** .

(V_2) *Subspaces of members of* **V** *are in* **V** .

(V_3) *If every finite subspace of a space* $(X, \mathcal{C})$ *is in* **V** *then* $(X, \mathcal{C})$ *is in* **V** .

When convexities $\mathcal{C}_i (i \in I)$ are taken in a variety **V** , the convex hull in their joint $\mathcal{J}$ has a simple expression ;

$$\langle A \rangle_{\mathcal{J}} = \bigcap_{i \in I} \langle A \rangle_{\mathcal{C}_i}$$

By (V_2), varieties can be characterized by excluded subspaces. A variety **V** is *finitely based* when **V** can be defined by a finite list of forbidden subspaces.

PROBLEM 4 (Jamison) Let V,W *be two finitely based varieties. Let* $\mathbf{V} \vee \mathbf{W}$ *be the class of all convexity spaces* $(X, \mathcal{C})$ *where* $\mathcal{C}$ *is a joint of a member of* V *with a member of* W (*). *Is* $\mathbf{V} \vee \mathbf{W}$ *finitely based ?*

4. COMBINATORIAL INVARIANTS

(4.1) Essential results in Axiomatic Convexity concern various parameters associated to combinatorial properties. Each of these parameters expresses a certain notion of dimension. All definition listed below are relative to a convexity space $(X, \mathcal{C})$.

(4.2) CONVEX DEPENDANCE : a set $A \subseteq X$ is said to be *free* if $a \notin \langle A \setminus a \rangle$, for every $a \in A$. The *rank* of $\mathcal{C}$ is the maximum cardinality (if finite) of a free set. The rank, denoted by *rank* $(\mathcal{C})$, equals the least integer k that fulfils the following property :

($\mathbf{F}_k$) *If* $C_1, \ldots, C_{k+1}$ *are convex sets of* $\mathcal{C}$ *, then one of them contains the intersection of the others.*

Real intervals of the real line form a convexity of rank 2 (see section 8). Convexities of rank n are obviously n-generated.

(4.3) HELLY PROPERTY :

(H_k) *If a finite family of convexes has an empty intersection, then this family contains at most* k *members with an empty intersection.*

Notice that (H_k) is a relaxation of (F_k) . The *Helly number* of denoted by $h(\mathcal{C})$ is the smallest integer k (if it exists) such that (H_k) holds. Ordinary convexity in $\mathbb{R}^d$ has Helly number $d+1$ (Helly, [1923]). Arithmetical progressions in $\mathbb{Z}$ satisfy (H_2) (Chinese Remainder Theorem). For further examples see Dantzer et al. [1963], Jamison [1982], Duchet, Quilliot [1986], Eckhoff [1986].

Berge, Duchet [1975] : $h(\mathcal{C}) = max\ (|A| \ ;\ A \subseteq X \ \text{ and } \bigcap_{a \in A} \langle A \setminus a \rangle = \emptyset)$;

We have also : $rank\ (\mathcal{C}) = max\ (h(\mathcal{C}_A) \ ;\ A \subseteq X)$.

Particular results on Helly number are given in other sections. Other results may be found in Sierksma [1975/76] and Soltan [1984].

(4.4) PARTITION PROPERTY

($\mathbf{P}_{k,n}$) *If* $(p_i)_{i \in I}$ *is a family of* $n = |I|$ *points, there exists a partition of* I *into* k *parts* $I_1 \ldots I_k$ *such that :*

$$\bigcap_{1 \leq \lambda \leq k} \{p_i \ ;\ i \in I_\lambda\} \neq \emptyset$$

Ordinary convexity in $\mathbb{R}^d$ has property ($\mathbf{P}_{k,(k-1)(d+1)+1}$) for $k,d \geq 1$ (Radon, [1921] for $k=2$, Birch [1960] for $d=2$, Tverberg

(*) It can be shown that $\mathbf{V} \vee \mathbf{W}$ is a variety.

[1966/82] for the general case). The *k-th partition number* denoted by $p_k(\mathcal{C})$ is the smallest integer n such that $(P_{k,n})$ holds. Number $p_2(\mathcal{C})$ is usually known as *Radon number* and is denoted by $r(\mathcal{C})$. An important inequality is $h(\mathcal{C}) \leq r(\mathcal{C}) - 1$ (Levi, [1951]): see (4.7). Eckhoff [1979] asked for a purely combinatorial proof of Tverberg's theorem.:

PROBLEM 5 (Eckhoff) Suppose a convexity space $\mathcal{C}$ has Radon number r. Is the following "partition inequality" : $p_k(\mathcal{C}) \leq (k-1)(r-1)+1$ *always true ?*

A first step towards a solution to this surprising problem is taken with the following results [Jamison-Waldner, 1981] ;

$$p_{kk'} \leqslant p_k \cdot p_{k'} \quad (\text{hence } p_k \leqslant p_2\, k^{\log_2 p_2}$$
$$p_{(k-i)k'+1} \leq (p_k - i)p_{k'}+1 \quad \text{for } 1 \leq i \leq k \ .$$

Jamison proved the partition inequality for order convexities (section 8), tree-convexities (section 9) and more generally for convexities that have the following property :

CIP(3,2): *for every point x, among three copoints relative to x two of them are disjoint.*

Roudneff [1986] proved the partition inequality for convexities of oriented matroids of rank ≤ 3 (section 7).

(4.5) CARATHEODORY PROPERTY

(C_k) *If* $x \in \langle A \rangle$, *then* $x \in \langle F \rangle$ *for some* $F \subseteq A$ *with* $|F| \leq k$.

Ordinary convexity in $\mathbb{R}^d$ has property (C_{d+1}) (Caratheodory, [1907]). For related results see Barany [1981]. The *Caratheodory number* denoted by $c(\mathcal{C})$ is the smallest integer k (if it exists) such that (C_k) holds. We have :

$$c(\mathcal{C}) = \max(|A|; A \subseteq X \text{ and } \langle A \rangle \not\subseteq \bigcup_{a \in A} \langle A \setminus a \rangle \ )$$

For *interval convexities* (3.6) more can be said [Duchet, 1986a] : *the Caratheodory number equals the least integer k such that every $k+1$-point set A satisfies the following property :*

$$\langle A \rangle = \bigcup_{a \in A} \langle A \setminus a \rangle \ .$$

(4.6) EXCHANGE NUMBER [Reay, 1965; Sierksma, 1976]

(E_k) *If* $x \in X$, $A \subseteq X$, $|A| \geq k$ *then :*

$$\langle A \rangle \subseteq \bigcup_{a \in A} \langle (A \cup x) \setminus a \rangle \ .$$

Ordinary convexity in $\mathbb{R}^d$ satisfies (F_{d+1}) Reay, 1965 . The *exchange number* [Sierskma, 1976], denoted by $e(\mathcal{C})$ is the smallest integer k (if it exists) such that (E_k) holds.

(4.7) INEQUALITIES

Levi [1951] : $h \leq r-1$

Sierksma [1976] : $e \leq c+1$

Sierksma [1977] : $r \leq (h-1)\ max(h,e-1)+2$

If $e \leq c$, $r \leq (c-1)(h-1)+3$

Jamison-Waldner [1981] : $P_k \leq (k-1)\ rank\ +1$

Ibid.(Kay Womble [1977] for k=2). $P_k \leq (k-1)ch\ -c\ +2$

For $i = 1,2,\ldots,n$ let $(X_i, \mathcal{C}_i)$ be a convexity space. Let $(X,\mathcal{C})$ denote their product space (3.8). Set $h_i = h(\mathcal{C}_i)$, $h = h(\mathcal{C})$; numbers r_i, c_i, e_i, r, c, e are defined analogously.

Sierksma [1975] : $h = \max_i h_i$

Eckhoff [1978,1979] : $\max_i r_i \leq r$

for $n = 2$: $r \leq r_1+r_2-1$

Sierksma [1976] : $r \leq \sum_i r_i - 2n + 2$

Soltan [1981](cf.Sierksma [1976]) : $e \quad 1 + \sum_i (c_i + sign(e_i-c_i-1))$

Soltan [1981], Sierksma [1975]*(n=2)* (also Reay [1970] if $\mathcal{C}_i$ are standard convexities in $\mathbb{R}^{d_i}$) $c = e-1+\varepsilon$ (*where* $\varepsilon = 0$ *if* $e_i = c_i+1$ *for every* i *or if* $c_i \geq 2$ *for every* i ; $\varepsilon = 1$ *in other cases).*

PROBLEM 6 (Kay, Womble) Characterize possible triples of the form $(h(\mathcal{C}), r(\mathcal{C}), c(\mathcal{C}))$ *for some convexity space* $(X,\mathcal{C})$.

5. CONVEX GEOMETRIES

(5.1) Convex Geometries were introduced independently by Edelman [1980a] and by Jamison [1980] . *Convex Geometries* are *finite* convexity spaces in which the *(finite) Krein-Milman property* holds :

(KM) *Every convex set is the convex hull of its extreme* (*) *points.*

Theory of Convex Geometries is interesting at least for two reasons. On the one hand, it is general enough to unify areas that were previously ill-matched : Convex Geometries were independently discovered under dual form by Korte and Lovász [1981,1984a] (shelling structures) and Crapo [1984] (selectors) in a completely different context; a similar structure appears also in Discrete Optimization [Hoffman 1979]. On the other hand the concept of Convex Geometry is rich enough to allow the growth of a theory.

If all singletons of a convex geometry are convex, then the

(*) A point x of a convex C is *extreme* in C iff $C \setminus x$ is convex.

convex geometry is completely determined by the lattice of its convexes. Edelman [1980] showed that *a finite convexity space is a convex geometry if and only if the lattice of its convexes is meet-distributive* (*). Hence one can say that the concept of Convex Geometry is implicit in Dilworth [1940]where meet-distributive lattices are characterized as lattices in which every element has a unique expression as a joint of joint-irreducible elements. These lattices were frequently rediscovered : see Monjardet [1985].

Although Edelman and Jamison restrict their attention to finite spaces, I suspect that no major difficulty arises about extending their results to infinite spaces (compare Soltan [1984]).

PROBLEM 7 (Duchet) Extend the theory of convex geometries to infinite spaces.

(5.2) EXAMPLES : Convexities of acyclic oriented matroids (section 7); partial order convexities and some related convexities (see section 8) 8); minimal path convexity in triangulated graphs (6.4); geodesic convexity in ptolemaic graphs (6.7); tree convexities (see section 9).

(5.3) CHARACTERIZATIONS AS CONVEXITY SPACES (Edelman, Jamison) A *finite* convexity space $(X,\mathcal{C})$ is a convex geometry if and only if each of the following equivalent axioms is satisfied.

(i) *(Anti-exchange axiom)*

$x,y \notin \langle A\rangle$, $x \neq y$ and $x \in \langle A \cup y\rangle$ imply $y \notin \langle A \cup x\rangle$

(ii) *(Augmentation axiom)*

If $C \in \mathcal{C}$ then $(C \cup x) \in \mathcal{C}$ for some $x \in X \setminus C$

(iii) *(Generation axiom)*

If $C = \langle A\rangle = \langle B\rangle$ then $C = \langle A \cap B\rangle$

(iv) $(X,\mathcal{C})$ *is a joint of monotone convexity spaces* (**)

(v) $(X,\mathcal{C})$ *has no minor isomorphic to* $\mathbf{Q}_0(2)$ *where* $\mathbf{Q}_0(2) = (\{1,2\} ; \{\emptyset,\{1,2\}\})$.

Antiexchange axiom (i) is to be compared with exchange axiom for closed sets (flats) of matroids. The equivalence between (i), (iv) and (v) holds for infinite spaces [Jamison, 1982]. A variant of anti-exchange axiom is :

(vi) *For every* $C \in \mathcal{C}$, *the relation* $x \in \langle C \cup y\rangle$ *between elements* x *and* y *of* $X \setminus C$ *is a partial order on* $X \setminus C$.

(*) A lattice is *meet-distributive* if and only if all intervals $[x,y]$ such that x is the meet of elements that y covers, are Boolean algebras.

(**) See (3.10) and (8.4) for definitions. See (5.8) for details

For other variants see Edelman, Jamison [1985], Thron [1985]. An equivalent form of (ii) is :

(vii) *Every maximal chain* $\emptyset = C_1 \subset C_2 \subset \ldots \subset X$ *of convexes comprises* $|X| + 1$ *convexes.*

Notice that the existence of extreme points for every convex set is not sufficient to imply (ii). The generation axiom (Hoffman, [1979]) is equivalent to the following axiom (cf. Korte, Lovász [1984b]) :

(viii) *Every convex* $C \in \mathscr{C}$ *contains a unique minimal subset* B *such that* $\langle B \rangle = C$.

(5.4) HELLY NUMBER. In a convexity space $(X, \mathscr{C})$, a set $K \subseteq X$ is *a clique* if K and all K-subsets are convex. Equivalently a clique is a convex set all vertices of which are extremal. *In a convex geometry the Helly number equals the maximum number of points in a clique.* (Edelman, Jamison [1985]). Compare with the Helly number of minimal path convexities in graphs (6.4).

(5.5) GREEDOIDS (Korte, Lovász) As a framework for greedy algorithms, Faigle [1979,1980] proposed an extension of matroidal structure to partially ordered sets. Greedoids (Korte, Lovász, [1981]) are a further generalization of "Faigle-geometries" (see Korte-Lovasz [1982]). A *Greedoid* $(X, \mathscr{F})$ is a finite set X endowed with a collection $\mathscr{F}$ of X-subsets such that the following axioms are satisfied.

(G_1) $\emptyset \in \mathscr{F}$

(G_2) *If* $A, B \in \mathscr{F}$, $|A| < |B|$ *then* $A \cup \{b\} \in \mathscr{F}$ *for some* $b \in B \setminus A$

Members of $\mathscr{F}$ are the *feasible* sets of the greedoid. The independent sets of a matroid are the feasible sets of a greedoid.

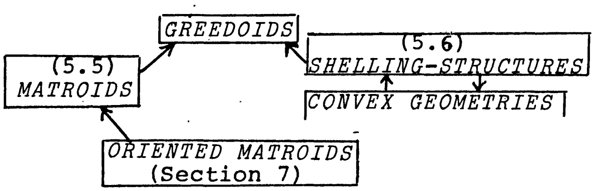
(5.6) SHELLING STRUCTURES (= Selectors, "Alternative precedence structures" or APS-greedoids) [Korte, Lovász, 1984b] are a special class of greedoids. Axioms are :

(SH_1) $\emptyset$, $X \in \mathscr{F}$

(SH_2) *If* $A, B \in \mathscr{F}$, $A \not\subseteq B$ *then* $A \cup \{b\} \in \mathscr{F}$ *for some* $b \in B \setminus A$.

Axiom (SH_2) is due to Björner [1983] who first pointed out the connection between greedoids and antiexchange axiom (implicit in Crapo [1984]) : let X be a finite set and let $\mathscr{C} \subseteq 2^X$. Set $\overline{\mathscr{C}} = \{X \setminus C \; ; \; C \in \mathscr{C}\}$. Then *$(X, \mathscr{C})$ is a convex geometry if and only if $(X, \overline{\mathscr{C}})$ is a shelling structure* : see (5.8) below.

(5.7)



(5.8) *COMPATIBLE ORDRES.* If τ is a total order on a set X and $x \in X$, we denote by τ_x the lower ideal $\{y \in X : y\tau x\}$. Let $\mathcal{C}$ be a family of total orders ("*criteria*") on X. A total order τ is said to be $\mathcal{C}$-compatible when every element $x \in X$ is for some criterias in $\mathcal{C}$, the best element of τ_x. *The set of all lower ideals (respectively of all upper ideals) of all $\mathcal{C}$-compatible orders are the convexes of a convex geometry (respectively the feasible sets of a shelling structure).* By (5.3)(iv) every convex geometry (hence every shelling structure) can be generated by this process. *The set of all $\mathcal{C}$-compatible orders induces a connected subgraph of the permutohedron on X* (Korte, Lovász, [1984c]; Edelman, Jamison, [1985]; compare Feldman-Högaasen [1969]. Compatible total orderings $x_1,\ldots,x_n$ of X are precisely those in which x_i is an extreme point of $\{x_1,x_2,\ldots,x_i\}$ for $i = 1,\ldots,n$. The sequences $x_n x_{n-1}\ldots x_i$ are precisely the words of the shelling structure when considered as a language over X (see Korte, Lovász [1984a]).

PROBLEM 8 (Edelman, Jamison) Determine the convex dimension of a convex geometry, i.e. the minimum number of criteria which are needed for the construction of that convex geometry.

Edelman and Jamison [1985] showed that the convex dimension is not less than the convex rank (4.2) but it may be greater for the construction of a given convex geometry : this number is at least equal to the convex rank (3.); see Edelman, Jamison [1985].

(5.9) *CHARACTERIZATIONS AS GREEDOIDS* (Korte, Lovász, Björner). A greedoid $(X,\mathcal{F})$ is a shelling structure if and only if $X \in \mathcal{F}$ *and* each of the following equivalent axiom is satisfied :

(i) *Every union of feasible sets is feasible.*

(ii) *(Interval property without upper bound)*

If $A, B, A \cup\{x\}$ belongs to $\mathcal{F}$ and $A \subseteq B$, then $(B \cup x) \in \mathcal{F}$

(iii) *If A, $A \cup\{x\}$, $A \cup\{y\}$ belong to $\mathcal{F}$ then $(A \cup\{x\}\cup\{y\}) \in \mathcal{F}$*

When replacing $A \subseteq B$ in (ii) by $B \subseteq A$, we obtain an axiom (Interval property without lower bound) which characterizes matroids among greedoids.

(5.10) *CIRCUITS* (Korte and Lovász [1984a]) Let $(X,\mathcal{C})$ be a convex geometry. Free sets of are defined as in paragraph (4.2). By analogy with Matroid Theory, a minimal non free set is named a *circuit*; a circuit R has a unique point r such that $r \in \langle R \setminus r\rangle$, named its *root*. Finite convexity subspaces of Euclidean convexity have the property ;

(CI) *If $A \cup\{x\}$ and $A \cup\{y\}$ are circuits with respective roots x*

and y *then there exists a unique circuit* R *with root* y *such that* $\{x,y\} \subseteq R \subseteq A \cup \{x,y\}$.

Say a convex geometry is good if it satisfies (CI) and if every 3-points subset if free.

Korte and Lovász [1984b] proved the following generalization of a famous Erdös-Szekeres theorem [1935] : for some function $f(n)$, every good convex geometry with at least $f(n)$ points contains a free set of n points.

Remark that, in a convex geometry, free sets are precisely the sets of extreme points of convex sets.

(5.11) ENUMERATION Let $(X, \mathcal{C})$ be a convex geometry. In the meet distributive lattice $(\mathcal{C}, \subseteq)$, the Möbius function (*) is easy to calculate (Edelman, Jamison [1985]) : for $C \subseteq D$ we have $\mu(C,D) = (-1)^{|D \setminus C|}$ if every point of $D \setminus C$ is extreme in D ; otherwise $\mu(C,D) = 0$. An (unpublished) theorem of Lawrence follows : we have $\sum\limits_k (-1)^k f_k = 0$ where f_k denotes the number of free sets with k points. In [1985] Edelman and Jamison (**) also give a nice interpretation of the values of the Zeta function (*) of $(\mathcal{C}, \subseteq)$. Let $Z(n)$ denote the number of nondecreasing sequences of n convexes of $\mathcal{C}$: $\emptyset \subseteq C_1 \subseteq \ldots \subseteq C_n \subseteq X$. Function $Z(n)$ is a polynomial in the variable n . Let us say a function $\varphi : X \to \{1,\ldots,n\}$ is *extremal* (respectively *strictly extremal*) if for every convex $C \in \mathcal{C}$, φ achieves its maximum on C at some extreme point of C (respectively f achieves its maximum only on extreme points). Then :

$Z(n)$ *and* $(-1)^{|X|} Z(-n)$ *are respectively the number of extremal functions* $X \to \{1,\ldots,n\}$ *and the number of strictly extremal functions* $X \to \{1,\ldots,n\}$.

6. CONVEXITY IN GRAPHS AND HYPERGRAPHS

(6.1) As far as I know, the first explicit use of convexity in graphs appears in Feldmann - Högassen [1969] where convexity in the permutahedron is investigated. Most results deal with geodesic convexity (6.5) A more general point of view appeared in Nebesky [1970,71], Sekanina [1975], Mulder [1978,1980], Harary, Nieminen [1981]. A systematic approach arises in Duchet, Meyniel [1983] and Farber, Jamison [1983]. All definitions and results below refer to a *connected* graph G on

(*) See Rota [1964]-or Aigner [1979] for details.
(**) See also Edelman [1980] and Stanley [1974, Prop. 2.1].

vertex set V (finite or infinite). See Berge [1985] for general notions of Graph Theory.

(6.2) A *graph convexity* (Duchet and Meyniel [1983]) is a pair $(G,\mathscr{C})$ formed with a (connected) graph G, with vertex set V, and a convexity $\mathscr{C}$ on V such that $(V,\mathscr{C})$ is a convexity space satisfying the additional axiom :

(GC) *Every convex set induces a connected subgraph.*

(6.3) *HADWIGER'S CONJECTURE* is a famous problem that generalizes the four colour problem. Let W be a V-subset that induces a connected subgraph of G. Add a new vertex x_W to the graph $G \setminus W$ and join it to every vertex y, $y \in V \setminus W$ for which G contains a $y - W$ edge. The resulting graph is the graph obtained from G by contraction of W. A *contraction* of G is every graph that can be obtained by a sequence of contractions of connected subgraphs. Hadwiger [1943] conjecture :

PROBLEM 9 (Hadwiger) If the complete graph with $p+1$ vertices is not a contraction of G then one can color the vertices of G in p colours such that adjacent vertices receive different colours.

A recent survey on this important conjecture is by Duchet, Xuong [1986].

The *Hadwiger number* $\eta(G)$ of a graph G is the largest integer p such that G has the complete graph with p vertices as a contraction. Helly (4.3) and Radon (4.4) numbers of a graph convexity $(G,\mathscr{C})$ are intimately related to the Hadwiger number (Duchet, Meyniel [1983]) :

(i) $h(\mathscr{C}) \leq \eta(G)$

(ii) $r(\mathscr{C}) \leq 2\eta(G)$

Equality is possible in (i).

PROBLEM 10 (Duchet) For every $\varepsilon > 0$ there is a graph convexity $(G,\mathscr{C})$ such that $r(\mathscr{C}) > (2-\varepsilon)\eta(G)$.

For further details see Duchet [1984].

(6.4) *MINIMAL PATHS.* Let $M(x,y)$ denote the set of all vertices of all chordless paths from x to y in a graph G. The convexity generated by the interval function M is called the minimal path-convexity (or M-convexity) on G (Jamison [1982] Duchet [1979,1986a]). Caratheodory, Helly and Radon numbers (see (4.5)(4.3) and (4.4)) are respectively denoted by $c_M(G)$, $h_M(G)$ and $r_M(G)$; we have (Duchet [1986a]; also Jamison and Nowakowski [1984] for h_M) :

$c_M(G) = 2$ *(if G is not complete),* $h_M(G) = \omega(G)$, $r_M(G) = \omega(G)+1$ *(if $\omega(G) \geq 3$). If $\omega=2$, $r_M(G) \leq 4$ and $r_M(G)=3$ iff G is a "path of blocks".*

Here above $\omega(G)$ denotes the maximum number of vertices of a complete subgraph in G .

The M-convexity of a graph G is a convex geometry if and only if G is triangulated — i.e. every cycle of length ≥ 4 has a chord — (Farber, Jamison [1983]). Extremal vertices correspond to simplicial vertices, i.e. vertices whose neighbourhood is complete (see (9.3)).

(6.5) GEODESIC CONVEXITY. Let $D(x,y)$ denote the set of all vertices of all shortest paths between x and y . The convexity generated by the interval function D is called the *geodesic convexity* (or *distance-convexity* or *D-convexity*) on G ; the D-convexity is the metric convexity (3.7) associated to the usual distance function $d(x,y)$ in graphs. First researches on D-convexities were motivated by the following question of Ore [1962] :

PROBLEM 11 (Ore) Characterize the geodetic graphs, i.e. the graphs in which every pair of vertices is joined by a unique shortest path.

Various constructions of geodetic graphs are known. See Stemple and Watkins [1968], Zelinka [1977], Stemple [1979], Parthasarathy and Srinivasan [1984] and Plesnik [1984]. Contrarily to M-convexities of graphs (6.4), the D-convexity is very general (see (6.6)) and became intensively studied since 1981 : see Jamison [1981a], Batten [1983], Soltan [1983], Soltan and Chepoi [1983,84,85], Farber [1985], Farber and Jamison [1986], Sampathkumar [1984].

(6.6) The geodesic convexity is in some sense universal with respect to Caratheodory, Helly and Radon properties : *given any finite convexity space $(X, \mathcal{C})$, there exists a finite graph G such that the Caratheodory, Helly and Radon numbers of the geodesic convexity in G coincide with those of $\mathcal{C}$* . (Duchet [1986b]).

PROBLEM 12 (Duchet) Does universal property above hold for partition numbers ?

(6.7) METRIC CHARACTERIZATIONS of some classes of graphs appear as a facet of the research on geodesic graph convexity. For instance, certain graphs are completely determined by the list of the isomorphism types of their convex subgraph ; see Egawa [1986], Van Cruyce [1984a,b]. In Bhaskara Rao and Rao Hebbare [1976] and Rao Hebbare [1979] the authors investigate the graphs having only trivial D-convex sets ($\emptyset$, V , singletons and edges). Graphs in which D-convexity determines a convex geometry are characterized (Farber, Jamison [1983]) by the *Ptolemaic* inequality :

$$d(x,y)d(z,t) \leq d(x,z)d(y,t) + d(y,z)d(x,t)$$

A related interesting class was introduced in Farber [1985] and Farber, Jamison [1986] : a graph is bridged if each cycle C of length ≥ 4 contains two vertices whose distance from each other in G is strictly less that in C.

A graph is bridged if and only if every closed neighbourhood of a D-convex set is again D-convex (Farber, Jamison [1986]; Soltan, Chepoi [1983]). In connection with the problem of the metric determination of a graph, the results of Graham and Winkler [1984] on isometric embeddings of graphs are to be considered.

(6.8) MEDIAN GRAPHS (Mulder [1978] are graphs in which the metric interval function D (see (6.5) has the following property :

$$|D(x,y) \cap D(y,z) \cap D(z,x)| = 1 \quad \text{for every} \quad x,y,z .$$

These graphs generalise hypercubes and are extensively studied : Mulder [1980a,b], Mulder, Schrijver [1979], Bandelt, Barthelemy [1984], Nieminen [1984].

Mulder [1980a] also considers *interval regular graphs* in which $D(x,y)$ contains exactly $d(x,y)$ neighbours of x, for any two vertices x,y of the graph.

PROBLEM 13 (Mulder) Conjecture : In any interval regular graph the sets $D(x,y)$ are geodesically convex.

Variants on the theme of betweeness or medians are numerous. See for instance Nebesky [1970,1971] (ternary algebras), Sekanina [1975] Barthelemy [1983] and Batten [1983]. Refer also to Fishburn [1971] and to sections 8, 9 for betweeness in partial orders and in trees.

(6.9) HYPERGRAPHS Natural generalizations of graph convexities may be considered in the context of abstract families of sets (hypergraphs : Berge [1986]); for instance Farber and Jamison [1983] characterized *strongly balanced hypergraphs* — in which every cycle possesses an edge that contains three vertices of the cycle — by the fact that a certain natural convexity associated to the hypergraph determines a convex geometry. One can also consider edges of a hypergraph as copoints of a convexity. Helly property has been studied with this point of view : Mulder, Schrijver [1979], Barthelemy [1985]. In [1979,1983] Bollobás and Duchet solved the corresponding extremal problem :

Let $h < p \leq n$ be integers: in a n-element set, a convexity with Helly number h has at most $\binom{n-1}{p-1}$ convexes of cardinality p. Moreover, this upper bound is reache only if the convexes of cardinality p are precisely the p-subsets containing a common element.

7. ACYCLIC ORIENTED MATROIDS

All matroids here are supposed to be simple. See Welsh [1976] for general definitions and properties of matroids.

(7.1) Let E be a finite set of points of $\mathbb{R}^d$. Minimal affinely dependent subsets of are the circuits of a matroid M on E *(affine* matroid). Every such circuit admits a unique Radon partition (see (4.4), Eckhoff [1975,79]). Oriented matroids constitute an elegant axiomatic setting for a combinatorial treatment of these Radon partitions ; in an oriented matroid M every circuit X of the underlying matroid $\underline{M}$ admits a unique partition into a positive part X^+ and a negative part X^-. These *signed circuits* satisfy certain axioms. For axioms and fundamental properties, see Bland, Las Vergnas [1978], Folkman, Lawrence [1978]. An important fact is the existence of a canonical way of orienting the orthogonal matroid $\underline{M}$: for affine matroids, the positive and the negative part of a cocircuit Y correspond to the separation the hyperplane $H = E \setminus Y$ determines through Y. The structure of Oriented Matroids is thus intimately related to convexity. A fundamental open question is the determination of the set of Radon partitions of finite sets of points in $\mathbb{R}^d$.

(7.2) *FACES* Affine matroids are *acyclic* : it means that no circuit X is entirely positive $(X = X^+)$ or entirely negative $(X = X^-)$. Convexity in acyclic oriented matroids was first investigated by Las Vergnas [1980] who proved the following facts : Let M be an acyclic oriented matroid with point set E. A *face* of M is a flat F such that the matroid $_{\overline{F}}$M obtained from M by sign interchange of all points in F is again acyclic. *Faces, when ordered by inclusion form a graded lattice* where the meet is the intersection. A point p is extreme if $\{p\}$ is a face. A rank r acyclic matroid admits at lease r extreme points.

PROBLEM 14 *(Las Vergnas) Characterize face lattices of acyclic oriented matroids.*

These lattices are meet-distributive (Edelman [1982]) and satisfy Euler's relations (Cordovil, Mandel, Las Vergnas [1982]). Further results may be found in Munson [1981], Lawrence [1983,84], Billera, Munson [1984].

(7.3) *CONVEX OPERATORS* Let M be an acyclic oriented matroid on a set E. All signed M circuits X whose negative part is a singleton $\{x\}$ may be viewed as operators $X^+ \to x$. The convexity on E determined (see (3.5)) by these operators is called the (canonical) *convexity of* M (Las Vergnas [1980]; cf. Goodman, Pollack [1982], Folkman, Lawrence [1978, p. 204].

PROBLEM 15 Characterize convexity spaces that arise from acyclic oriented matroids.

7.4) SEPARATIONS Let M be an acyclic oriented matroid on a point set E . Two *disjoint* E-subsets A,B are said to be separable if no signed circuit (X^+,X^-) of M both satisfy $X^+ \subseteq A$ and $X^- \subseteq B$ A hyperplane $H \subseteq E$ *separates* A and B when the signed parts of the cocircuit $A \setminus B$ contain respectively A and B . A *separation* of M (or *"non-Radon partition"*, see Brylawski [1976], Cordovil [1985]) is a partition of E into separable sets. Bienia and Cordovil [1985] gave a *characterization of the set of separations* of an oriented matroid.

(7.5) SEPARATION THEOREMS Let M denote an acyclic oriented matroid on a point set E . For $A \subseteq E$, the restriction of M to A is denoted by $M(A)$, the convex hull of A is denoted by A_M . Let us consider the following properties relative to separation of two subsets A,B of E :

(i) *A and B are separable.*

(ii) *$\{A,B\}$ is a separation of* $M(A \cup B)$

(iii) *There exists a separation $\{A',B'\}$ of M such that $A \subseteq A'$ and $B \subseteq B'$*

(iv) *There exists an extension M' of M where A and B are separated by a hyperplane*

(v) *In every extension N of $M(A \cup B)$, we have:*

$$A_N \cap B_N = \emptyset$$

(vi) *In every extension M' of M , we have :*

$$A_{M'} \cap B_{M'} = \emptyset .$$

Properties (i), (ii), (iii), (iv) are equivalent. Furthermore (iv) ⇒ (v) ⇒ (vi) (Las Vergnas [1980], Mandel [1982]). The implication (v) ⇒ (iv) is conjectured by Mandel who found a rank 4 oriented matroid that satisfies (vi) but not (v).

PROBLEM 16 (Mandel) Conjecture : A pair A,B of sets of points in an oriented matroid M *is separable if and only if condition* (v) *holds.*

Equivalence between (iv), (v) and (vi) holds for certain class of oriented matroids : see Bachem, Wanka [1986]. Further results relative to separation may be found in Goodman, Pollack [1982], Cordovil [1982], Cordovil, Duchet [1986].

(7.6) Further results relative to convexity in oriented matroids may be found in Méi [1971], Cheung [1974], Goodman, Pollack [1982], Edelman [1984], Bachem, Kern [1982], Billera, Munson [1984b].

8. PARTIALLY ORDERED SETS

(8.1) Let P denote a finite partially ordered set. Define an interval-function by $I(x,y) = \{z \in P\ ;\ x \underset{P}{\leq} z \underset{P}{\leq} y\}$. The interval convexity (see (3.6)) generated by I is named the *order convexity* of P (Franklin [1962]) and is denoted by $\mathcal{C}_P$.

For any $x \in P$, at most two copoints relative to x may exist in $\mathcal{C}_P$, namely the sets $C^+(x) = \{y \in P\ ;\ x \leq y\}$ and $C^-(x) = \{y \in P\ ;\ x \nleq y\}$. Thus, property CIP(3,2) is satisfied (4.4): *the partitions inequality holds in every order convexity* (Jamison-Waldner [1981a]). Order convexities form a variety (3.10) which is "sum-closed" but *not finitely based* (see Jamison [1979,82]).

A convexity space $(X,\mathcal{C})$ *is the oder convexity of some partial order* P *on* X *if and only if it has the three following properties* (Jamison [1979]) :

(i) *Every free set is a union of two convex sets.*
(ii) $\mathcal{C}$ *has Caratheodory number* ≤ 2
(iii) *Every convexity subspace with at most 5 points is an order convexity space.*

The first axiomatic of partial orders in terms of betweeness was given by Altwegg [1950]. See Fishburn [1971] for further information.

(8.2) TOTAL ORDER CONVEXITIES : order convexities of totally ordered sets. *Every order convexity of a poset* P *is the joint* (3.10) *of the total order convexities of linear extensions* (*) *of* P. (Edelman, Jamison [1985]); hence, finite order convexities are convex geometries; remark that linear extensions and their reversals are *compatible* orders with the meaning of (5.8). Arbitrary joints of total order convexities form an interesting variety we denote by $\bigvee$TO , indeed they include partial order convexities, ordinary convexities in Euclidean spaces (Jamison [1982]) and convexities of acyclic oriented matroids (Section 7). Variety $\bigvee$ TO is the smallest variety containing all varieties $\bigvee^n$ TO constituted of joint of n total convexities for $n \in \mathbb{N}$.

PROBLEM 17 (Jamison) Characterize the variety $\bigvee$TO . *Is* $\bigvee^n$TO *finitely based for each* $n \in \mathbb{N}$ *?*

Variety TO of total order convexities is finitely based since we have : *a convexity space* $(X,\mathcal{C})$ *is the order convexity space associated to some total order on* X *if and only if one of the*

(*) A *linear extension* of P is a total order τ such that $x \underset{\tau}{\leq} y$ whenever $x \underset{P}{\leq} y$

following equivalent conditions is satisfied.

(i) *Every subconvexity of* $\mathscr{C}$ *induced by a X-subset of at most 4 points is a total order convexity.*

(ii) $\mathscr{C}$ *has rank* ≤ 2 *, has the antiexchange property and has the following separation property : for every pair* $a,b \in X$ *there exists a partition of* X *into convex sets* A *and* B *so that* $a \in A$ *and* $b \in B$.

Various alternative characterizations of total orders in terms of betweeness are known. See Fischburn [1971] and its references; also (9.2).

(8.3) Finite partially ordered sets may be viewed themselves as convex sets : Feldman-Högaasen [1969] exhibit a Galois connection between partial orders on a finite set X and the geodesically-convex subsets of the permutohedron with basic set X .

(8.4) In the *downset convexity* determined by a poset P , convexes are lower ideals of P , i.e. P-subsets I such that $y \in I$ whenever $y \leq x$ for some $x \in I$. *A finite convexity space* $(X, \mathscr{C})$ *is the downset convexity of some poset on* X *if and only if it is a convex geometry in which every union of convex sets is again convex* (*) (Edelman, Jamison [1984], Korte, Lovasz [1984b]). Remark that downset convexities of total orders (*monotone convexities* in Jamison [1982]) may be considered as bricks in the construction of convex geometries : see (5.2(iv)).

(8.5) A *(meet) semi-lattice* L is a poset in which each pair of elements has an infimum. The *semilattice convexity* associated with L consists of all subsemilattices of L . It can be viewed as a "refinement" of the downset convexity of L . Finite semilattice convexities are convex geometries. The Caratheodory number of the semilattice convexity coincides with the "breadth" (see Birkhoff [1967] p. 99, Crawley, Dilworth [1973] p. 38) of the underlying semilattice (Jamison [1982]). Note that the rank of a finite convexity space $(X, \mathscr{C})$ is the breadth of the semilattice $(\mathscr{C}, \subseteq)$ (Ibid.)

(8.6) Various other examples of convex geometries arise in the context of partially ordered sets. In [1983] Cochand and Duchet investigated the order convexity of a product of total orders. For every positive integer k , Saks defines a closure operator Ω_k (see (3.5)) in a poset P by :

$$\Omega_k(A) = A \cup \{y \in P \; ; \; y < a_1 < a_2 < \dots < a_k \text{ for some chain } a_1, a_2, \dots, a_k \text{ of } A\}$$

(*) This result can easily be extended to the infinite case.

Ω_k-closed sets form a convex geometry $\mathcal{D}_p^k$. Free sets (*) of $\mathcal{D}_p^k$ are named *k-families*. A k-family A admits a unique partition into disjoint, possibly empty antichains $A_1, A_2, \ldots, A_k$ such that $A_k \leq A_{k-1} \leq \ldots \leq A_1$ where $A_i \leq A_j$ means that for every $a_i \in A_i$ there exist some $a_j \in A_j$ such that $a_i \leq a_j$. Given two k-families A, B of P, define $A \leq B$ if and only if $A_i \leq B_i$ for all $i = 1, \ldots, k$. *The set of all k-families ordered in this way is a join-distributive lattice* (Greene, Kleitman [1976]), hence may be considered as the dual lattice of convex sets of some convex geometry : for a discussion about these two last examples see Edelman, Jamison [1985].

9. TREE-LIKE STRUCTURES

(9.1) TREE CONVEXITIES. In a tree T the M-convex sets (6.4) coincide with the D-convex sets (6.5) : they form the convexes of the *tree-convexity* determined by T. Tree-convexities are convex geometries (section 5) and are *coherent convexities* (**). Results by Duchet [1978], Flament [1978], Slater [1978] imply that a given family $\mathcal{F}$ of subsets of a finite set X is a family of connected subsets of some tree on X if and only if the smallest coherent convexity that contains $\mathcal{F}$ has Helly property (H_2) (4.3). Consequently : *finite tree convexities are characterized as the convex geometries with Helly number ≤ 2 in which any two different points are contained in disjoint convexes whose union is the whole space* ("hemispaces"; compare with (9.2). In this characterization the assumption of being a convex geometry can be replaced by the property of being coherent. Compare to Skolander [1952].

Tree convexities satisfy the partitions inequality since they have property CIP(3.2) (see (4.4)).

A metric characterization of trees was provided by Colonius and Schutze [1981] where related references may be found. Further developments may be found in Barthelemy [1983], Duchet [1986] and Duchet, Quilliot [1986].

(9.2) TOTAL ORDER CONVEXITIES (See (8.6)) are tree convexities of a special kind. Using structural characterizations of interval hypergraphs (Tucker [1972], Trotter, Moore [1976], Duchet [1978], Nebesky [1984]) one can obtain : a family $\mathcal{F}$ of subsets of a finite

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(*) As defined in (5.10).

(**) A convexity space is *coherent* when the union of two intersecting convexes is again convex.

set X is a family of intervals of a total order on X if and only if the smallest coherent convexity containing $\mathcal{F}$ has Radon number $\leq$ 3 (see (4.4)). Consequently, *a finite convexity space is a total order convexity if and only if it is a convex geometry with Radon number ≤ 3 and in which any two different points are contained in disjoint convexes whose union is the whole space.* Again, the assumption "convex geometry" may be replaced by "coherent convexity". Compare with (8.6).

(9.3) ARBORESCENT CONVEXITIES. Let Arb denote the smallest variety (3.10) that contains tree convexities. Members of Arb ("*arborescent* convexity spaces") may be considered as a pertinent abstraction of what a tree-like structure should be. Downset convexities (8.4) and M-convexities of triangulated graphs (6.4) are examples of arborescent convexities which are not tree convexities.

PROBLEM 18 (Duchet) : Develop the theory of arborescent convexities (combinatorial invariants, partition problem, separation properties ..)

REFERENCES

AIGNER M. Combinatorial Theory, Springer Verlag, New York 1979.

ALEXANDROFF A.D. Convexe Polyeder, Akad. Verlag, Berlin 1950.

ALTWEGG M. "Zur Axiomatic der teilweise geordneten Mengen" Commen. Math. Helv. 24, 1950, 149-155.

BACHEM A., KERN W. "Convexity and optimization in discrete structures" in Mathematical Programming, Proc. Conf. Rio de Janeiro, 1981 (R.W. Cottle, M.L. Kelwanson, B. Korte eds) North Holland 1984 [1984a]

BACHEM A., KERN W. "Adjoints of oriented matroids", preprint, [1984b]

BACHEM A., WANKA A. "Separation theorems for oriented matroids" Res. Rep. Math. Inst., Univ. Köln (W.Germany) 1986.

BAIR J., FOURNEAU R. Etude géométrique des espaces vectoriels I Une introduction, Lecture Notes in Math., 489, Springer Verlag, New York, Heidelberg, Berlin 1975.

BAIR J., FOURNEAU R. Etude géométrique des espaces vectoriels II Polyèdres et polytopes convexes, Lecture Notes in Math., 802, Springer Verlag, New York, Heidelberg, Berlin 1980.

BANDELT H.J., BARTHELEMY J.P. "Medians in median graphs" Disc. Appl. Math. 8, 1984, 131-142.

BARANY I. "A generalization of Caratheodory's theorem", preprint 1981.

BARNETTE D. "A proof of the lower bound conjecture for convex poly-

topes", Pacific J. Math. 46, 1973, 349-354.
BARTHELEMY J.P. "Caractérisations médianes des arbres", Ann. Disc Math. 17, 1983, 39-46.
BARTHELEMY J.P. "From copair hypergraphs to median graphs with latent vertices", submitted 1985.
BATTEN L.M. "Geodesic subgraphs", J. of Graph Theory 7, 1983, 159-163
BEAN P.W. "Helly and Radon type theorems in interval convexity spaces", Pacific J. Math. 51, 1974, 363-368.
BENNETT M.K., "Topological convexity lattices" Algebra Universalis 4, 1974, 220-225.
BENNETT M.K. "A lattice characterization of convexity" J. Comb. Th. 5 1968, 198-202.
BENNETT M.K. "Generalized convexity lattices" J. Comb. Th. 10, 1971, 140-144.
BERGE C., Graphs and Hypergraphs, North Holland, Amsterdam 1973, rev. ed.1976; Graphs Ibid. 1985; Hypergraphs Ibid. in preparation [1986].
BERGE C., DUCHET P., "A generalization of Gilmore's theorem" in Recent Advances in Graph Theory, Proc. 2nd Czekoslovak Symp., Prague Academia, Prague 1975, 49-55.
BHASKARA RAO S., RAO HEBBARE S.P. "Characterization of planar distance convex simple graphs" Proc. Symp. on Graph Theory, ISI, Calcutta 1976.
BIENIA W., CORDOVIL R. "An axiomatic of non Radon partitions" Eur. J. Combin., 1986, to appear.
BILLERA L.J., LEE C.W., "A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial polytopes" Bull. Amer. Math. Soc. (N.S.) 2, 1981, 181-185.
BILLERA L.J., MUNSON B.S., "Triangulations of oriented matroids and convex polytopes" SIAM J. Alg. Disc. Methods 5, 1984, 515-525 [1984a]
BILLERA L.J., MUNSON B.S. "Polarity and inner products in oriented matroids" Eur. J. Combin. 5, 1984, 293-308 [1984b].
BIRCH , "On 3N points in a plane" Proc. Camb. Phil. Soc. 55, 1960, 289-293.
BIRKHOFF G. Lattice Theory, 3rd ed. Vol. XXV, AMS Coll. Publ. Providence, R.I. 1967.
BJÖRNER A. "Shellable and Cohen-Macaulay partially ordered sets" Trans. Amer. Math. Soc. 260, 1980, 159-183.
BJÖRNER A. "On matroids, groups and exchange langages", preprint, Dept. of Math. Univ. Stockholm 1983.
BLAND R.G., LAS VERGNAS M. "Orientability of matroids" J. Comb. Th. B 24, 1978, 94-123.

BOLLOBAS B., DUCHET P. "Helly family of maximal size" J. Comb. Th. A, 26, 1979, 197-200.

BOLLOBAS B., DUCHET P. "On Helly families of maximal size" J. Comb. Th. B 35, 1983, 290-296.

BOLTIANSKII V.G., SOLTAN P.S. Combinatorial Geometry of Various Classes of Convex Sets, Chtiintsa 1978, 280 pages (in russian)

BONNESEN T. Les problèmes des isopérimètres et des isépiphanes, Gauthier-Villars, Paris 1929.

BONNESEN T., FENCHEL W. Théorie der Konvex Körper, Ergebnisse der Mathematik und ihrer Grenzgelicke, Bd. 3, nr. 1, Springer, Berlin 1934 (Berichtigter reprint 1974, Springer, New York, Heidelberg, Berlin).

BOURBAKI N. Theorie des Ensembles, ch. 4 - Structures (2nd edition) Hermann, Paris 1966.

BRUGGESSER H., MANI P. "Shellable decompositions of cells and spheres" Math. Scand. 29, 1971, 197-205.

BRUNN H. Über Ovale und Eifläche, Doct. Diss., München 1887.

BRYANT V.W., WEBSTER R.J. "Convexity spaces I. The basic properties" J. Math. Anal. and Appl. 37, 1972, 206-273.

BRYANT V.W., WEBSTER R.J. "Convexity spaces II. Separation" J. Math. Anal. and Appl. 43, 1973, 321-327.

BRYANT V.W., WEBSTER R.J. "Convexity spaces III. Dimension" J. Math. Anal. and Appl. 57, 1977, 382-392.

BRYLAWSKI T.H. "A combinatorial perspective on the Radon convexity theorem" Geom. Dedicata 5, 1976, 459-466.

BURRIS S. "Representation theorems for closure spaces" Colloq. Math. 19, 1968, 187-193.

BURRIS S. "Embedding algebraic closure spaces in 2-ary closure spaces", Portug. Math. 31, 1972, 183-185.

BUSEMANN H. The geometry of geodesics, Acad. Press, New York 1955.

CANTWELL J. "Geometric convexity I" Bull. Inst. Math. Acad. Sinica 2, 1974, 289-307.

CANTWELL J. "Geometric Convexity II. Topology" Bull. Inst. Math. Acad. Sinica 6, 1978, 303-311.

CANTWELL J., KAY D.C. "Geometric convexity III. Embedding " Trans. Amer. Math. Soc. 246, 1978, 211-230.

CARATHEODORY C. "Über den Variabilitäts bereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen" Math. Ann. 64, 1907, 95-115.

CHVÁTAL V. "Unsolved problem n°7" in Hypergraph Seminar (C. Berge, D.R. Ray-Chauduri eds) Lecture Notes in Math. 411, Springer Verlag,

Berlin 1974.
CHVÁTAL V. "Intersecting families of edges in hypergraphs having the hereditary property" in Hypergraph Seminar (C. Berge, O.R. Ray-Chauduri eds) Lecture Notes in Math. 411, Springer Verlag, Berlin 1974.
COCHAND M., DUCHET P. "Sous les pavés ...", Ann. Discr. Math. 17, 1983 191-202.
COHN P.M. Universal Algebra, Harper and Row, New York 1965. New ed., D. Reidel Pub. Co. Dordrecht-Boston-London, 1981,413 pages.
COLONIUS H., SCHULZE H.H. "Tree structure for proximity data" British J. Math. & Stat. Psychology 34, 1981, 167-180.
CORDOVIL R. "Sur un théorème de séparation des matroïdes orientés de rang 3" Disc. Math. 40, 1982, 163-169.
CORDOVIL R. "A combinatorial perspective on the non-Radon partitions" J. Comb. Theory A 38, 1985, 38-47.
CORDOVIL R., DUCHET P. "Séparation par une droite dans les matroïdes orientés de rang 3 " Disc. Math. 1986, to appear.
CORDOVIL R., LAS VERGNAS M., MANDEL A. "Euler's relations, Möbius functions and matroid identities" Geom. Dedicata 12, 1932, 147-162.
CRAPO H.H. "Selectors" Adv. in Math. 54, 1984, 233-277.
CRAWLEY P., DILWORTH R.P. Algebraic Theory of Lattices, Princeton Hall, London 1973.
DANZER L., GRÜNBAUM B., KLEE V. "Helly's theorem and its relatives" Proceedings of Symposia in Pure Mathematics, vol. VII, Convexity, 1963, 101-180.
DEGREEF E., "Pasting and folding convexity spaces" Tijelschrift van het Belgisch Wiskundig Genootshap 31, 1979, Ser.B, 215-230.
DILWORTH R.P. "Lattices with unique irreducible decompositions" Am. Math. 41, 1940, 771-777.
DOIGNON J.-P. / See the erratum.
DUCHET P. "Schemes of graphs" in Graph Theory and Combinatorics (Cambridge Combin. Conf. in honour of Paul Erdös) (B. Bollobas ed.) Academic Press, 1984, 145-153.
DUCHET P. "Convex sets in Graphs II : minimal path convextiy" 1985, to appear in J. Comb. Th. B.
DUCHET P. "Convex sets in Graphs III : geodesic convexity", in preparation [1986a].
DUCHET P. "Tree-hypergraphs and their representative trees", 1986, submitted. [1986b]
DUCHET P., MEYNIEL H. Ensembles convexes dans les graphes I : théorèmes de Helly et de Radon pour graphes et surfaces, Eur. J. Comb. 4, 1983, 127-132.
DUCHET P., QUILLIOT A. "Tree-like representations of Helly families,

[1986], in preparation.
DUCHET P., XUONG N. "Hadwiger's conjecture. A survey" Proc. of Graph Theory Conf. at the memory of G. Dirac (Sandjberg 1985), to appear
ECKHOFF J. "Radon partitionen und Konvexe Polyeder" J. Reine Angew Math. 277, 1975, 120-129.
ECKHOFF J. "Radon's theorem revisited" in Contributions to Geometry, Proc. Geom. Symp., Siegen 1978 (J. Toelke and J.M. Wills eds) Birkhäuser, Basel 1979, 164-185./See also the erratum.
EDELMAN P.H. "Meet distributive lattices and the anti-exchange closure" Alg. Univ. 10, 1980, 290-299.
EDELMAN P.H. "Zeta polynomials and the Möbius function" Eur. J. Comb. 1, 1980, 335-340.
EDELMAN P.H. "The lattice of convex sets of an oriented matroid" J. Comb. Th. B 33, 1982, 239-244.
EDELMAN P.H. "The acyclic sets of an oriented matroid" J. Comb. Th. B 36, 1984, 26-31.
EDELMAN P.H., JAMISON R.E. "The theory of convex geometries" Geometriae Dedicata 19, 1985, 247-270.
EGAWA Y. "Characterization of the cartesian product of complete graphs by convex subgraphs" Discr. Math. 58, 1986, 307-309.
ELLIS "A general set-separation theorem" Duke Math. J. 19, 1952, 417-421.
FAIGLE U. "The greedy algorithm for partially ordered sets" Discr. Math. 28, 1979, 153-159.
FAIGLE U. "Geometries on partially ordered sets" J. Comb. Th. B 28, 1980, 26-51.
FARBER M. "Bridged graphs and geodesic convexity" submitted [1985].
FARBER M., JAMISON R.E. "Convexity in graphs and hypergraphs" Report CORR. 83-46, Univ. of Waterloo, 1983.
FARBER M., JAMISON R.E. "On local convexity in graphs" to appear in Disc. Math.[1986].
FELDMAN-HÖGAASEN J. "Ordres partiels et permutoèdre" Revue de la Maison des Sciences de l'Homme 28, 1969, 27-38.
FISHBURN P.C. "Betweeness, orders and interval graphs" J. Pure & Appl. Algebra 1, 1971, 159-178.
FLAMENT C. "Hypergraphes arborés" Disc. Math. 21, 1978, 223-226.
FOLKMAN J., LAWRENCE J. "Oriented matroids" J. Comb. Th. B 25, 1978, 199-236.
FRANKLIN S.P. "Some results on order-convexity" Amer. Math. Mon. 69, 1962, 357-359.
GOODMAN J.E., POLLACK R. "Helly-type theorems for pseudoline arrange-

ments in $\mathbb{P}^2$ " J. Comb. Th. A 32, 1-19 [1982a]
GOODMAN J.E., POLLACK R. "Convexity theorems for generalized planar configurations" in Convexity and related combinatorial geometry (Proc. 2nd Oklahoma Conf.) M. Dekker Inc., New York - Basel 1982 73-80 [1982b]
GRAETZER G. Universal Algebra, Van Nostrand, Princeton 1968.
GRAHAM R.L., WINKLER P.M. "Isometric embeddings of graphs" Proc. Nat. Acad. Sci. U.S.A. 81, 1984, n° 22, Phys. Sci. 7259-7260
GROTSCHEL M., LOVÁSZ L., SCHRIJVER A., "The ellipsoid method and its consequences in combinatorial optimization" Combinatorica 1, 1981, 169-197.
GRÜNBAUM B. Convex Polytopes, London-New York-Sydney, Wiley 1967.
HADWIGER B. "Über eine Klassifikation der Streckenkomplexe" Vierteljahrschr. naturf. Ges. Zürich 88, 1943, 133-142.
HADWIGER B., DEBRUNNER H. Combinatorial Geometry in the Plane (translated from German by V. KLEE) (Holt, Renehart and Winster eds.), New-York - Chicago - San Francisco - Toronto - London 1964.
HAMMER P.C. "Extended topology : domain finitness" Indag. Math. 25, 1963, 200-212.
HAMMER P.C. "Semispaces and the topology of convexity" in Proc.Symp. Pure Math., Am. Math. Soc. 7, 1963, 305-316.
HAMMER R. "Beziehungen zwischen den Saetzen von Radon, Helly und Caratheodory bei axiomatischen Konvexitaeten" Abb. Math. Sem. Univ. Hamburg 46, 1977, 3-24.
HAMMER P.L., JOHNSON E.L., KORTE B.H.(eds) Discrete Optimization I, Ann. Disc. Math. 4, North Holland Pub. Co., Amsterdam - New York - Oxford 1979.
HARARY F., NIEMINEN J. "Convexity in graphs" J. Differ. Geom. 16, 1981, 185-190.
HEBBARE S.P.R. "A class of distance convex simple graphs" Ars Combinatoria 7, 1979, 19-26.
HELLY E. "Über Mengen konvexer korper mit gemeinschaftlichten punkten" Iber. Deutsch. Math. Verein. 32, 1923, 175-176.
HOFFMAN A. "Binding constraints and Helly numbers" Ann. of N.Y. Acad. Sci., 2nd Int. Conf. on Comb. Math. 19, 1979, 284-288.
JAMISON R.E. "A general theory of convexity, Ph.D. Univ. Washington 1974.
JAMISON R.E. "Copoints in antimatroids" Proc. 11th S.E. Conf. Comb. Graph. Th. and Computing, Congressus Numerantium 29, 1980, 535-544.
JAMISON R.E. "A perspective on abstract convexity : classifying alignments by varietes" in Convexity and Related Combinatorial

Geometry, Proc. 2nd Univ. of Oklahoma Conf. (D.C. Ray, M. Breens eds) Dekker, New York 1982, 113-150.

JAMISON-WALDNER R.E. "A convexity characterization of ordered sets", in Proc. Tenth South Conf. Comb: Graph Theory and Computing, Florida Atlantic Univ., 1979, 529-540.

JAMISON-WALDNER R.E. "Partition numbers for trees and ordered sets" Pacific J. Math. 96, 1981, 115-140 [1981a]

JAMISON-WALDNER R.E. "Convexity and block graphs" Congr. Numerantium 33, 1981, 129-142. [1981b]

JAMISON R.E., NOWAKOWSKI R. "A Helly theorem for convexity on graphs" Disc. Math. 51, 1984, 35-39.

KAKUTANI "Ein Beweis des Sätzes von Eidelheit über konvexe Mengen" Proc. Imp. Acad. Tokyo 13, 1937, 93-94.

KALAI G. "Characterization of f-vectors of families of convex sets in $\mathbb{R}^d$ - Part I : Necessity of Eckhoff's conditions, Israel J. Math. to appear. [1985a]

KALAI G. "Characterization of f-vectors of families of convex sets in $\mathbb{R}^d$ - Part II : sufficiency of Eckhoff's conditions, J. Comb. Th. A, to appear. [1985b]

KARMARKAR N. "A new polynomial time algorithm for linear programming" Combinatorica 4, 1984, 373-395.

KORTE B., LOVASZ L."Mathematical Structures underlying greedy algorithms" in Fundamentals of Computation Theory, Lectures Notes in Computer Science 117, Springer 1981, 205-209.

KORTE B., LOVASZ L. "A note on selectors and greedoid" Report n° 82244-OR, Inst. Operations Research, Bonn Univ. 1982.

KORTE B., LOVASZ L. "Poset, matroids and greedoids" Report n° 83278-OR, Inst. Operations Research, Bonn Univ. 1983.[1983a]

KORTE B., LOVASZ L. "Structural properties of Greedoids" Combinatorica 3, 1983, 359-374 [1983b]

KORTE B., LOVASZ L. "Greedoids - a structural framework for the greedy algorithm" in Progress in Combinatorial Optimization (Silver Jubilee Conf. on Combinatorics, Waterloo 1982) (W.R. Pulleyblank ed) Acad. Press, New York 1984 [1984a]

KORTE B., LOVASZ L. "Shelling structures, convexity and a happy end" in Graph Theory and Combinatorics (Cambridge Combinatorial Conf. in honour of Paul Erdös) (B. Bollobas ed) Acad. Press, 1984, 219-232 [1984b].

LAS VERGNAS M. "Convexity in oriented matroids" J. Comb. Th. B 29, 1980, 231-243.

LAWLER E.L. Combinatorial Optimization : networks and matroids, Holt,

Rinehart and Windston, New York 1976.
LAWRENCE J. "Lopsided sets and orthant intersection by convex sets" Pacific J. Math. 104, 1983, 155-173.
LAWRENCE J. "Shellability of oriented matroid complexes, 1984, preprint.
LEVI F.W. "On Helly's theorem and the axioms of convexity" J. Indian Math. Soc. 15, 1951, Part A, 65-76.
LOVASZ L. "Geometrical methods in combinatorial methods" in Progress in Combinatorial Optimization (Silver Jubilee Conf. on Combinatorics, Waterloo 1982) (W.R. Pulleyblank ed.) Academic Press, New York 1984.
McMULLEN P. "The maximum number of faces of a convex polytope" Matematika 17, 1971, 179-184.
MANDEL A. The topology of oriented matroids, Ph.D. Univ. Waterloo, 1982.
MEI P.S. Axiomatic theory of linear and convex closure, Thesis, P Purdue Univ. 1971, 152 pages.
MONJARDET B. "A use for frequently rediscovering a concept" Order 1, 1985, 415-417.
MOORE Introduction to a form of general analysis, New Haven Math. Coll., Yale Univ. Press, New Haven 1910.
MOTZKIN S.T.S. "Linear inequalities" Mimeographed Lecture Notes, Univ. of California, Los Angeles 1951.
MULDER H.M. "The structure of median graphs" Disc. Math. 24, 1978, 197-204.
MULDER H.M. The interval function of a graph, Doct. Diss. Vrije Universiteit Amsterdam, Math. Center tracts 132, 1980.[1980a]
MULDER H.M. "n-cubes and median graphs" J. Graph Th. 4, 1980, 197 [1980b]
MULDER H.M., SCHRIJVER A. "Median graphs and Helly hypergraphs" Disc. Math. 25, 1979, 41-50.
MUNSON B.S. Face lattices of oriented matroids, Thesis, Cornell Univ. 1981.
NEBESKY L. "Graphic algebras" Comment. Math. Univ. Carolinae, 1970, 533-544.
NEBESKY L. "Median graphs" Comment. Math. Univ. Carolinae 12, 1971, 317-325
NEBESKY L. "Hypergraphs and intervals II " (Russian and Czech supparies) Casopis Pest. Mat. 109, 1984, n°3, 286-289.
NIEMINEN J. "A characterization of median graphs" Indian J. Pure Appl. Math. 15, 1984, n°7, 727-730.
ORE O. Theory of Graphs, Amer.Math. Soc. Providence, R.I. 1962.

PARTHASARATHY K.R., SRINIVASAN N. "Geodetic block of diameter three" Combinatorica 4, 1984, 197-206.

PLESNIK J. "A construction of geodetic graphs based on pulling subgraphs homeomorphic to complete graphs" J. Comb. Th. B. 36, 1984, 284-297.

PRENOWITZ W. Geometry of Join Systems, Holt, Rinehard and Winston, New York 1969.

PRENOWITZ W., JANTOSCIAK J. Joint geometries. A theory of convex sets and linear geometry, Springer 1979, 534 pages.

RADO "Theorems on the intersection of convex-sets of points" J. London Math. Soc. 27, 1952, 320-328.

RADON "Mengen Konvexer Körper, die einen geineinsamen Punkt enthalten" Math. Ann. 83, 1921, 113-115.

RAO HEBBARE S.P. "A class of distance convex simple graphs" Ars Combinatoria 7, 1979, 19-26.

REAY J.R. "Caratheodory theorems in convex product structures" Pacif. J. Math. 35, 1970, 227-230. /See also the erratum

ROCKAFELLAR R.T. Convex analysis, Princeton N.J., Princeton Univ. Press 1970.

ROTA G.C. "On the foundations of combinatorial theory : 1 Theory of Möbius function" Z. Warsch. Verw. Gebeit 2, 1964, 340-368.

ROUDNEFF J.P. Matroïdes orientés et arrangements de pseudodroites, Thèse de 3ème cycle, Univ. Paris 6, 1986. [1986a]

ROUDNEFF J.P. "Tverberg-type theorems for pseudoconfigurations of points in the plane" to appear in Eur. J. Combin.[1986b]

SAMPATHKUMAR E. "Convex sets in a graph" Indian J. Pure Appl. Math. 15, 1984, n° 10, 1065-1071.

SCHMIDT J. "Veber die Rolle der transfiniten Schlussweisen in einer allgeineinen Ideal-theorie" Math. Nachr. 7, 1952, 165-182.

SCHMIDT J. "Einige grundelegende Begriffe and Saetze aus der Theorie der Huellen-operatoren" Bericht ueber die Mathematiker-Tagung in Berlin, Januar 1953, Deutscher Verlag der Wissenschaften, Berlin 1953, 21-48.

SCHRIJVER A. "Packing and covering in combinatorics" Math. Centre Tracts n° 106, Mathematisch Centrum, Amsterdam 1979.

SEKANINA M. "Graphs and betweeness" Mat. Casopis Sloven. Akad. Vied. 25, 1975, n° 1, 41-47.

SIERKSMA G. "Caratheodory and Helly numbers of convex-product-structures" Pac. J. Math. 61, 1975, 275-282.

SIERKSMA G. Axiomatic convexity theory and the convex product space, Doct. Diss. Univ. Groningen, 1976, 113 pages.

SIERKSMA G. "Convexity on unions of sets" Comp. Math. 42(3), 1981, 391-400.

SIERKSMA G., BOLAND J. "On Eckhoff's conjecture for Radon numbers; or how far the proof is still away" J. Geom. 20, 1983, n° 2, 116-121

SLATER P.J. "A characterization of soft hypergraphs" Canad. Math. Bull. 21, 1978, n° 3, 335-337.

SOLTAN V.P."Exchange and Caratheodory numbers in the Cartesian product of convexity structures" (en russe), Oukr. Geom. Sb. 24, 1981, 104-108.

SOLTAN V.P. "d-convexity in graphs" Soviet Math. Dokl. 28, 1983, 419-421.

SOLTAN V.P. Introduction to the theory of axiomatic convexity (russian), Chiintsa, Kichiniev 1984.

SOLTAN V.P., CHEPOI V.D. "Conditions for invariance of set diameters under d-convexification in a graph" Cybernetics 19, 1983, 750-756.

SOLTAN V.P., CHEPOI V.D. "The number of d-convex sets in a graph" (Russian) Izv. Akad. Nauk Moldav. SSR Ser. Fiz. tekn. Mat. Nauk, 1984 1984 n° 2, 19-23, 78 [1984a]

SOLTAN V.P., CHEPOI V.D. "d-convex sets in triangulated graphs (Russian) Mathematical modeling of processes in economics, Mat. Issled. n° 78, 1984, 105-124. [1984b]

SOLTAN V.P., CHEPOI V.D. "d-convexity and Steiner functions on a graph" (Russian,English summary) Dokl/ Akad. Nauk. BSSR 28, 1985 n° 2, 407-408, 476.

STANLEY R.P. "Combinatorial reciprocity theorems" Adv.in Math. 14, 1974, 194-253.

STANLEY R.P. "The upper bound conjecture and Cohen Macaulay rings" Studies in Appl. Math. 54, 1975, 135-142.

STANLEY R.P. "The number of faces of a simplicial convex polytope", Adv. Math. 35, 1980, 236-238.

STEMPLE J.G. "Geodetic graphs homeomorphic to a complete graph" Second Intern. Conf. on Comb. Math., New York 1978, 512-517; Ann. New York Acad; Sci. 319, New York Acad. Sci., New York 1979.

STOER J. WITZGALL C,J. Convexity and optimization in finite dimension I, Springer, Berlin 1970.

TARSKI A. "Fondamentale Begriffe der Methodologie der deduktiven Wissenschoften" Monatsh., Math. Phys. 37, 1930, 360-404.

THRON R. "Characterizations of the anti-exchange closure" Beiträge Algebra Geom n° 19, 1985, 65-74.

TROTTER W.T., MOORE J.I. "Characterization problems for graphs, partially ordered sets, lattices and families of sets, Disc. Math. 16, 1976, 361-381.

TUCKER A. "A structure theorem for the consecutive's property" J. Comb. Th. 12, 1972, 153-162.

TVERBERG H. "A generalization of Radon's theorem" J. London Math. Soc. 41, 1966, 123-128.

TVERBERG H. "A generalization of Radon's theorem II" Bull. Aust. Math. Soc. 24, 1981, 321-325.

VAN DER CRUYCE P. "A convexity problem in 3-polytopal graphs" Arch. Math. 43, 1984, 84-88.

VAN DER CRUYCE P. "A convex characterization of the graphs of the dodecahedron and icasahedron" Disc. Math. 50, 1984, 99-105.

VALENTINE F.A., Convex sets, Mc Graw Hill, New York 1964.

WEGNER G. "d-collapsing and nerves of families of convex sets" Arch. Math. 26, 1975, 317-321.

WELSH D.J.A. Matroid Theory, Acad. Press 1976.

WILLE R. Kongruenzklassengeometrien, Lecture Notes in Math. 113, Springer Verlag, Berlin 1970.

YAGLOM I.M., BOLTIANSKI V.G. Convexe Figuren, D.V. Wissensheften, Berlin 1956.

YAGLOM I.M., BOLTIANSKI V.G. Convex Figures, Holt, New York 1961.

YEMELICHEV V.A., KOVALEV M.M., KRAVTSOV M.K. Polytopes, Graphs and Optimization, Cambridge Univ. Press, Cambridge - London - New York - New Rochelle - Melbourne - Sydney 1984.

ZELINKA B. "Geodetic graphs which are homeomorphic to complete graphs", Math. Slovaca 27, 1977, 129-132.

ERRATUM

DOIGNON J.P. "Convexity in cristallographical lattices" Journ. of Geom. 3, 1973, 71-85.

DOIGNON J.P. "Caractérisations d'espaces de Pasch-Peano" Bull. Acad. Royale de Belgique (classes des Sciences) 62, 1976, 679-699.

DOIGNON J.P., REAY J.R., SIERKSMA G. "A Tverberg type generalization of the Helly number of a convexity space" J. of Geom. 16, 1981,118-125

ECKHOFF J."Helly theorem revisited", in preparation [1986].

REAY J.R. "Generalizations of a theorem of Caratheodory, Mem. Amer. Math. Soc. 54, 1965, 1-50.

SHOLANDER "Trees, Lattices, order and betweeness" Proc. A.M.S. 3, 1952, 369-381.

WHITFIELD J.H.M., YONG S. "A characterization of line spaces" Canad. Math. Bull. 24, 1981, 273-277.

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