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In: Zdeněk Frolík and Vladimír Souček and Marián J. Fabián (eds.): Proceedings of the 14th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1987. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 14. pp. [385]--389.

Persistent URL: http://dml.cz/dmlcz/701910

ON SETS OF SMALL MEASURE

Oľga Kulcsárová and Beloslav Riečan

In many applications of measure theory it is not necessary to know the precise value m(E), but only the fact, wheather $m(E) = 0$ or $m(E) \neq 0$. In another area of problems it is necessary to know only wheather E has a "small" measure or not. One possibility of precising the notion of a small element is contained in the following definition.

Definition 1. Let S be a lattice with the least element 0. By a small system we shall understand a sequence $(N_n)_n$, $N_n \subset S$ satisfying the following conditions:

1. $0 \in N_n$, $N_{n+1} \subset N_n$ for every $n \in N$.
2. If $a \in N_n$, $b \in S$ and $b \leqq a$, then $b \in N_n$.
3. If a, b, $c \in N_n$, then $a \vee b \vee c \in N_{n-1}$.
4. If $a_i \geqq a_{i+1}$ $(i = 1, 2, \dots)$ $\bigwedge a_i = 0$, then to every $n \in N$ there is i such that $a_i \in N_n$.

As an example one can consider a finite measure space (X,S,m) and put $N_n = \{E \in S: m(E) < 3^{-n}\}$. As another example one can consider the set S of all integrable functions and put $N_n = \{f \in S: \int |f| \, d\mu < 3^{-n}\}$. The notion of a small system was introduced in [4] (for σ-rings S of sets only) and a review of the theory and applications is contained in [5] and [3]. In this note we shall present another characterizations of small systems by the help of real functions. This paper is in final form and no version of it will be submitted for publication elsewhere.

Definition 2. Let S be a lattice with the least element 0. A function $m: S \longrightarrow R$ will be called a submeasure, if the following properties are satisfied:

1. $m(0) = 0$.
2. If $a \leqq \bigvee_{i=1}^{n} a_i$, then $m(a) \leqq \sum_{i=1}^{n} m(a_i)$.
3. If $a_i \geqq a_{i+1}$ $(i = 1, 2, \dots)$ and $\bigwedge a_i = 0$, then

$\lim_{i\to\infty} m(a_i) = 0.$

Our main result states that the two concepts are equivalent in the following way.

Definition 3. A sequence $(N_n)_n$ of subsets of S and a submeasure m: $S \to R$ are called to be equivalent if the following two properties are satisfied:

(i) To every $\varepsilon > 0$ there exists $n \in N$ such that $a \in N_n$ implies $m(a) < \varepsilon$.

(ii) To every $n \in N$ there exists $\varepsilon > 0$ such that $m(a) < \varepsilon$ implies $a \in N_n$.

Theorem 1. Let S be a distributive lattice with the least element 0. Then to every submeasure m: $S \to R$ there exists a small system $(N_n)_n$ equivalent with m and to every small system $(N_n)_n$ there exists a submeasure m: $S \to R$ equivalent with $(N_n)_n$.

Proof. If m is a submeasure, then it is sufficient to put $N_n = \{a \in S;\ m(a) < 3^{-n}\}$. On the other hand, to given $(N_n)_n$ we put

$h(x) = \sup\{n \in N;\ x \in N_n\}$, $f(x) = e^{-h(x)}$,

$$m(x) = \inf\left\{\sum_{i=1}^{n} f(x_i);\ x = \bigvee_{i=1}^{n} x_i,\ x_i \in S,\ n \in N\right\}.$$

Evidently $h(0) = \infty$, $f(0) = 0$, $m(0) = 0$. Now we prove the conditions 2 and 3 of Definition 2. First let $b, c \in S$, $b \leqq c$. Then to every $\varepsilon > 0$ there are c_i such that $\bigvee c_i = c$, $m(c) + \varepsilon \geqq \sum f(c_i) \geqq \sum f(c_i \wedge b)$, where $\bigvee (c_i \wedge b) = b \wedge \bigvee c_i = b \wedge c = b$, so that $m(c) + \varepsilon > \sum f(c_i \wedge b) \geqq m(b)$, hence $m(b) \leqq m(c)$. Further for every $x, y \in S$ and $\varepsilon > 0$ there are x_i, $y_j \in S$ such that $x = \bigvee x_i$, $y = \bigvee y_j$ and $m(x) + \varepsilon > \sum f(x_i)$, $m(y) + \varepsilon > \sum f(y_j)$, hence

$$m(x) + m(y) + 2\varepsilon > \sum f(x_i) + \sum f(y_j) \geqq m(x \vee y)$$

because of $x \vee y = \bigvee x_i \vee \bigvee y_j$. Therefore $m(x) + m(y) \geqq m(x \vee y)$, so that the condition 2 is satisfied. If $a_i \geqq a_{i+1}$ $(i = 1, 2, \dots)$ and $\bigwedge a_i = 0$, then to every $n \in N$ (with $2^{-n} < \varepsilon$) there is such i that $a_i \in N_n$. Then $h(a_i) \geqq n$, $m(a_i) \leqq f(a_i) = 2^{-h(a_i)} \leqq 2^{-n} < \varepsilon$. Hence also the condition 3 is satisfied. The fact that $(N_n)_n$ and m are equivalent follows from the inequalities (see [2])

$$m(x) \leqq f(x) \leqq 2\, m(x) \tag{1}$$

for all $x \in S$. Since $m(x) \leqq f(x)$ is evident, we shall prove $f(x) \leqq 2m(x)$ only. Let

$x = \bigvee_{i=1}^{n} x_i$. Put $a = \sum_{i=1}^{n} f(x_i)$. We shall prove by induction

$$f(x) \leq 2 \sum_{i=1}^{n} f(x_i) = 2a.$$

If $a < \infty$ there are two possibilities: 1. $f(x_i) < a/2$ for all i. 2. There is i such that $f(x_i) \geq a/2$. In the first case choose maximal k such that $\sum_{i=1}^{k-1} f(x_i) < a/2$. Since $\sum_{i=1}^{k} f(x_i) \geq a/2$, we obtain $\sum_{i=k+1}^{n} f(x_i) = a - \sum_{i=1}^{k} f(x_i) \leq a/2$. Because of the inductive assumption we have

$$f(\bigvee_{i=1}^{k-1} x_i) \leq 2 \sum_{i=1}^{k-1} f(x_i) \leq a, \quad f(\bigvee_{i=k+1}^{n} x_i) \leq a \tag{2}$$

and moreover $f(x_k) \leq \sum_{i=1}^{n} f(x_i) = a$. Now, if we put $r(a) = \inf \{ n: 2^{-n} \leq a\}$, then

$$f(y) \leq a \Rightarrow y \in N_{r(a)} \tag{3}$$

Indeed, $2^{-h(y)} = f(y) \leq a$ implies $r(a) \leq h(y)$, so that $y \in N_{r(a)}$.

Now (2) and (3) imply $\bigvee_{i=1}^{k-1} x_i \in N_{r(a)}$, $x_k \in N_{r(a)}$, $\bigvee_{i=k+1}^{n} x_i \in N_{r(a)}$ hence by the axiom 3 of small systems $x = \bigvee_{i=1}^{n} x_i \in N_{r(a)-1}$, $f(x) \leq 2.2^{-r(a)} \leq 2a = 2 \sum_{i=1}^{n} f(x_i)$. If there is i such that $f(x_i) \geq a/2$ (say, $f(x_n) \geq a/2$), then $\sum_{i=1}^{n-1} f(x_i) = a - f(x_n) \leq a/2$ so by induction assumption $f(\bigvee_{i=1}^{n-1} x_i) \leq 2 \frac{a}{2} = a$. By (3) we obtain $\bigvee_{i=1}^{n-1} x_i \in N_{r(a)}$, $x_n \in N_{r(a)}$, hence $x = \bigvee_{i=1}^{n} x_i \in N_{r(a)-1}$ and $f(x) \leq 2.2^{-r(a)} \leq 2a = 2 \sum_{i=1}^{n} f(x_i)$. So we have proved

$$x = \bigvee_{i=1}^{n} x_i \Rightarrow f(x) \leq 2 \sum_{i=1}^{n} f(x_i). \tag{4}$$

The implication (4) implies $f(x) \leq 2m(x)$, so that (1) is proved.

Now to every $\varepsilon > 0$ choose $n > -\log_2 \varepsilon$. Then $x \in N_n$ implies $h(x) \geq n$, $f(x) \leq 2^{-n}$, hence by (1) $m(x) \leq f(x) \leq 2^{-n} < \varepsilon$. On the other hand, for every $n \in N$ choose $\varepsilon < 2^{-n-1}$. Then $m(x) < \varepsilon$ implies $2^{-h(x)} = f(x) \leq 2\, m(x) < 2\varepsilon < 2^{-n}$ so that $h(x) \geq n$ and $x \in N_n$. We have proved that $(N_n)_n$ and m are equivalent.

Another possibility for characterizing the family of elements of small measure gives the fuzzy set theory ([1], [6]). By a fuzzy subset of a given space X we mean any mapping $u: X \longrightarrow \langle 0, 1 \rangle$. The number $u(x)$ represents the degree in which an element x has given property.

Definition 4. Let S be a lattice with the least element 0. We shall say that a real function $u: S \longrightarrow \langle 0, 1 \rangle$ is a fuzzy set of small elements, if the following properties are satisfied:

1. $u(0) = 0$.
2. If $b \leq \bigvee_{i=1}^{n} a_i$, then $u(b) \geq \prod_{i=1}^{n} u(a_i)$.
3. If $a_i \geq a_{i+1}$ $(i = 1, 2, \dots)$ and $\bigwedge a_i = 0$, then $\lim_{i \to \infty} u(a_i) = 1$.

Theorem 2. Let S be a distributive lattice with the least element 0. Then to every fuzzy set u of small elements there is a small system $(N_n)_n$ equivalent to u (i.e. to every n there is $\varepsilon > 0$ such that $u(x) > 1 - \varepsilon \Longrightarrow x \in N_n$ and to every $\varepsilon > 0$ there is $n \in N$ such that $x \in N_n \Longrightarrow u(x) > 1 - \varepsilon$) and to every small system $(N_n)_n$ there is a fuzzy set u of small elements equivalent to $(N_n)_n$.

Proof. It is an immediate consequence of Theorem 1.

REFERENCES

[1] DUBOIS D. - PRADE H. "Fuzzy sets and systems. Theory and applications", Acad. Press, New York 1980.

[2] KOMORNÍK J. "Correspondence between semi-measures and small systems", Mat. čas. SAV, 25 (1975), 59-62.

[3] NEUBRUNN T. - RIEČAN B. "Measure and integral", VEDA, Bratislava 1981.

[4] RIEČAN B. "Abstract formulation of some theorems of measure theory", Mat.-fyz. čas. SAV, 16 (1966), 286-273.

[5] RIEČAN B. "Subadditive measure and small systems", Čas. pěst. mat., 99 (1974), 284-290.

[6] ZADEH L. A. "Fuzzy sets", Information and control, 8 (1965), 338-353.

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