Jan Paseka; Bohumil Šmarda

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On Some Notions Related to Compactness for Locales

JAN PASEKA, BOHUMIL ŠMARDA*)

Brno, Czechoslovakia

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There are four possible ways of saying what it means for a topological space X to be locally compact:

(1) Every point of X has a compact closed neighbourhood (or, a neighbourhood whose closure is compact).

(2) Every point of X has a compact neighbourhood.

(3) Every point of X has a base of compact neighbourhoods (i.e., given $x \in U$ open in X, there exists a compact K with $x \in K \subseteq U$).

(4) Every point of X has a base of compact closed neighbourhoods.

For Hausdorff space X, there are all equivalent, of course; and many textbooks on topology, whose authors aren't particularly interested in compactness in non-Hausdorff spaces, tent to give (1) or (2) as the definition of local compactness. The condition (3) is the correct and usual notion of local compactness for not-necessarily-Hausdorff spaces, because it conforms to the general scheme for defining local version of topological properties and, as it is well known (see e.g. [4]), locally compact locales in this sense are exactly the distributive continuous lattices. In this paper we will study the locale-theoretic analogue of the condition (1) called weak local compactness.

A locale L is compact iff L is weakly locally compact and almost compact. Weakly locally compact locales are closed under closed sublocales and finite products. An arbitrary product ΠL_γ of locales is weakly locally compact iff each L_γ is weakly locally compact and L_γ is compact for all but finitely many γ. A sum ΣL_γ is weakly locally compact iff each L_γ is weakly locally compact.

In the second part we investigate almost compact locales. A product ΠL_γ is almost compact iff any L_γ is almost compact. A Hausdorff locale L is compact iff $\uparrow a$ is almost compact for all $a \in L$. If L is a regular locally almost compact locale then L is weakly locally compact.

The notion of the one-point extension may be adapted to locales (for spaces see [1]) and we consider some connections between locales and their one-point extensions

*) Katedra algebry a geometrie PF UJEP, Janáčkovo nám. 2a, 662 95 Brno, Czechoslovakia.

concerning separation axioms. We investigate also the one-point compactification of locales, which coincides with the Alexandroff extension on topological spaces. Using the one-point compactification, we can prove that every weakly locally compact regular locale is spatial. Some of these results are generalized from known results for spaces (for example, see [1] and [12]).

All unexplained facts concerning locales can be found in P. T. Johnstone [5]. Recall that *a frame* is a complete lattice L in which the infinite distributive law $a \wedge \bigvee S = \bigvee\{a \wedge s: s \in S\}$ holds for all $a \in L$, $S \subseteq L$. A frame homomorphism $K \to L$ is a map preserving finite meets and arbitrary joins. Let *Frm* be the category of frames. Many facts (see [5]) indicate the importance of the opposite category $Loc = Frm^{op}$. Objects of *Loc* are called *locales*. Of course, sublocales correspond to quotient frames and products of locales correspond to sums of frames. If T is a topological space then the lattice $O(T)$ of all open sets of T is a locale. These locales and locales isomorphic with them are called spatial or topologies. A continuous map $f: S \to T$ of topological spaces determines a frame homomorphism $O(f): O(T) \to$ $\to O(S)$ sending $V \in O(T)$ to $f^{-1}(V)$. We get a functor $O: Top \to Loc$, where Top is the category of topological spaces and continuous maps. O has a right adjoint $P: Loc \to Top$ assigning to a locale L the topological space $P(L)$ of prime (i.e. $\wedge$-irreducible and $\neq 1$) elements of L. Open sets of $P(L)$ are $\hat{x} = \{a \in P(L): x \nleqq a\}$, where $x \in L$.

From the topological point of view, we will formulate results in the category *Loc*, but proofs, which are mostly carried out in lattice-theoretic terms, in the category *Frm*.

Let L be a locale. L is *regular* ([3]) if $a = \bigvee(x \in L: x \lhd a)$ for all $a \in L$, where $x \lhd a$ means $x^* \vee a = 1$ (where x^* is the pseudocomplement of x). L is *Hausdorff* ([6]) if $a, b \in L$, $1 \neq a \nleqq b$ implies that there exists $c \in L$ such that $c^* \nleqq a$, $c \nleqq b$. It was proved in [6] that L is a Hausdorff locale iff $a = \bigvee \Box a$ for each $a \in L \setminus \{1\}$, where $\Box a = \{x \in L: x \leqq a, x^* \nleqq a\}$. L is a *T_2'-locale* ([10]) if, for each $a \in L \setminus \{1\}$, there exists an ideal $A \subseteq \Box a$ such that $a = \bigvee A$. L is conjunctive if for each two elements $a, b \in L$ with $a \nleqq b$ there is an element $c \in L$ such that $a \vee c = 1$ and $b \vee c \neq 1$. We put $\Box 1 = L$.

We say that an element $a \in L$, $a \neq 1$ of a locale L is prime (*semiprime*, resp.) if $x \wedge y \leqq a \Rightarrow x \leqq a$ or $y \leqq a$ ($x \wedge y = 0 \Rightarrow x \leqq a$ or $y \leqq a$, resp.) holds, for each $x, y \in L$. If we denote $D(L)$ ($P(L)$ resp., $S(L)$ resp.) the set of all dual atoms (prime elements resp., semiprime elements resp.) in L then $D(L) \subseteq P(L) \subseteq S(L)$. We say that L is a *T_1-locale* (an S-locale resp.) if $P(L) = D(L)$ ($S(L) = D(L)$ resp.) – see [9]. Spatial Hausdorff locales (or T_2'-locales or S-locales) are topologies of usual Hausdorff topological spaces. A locale L is *dually atomic* if for any $1 \neq a \in L$ there is a dual atom $d \in D(L)$ such that $d \geqq a$.

Recall that sublocals of L correspond to *nuclei* on L, i.e., to maps $j: L \to L$ such that $a \leqq j(a)$, $jj(a) = j(a)$ and $j(a \wedge b) = j(a) \wedge j(b)$ for all $a, b \in L$. A surjective homomorphism $f: K \to L$ of frames is *closed* if $f(a) = f(b) \Rightarrow a \vee f^0(0) = b \vee$

$\vee f^0(0)$ for each $a, b \in K$, where $f^0(0) = \bigvee(x \in K: f(x) = 0)$. We denote $L_r = \{l \in L: l = l^{**}\}$.

1. Weakly locally compact locales

Let us recall that a locale L is *almost compact* if each covering of L has a finite dense subset. For a locale L we will denote $S_L = \{l \in L: l^* \neq 0\}$. Then the following are equivalent:

1. L is not almost compact.
2. An ideal Q in L exists such that $Q \subseteq S_L$, $\bigvee Q = 1$.
3. A proper filter F in L exists such that $\bigvee(a^*: a \in F) = 1$.

Such a filter is called an *α-filter*.

Some properties of almost compact locales are in [10]. Recall that a topological space T is *locally compact* iff for each $x \in T$ there exists an open set U such that $x \in U$, $\bar{U}$ is compact. If L is a locale then we put $F_C = \{a \in L: \uparrow a \text{ is compact}\}$.

1.1. Proposition. Let T be a topological space, $O(T)$ be the locale of all open sets of T. Then T is locally compact iff $\bigvee(a^*: a \in F_C) = 1$.

Proof. $\Rightarrow$: If $x \in T$ then an open set U exists such that $x \in U$, $\bar{U}$ is compact, i.e., $T \setminus \bar{U}$ is open, $T \setminus \bar{U} \in F_C$. Clearly, $x \in U \subseteq (T \setminus \bar{U})^*$, i.e. $\bigvee(a^*: a \in F_C) = 1$.

$\Rightarrow$: If $x \in T$ then $a \in F_C$ exists such that $x \in a^*$. Clearly, $T \setminus a$ is compact and closed. Now, we have $a^* \subseteq T \setminus a$, i.e., $\bar{a}^* \subseteq T \setminus a$. Evidently, $\bar{a}^*$ is compact.

Motivated by 1.1, we adopt the following

Definition. Let L be a locale. We say that L is weakly locally compact or wl-compact if $\bigvee(a^*: a \in F_C) = 1$.

Clearly, compact locales are wl-compact. Namely, if L is compact then $0 \in F_C$, i.e., $1 = 0^* = \bigvee(a^*: a \in F_C)$.

1.2. Proposition. Let L be a locale which is not compact. Then L is wl-compact iff F_C is an α-filter.

Proof. $\Rightarrow$: Since $\bigvee(a^*: a \in F_C) = 1$ we have to show that F_C is a filter. Evidently. $0 \notin F_C$ and $b \geqq a$, $a \in F_C \Rightarrow b \in F_C$. Let $a, b \in F_C$, $\bigvee_{i \in I} x_i = 1$, $x_i \geqq a \wedge b$ for any $i \in I$. Since $\uparrow a, \uparrow b$ are compact we have $\bigvee(x_i: i \in K) \vee a = 1 = \bigvee(x_i: i \in K) \vee b$ for some finite $K \subseteq I$. Now, we have $1 = [\bigvee(x_i: i \in K) \vee a] \wedge [\bigvee(x_i: i \in K) \vee b] = \bigvee(x_i: i \in K) \vee (a \wedge b)$, i.e., $a \wedge b \in F_C$. The rest of the proof is obvious.

As an appl'cation of 1.2 we have the following characterization of compact locales.

1.3. Theorem. A locale L is compact iff L is wl-compact and almost compact.

Proof. $\Rightarrow$: It is evident.
$\Leftarrow$: This results immediately from 1.2 by the fact that a frame L is not almost compact iff there exists an α-filter in L (see [10]).

1.4. Lemma. Let L be a locale, $a \in L$. If $\uparrow x$ is compact in L then $\uparrow(x \vee a)$ is compact in $\uparrow a$.

1.5. Proposition. Every closed sublocale of a wl-compact locale is a wl-compact locale.

Proof. Let L be a frame, $a \in L$. Now, we have $1 = \bigvee(x^*: \uparrow x \text{ is compact in } L) = \bigvee(x^* \vee a: \uparrow(x \vee a) \text{ is compact in } \uparrow a) \leqq \bigvee(y^{\otimes} \geqq a: \uparrow y \text{ is compact in } \uparrow a)$, where $y^{\otimes}$ is the pseudocomplement in $\uparrow a$. In all we obtain that $\uparrow a$ is wl-compact.

1.6. Proposition. Let L be a wl-compact locale. Then for each $1 \neq a \in F_C$ there exists $d \in D(L)$ such that $d \geqq a$. Moreover, L is dually atomic.

Proof. If $1 \neq a \in F_C$ then $\uparrow a$ is dually atomic because $\uparrow a$ is compact. Clearly, $D(\uparrow a) \subseteq D(L)$. Namely, if d is a dual atom in $\uparrow a$ and $x > d$, $x \in L$ then $x \in \uparrow a$, i.e., $x = 1$. The rest follows from the fact that there exists $a \in F_C$, $a \neq 1$. Evidently, if $F_C \setminus \{1\} = \emptyset$ then $1 = \bigvee(a^*: a \in F_C) = \bigvee(a^*: a \in F_C \setminus \{1\}) = 0$, a contradiction. If $1 \neq b \in L$ then $\uparrow b$ is wl-compact, i.e., there is an element $m \in D(\uparrow b) \subseteq D(L)$.

1.7. Proposition. Let L be a frame, $a, b \in L$ such that $\uparrow a$, $\uparrow b$ be wl-compact. Then $\uparrow(a \wedge b)$ is wl-compact.

Proof. If $\uparrow x$ is compact in $\uparrow a$, $\uparrow y$ is compact in $\uparrow b$ then $\uparrow(x \wedge y)$ is compact in $\uparrow(a \wedge b)$. Now, we have $\bigvee(x^{\otimes_1}: \uparrow x \text{ is compact in } \uparrow a) = 1 = \bigvee(y^{\otimes_2}: \uparrow y \text{ is compact in } \uparrow b)$, where $x^{\otimes_1}$, $(y^{\otimes_2})$ is the pseudocomplement in $\uparrow a$ $(\uparrow b)$. Clearly, $x^{\otimes_1} \wedge y^{\otimes_2} \leqq \leqq (x \wedge y)^{\otimes}$, where $(x \wedge y)^{\otimes}$ is the pseudocomplement in $\uparrow(a \wedge b)$. Evidently, $1 = \bigvee(x^{\otimes_1} \wedge y^{\otimes_2}: \uparrow x \text{ is compact in } \uparrow a, \uparrow y \text{ is compact in } \uparrow b) \leqq \bigvee(z^{\otimes}: \uparrow z \text{ is compact in } \uparrow(a \wedge b))$, i.e., $\uparrow(a \wedge b)$ is wl-compact.

1.8. Remark. It is interesting to note that wl-compact Hausdorff spaces are regular but there exists a wl-compact Hasudorff locale which is not regular (see [10], Prop. 2.4).

1.9. Proposition. If L is a wl-compact regular locale then $a = \bigvee(x \lhd a: x^* \in F_C)$ for each $a \in L$.

Proof. Let $a \in L$. Now, we have $a = \bigvee(x\colon x \lhd a)$, $1 = \bigvee(y\colon y^* \in F_C)$. Clearly, $a = \bigvee(x \wedge y\colon x \lhd a,\, y^* \in F_C) = \bigvee(z\colon z \lhd a,\, z^* \in F_C)$.

This suggests the following

1.10. Lemma. Let L be a locale. Then it holds:

(i) $x \lhd a,\, x^* \in F_C \Rightarrow x \ll a$ (x is way below a – see [4]).

(ii) If L is a regular wl-compact locale then $x \ll a$ iff $x \lhd a,\, x^* \in F_C$.

Proof. (i) Let $x \lhd a$, $x^* \in F_C$ and $S \subseteq L$ be a directed set such that $a \leqq \bigvee S$. Then $x^* \vee \bigvee S = 1$, i.e., there is $s \in S$ such that $x^* \vee s = 1$ and we have $x \leqq s$.

(ii) Since L is a regular wl-compact frame we have from 1.9 and 1.10 (i) that L is continuous, i.e., the space $(P(L), O(P(L)))$ is a locally compact Hausdorff space. Now, let $x \ll a$. Then there exists by [5], 4.2 a compact set $K \subseteq P(L)$ such that $x \subseteq K \subseteq a$. Clearly, it is easy to check that $P(L) \setminus K \in F_C$ and we have $P(L) \setminus K \subseteq x$, i.e., $x \lhd a$, $x^* \in F_C$.

1.11. Corollary. Let L be a regular locale. Then L is continuous iff L is a wl-compact locale.

Proof. It follows from 1.10 and 1.9.

1.12. Lemma. If L is a wl-compact locale then for each $a \in F_C$ there exists $x \in F_C$ such that $x \lhd a$.

Proof. Evidently, $\bigvee(x^*\colon x \in F_C) = 1$. Since $\uparrow a$ is compact in L then there exists $x \in F_C$ such that $x^* \vee a = 1$, i.e., $x \lhd a$, $x \in F_C$.

We call the attention to the fact that the proofs are in the category *Frm* of frames.

1.13. Proposition. If L is a locale then $L \cong L \times 2$, where 2 denotes the dyadic locale which has precisely two elements 0 and 1.

Proof. If $i_1\colon L \to L + 2$, $i_2\colon 2 \to L + 2$ are the canonical injections then each element in $L + 2$ has the form $i_1(x)$ for some $x \in L$. Namely, if $\bar{x} \in L + 2$ then $\bar{x} = \bigvee_j i_1(x_j) \wedge i_2(y_j)$, $x_j \in L$, $y_j \in 2$. Now, we have $\bar{x} = \bigvee(i_1(x_j) \wedge i_2(y_j)\colon y_j = 0) \vee \bigvee(i_1(x_j) \wedge i_2(y_j)\colon y_j = 1) = \bigvee i_1(x_j) = i_1(\bigvee x_j) = i_1(x)$ for some $x \in L$. The rest is obvious.

1.14. Proposition. A finite product of wl-compact locales is wl-compact.

Proof. It is enough to prove that a sum of two wl-compact frames is wl-compact. The rest follows by an obvious induction.

Let L, K be wl-compact frames, $i_1: L \to L + K$, $i_2: K \to L + K$ be the canonical injections. Let $x \in L$, $y \in K$, $\uparrow x$ be compact in L, $\uparrow y$ be compact in K. Now, we have $\uparrow x + \uparrow y \cong \uparrow(i_1(x) \wedge i_2(y))$, i.e., $\uparrow(i_1(x) \vee i_2(y))$ is compact because a sum of compact frames is compact. Evidently, $\bigvee(a^*: \uparrow a$ is compact in $L + K) \geqq \bigvee((i_1(x) \vee \vee\, i_2(y))^*: \uparrow x$ is compact in L, $\uparrow y$ is compact in $K) = \bigvee(i_1(x^*) \wedge i_2(y^*): \uparrow x$ is compact in L, $\uparrow y$ is compact in $K) = i_1(\bigvee(x^*: \uparrow x$ is compact in $L)) \wedge i_2(\bigvee(y^*: \uparrow y$ is compact in $K)) = 1$ because L and K are wl-compact.

1.15. Theorem. Let L_γ, $\gamma \in \Gamma$ be locales. Then the product $\Pi(L_\gamma: \gamma \in \Gamma)$ is wl-compact iff all L_γ are wl-compact and L_γ are compact for all but finitely many $\gamma \in \Gamma$.

Proof. $\Rightarrow$: a) Let $\gamma_0 \in \Gamma$. Since ΣL_γ is wl-compact then there exists a dual atom D in ΣL_γ which has the form $D = \bigvee(i_\gamma(d_\gamma): d_\gamma$ is a dual atom in L_γ, $\gamma \in \Gamma)$. If we put $x = i_{\gamma_0}(0) \vee \bigvee(i_\gamma(d_\gamma): \gamma \neq \gamma_0)$ then $\uparrow x$ is wl-compact (see 1.6), $\uparrow x \cong L_{\gamma_0} + 2$, where $\sum_{\gamma \neq \gamma_0} \uparrow d_\gamma \cong 2$, i.e., L_{γ_0} is wl-compact.

b) Let D be the dual atom from the part a). Since ΣL_γ is wl-compact we have $1 = \bigvee(a^*: \uparrow a$ is compact in $\Sigma L_\gamma)$. Now, there exists some $a \in \Sigma L_\gamma$, $\uparrow a$ is compact in ΣL_γ such that $a^* \nleqq D$, i.e., there exist indices $\gamma_1, \ldots, \gamma_n \in \Gamma$ and elements $x_i \in L_{\gamma_i}$ $(i = 1, \ldots, n)$ such that $i_{\gamma_1}(x_1) \wedge \ldots \wedge i_{\gamma_n}(x_n) \nleqq d$, $i_{\gamma_1}(x_1) \wedge \ldots \wedge i_{\gamma_n}(x_n) \leqq a^*$. Clearly, $[i_{\gamma_1}(x_1) \wedge \ldots \wedge i_{\gamma_n}(x_n)]^* = i_{\gamma_1}(x_1^*) \vee \ldots \vee i_{\gamma_n}(x_n^*) = b \neq 1$, $b \geqq a$, i.e., $\uparrow b$ is compact in ΣL_γ.

Let $\gamma \neq \gamma_i$ $(i = 1, \ldots, n)$. We show that L_γ is compact. If $y_j \in L_\gamma$, $\bigvee y_j = 1$ then $\bigvee i_\gamma(y_j) = 1$, i.e., $\bigvee_{k=1}^m i_\gamma(y_{jk}) \vee b = 1$. Now, we have $1 = i_\gamma(\bigvee_{k=1}^m y_{jk}) \vee i_{\gamma_1}(x_1^*) \vee \ldots \ldots \vee i_{\gamma_n}(x_n^*)$. Since $\gamma \neq \gamma_i$ $(i = 1, \ldots, n)$, we have that $1 = \bigvee_{k=1}^m y_{jk}$, i.e., L_γ is compact.

$\Leftarrow$: Let each L_γ be wl-compact. We denote Γ_0 the set of indices of all non-compact L_γ. Clearly, Γ_0 is finite and we have $\sum_{\gamma \in \Gamma} L_\gamma \cong \sum_{\gamma \in \Gamma_0} L_\gamma + \sum_{\gamma \notin \Gamma_0} L_\gamma$. From 1.14 we know that $\sum_{\gamma \in \Gamma_0} L_\gamma$ is wl-compact and from Tychonoff theorem we have that $\sum_{\gamma \notin \Gamma_0} L_\gamma$ is compact and hence wl-compact. Finally, ΣL_γ is again wl-compact.

1.16. Theorem. Let L_γ $(\gamma \in \Gamma)$ be locales. Then the sum ΣL_γ is wl-compact iff L_γ are wl-compact for all $\gamma \in \Gamma$.

Proof. $\Rightarrow$: Let $\pi_\gamma: \Pi L_\gamma \to L_\gamma$ be the canonical projections (in the category *Frm*) and let us put $x_{\gamma_0} = \bigvee(y \in \Pi L_\gamma: \pi_{\gamma_0}(y) = 0)$ for each $\gamma_0 \in \Gamma$. Then $\uparrow x_{\gamma_0} \cong L_{\gamma_0}$ and $\uparrow x_{\gamma_0}$ is wl-compact (see 1.6).

$\Leftarrow$: Let each L_γ be wl-compact and $\uparrow y_\gamma$ be compact in L_γ. Then $\bar{y}_\gamma = \bigvee(y \in \Pi L_\gamma: \pi_\gamma(y) = y_\gamma)$ is such that $\uparrow \bar{y}_\gamma$ is compact in ΠL_γ which can be easily verified. Now, we have $\pi_\beta(\bar{y}_\gamma^*) = 0$ for $\beta \neq \gamma$, $\pi_\gamma(\bar{y}_\gamma^*) = y_\gamma^*$. Evidently, $\bigvee(y^*: \uparrow y$ is compact in $\Pi L_\gamma) \geqq \geqq \bigvee(\bar{y}_\gamma^*: \uparrow y_\gamma$ is compact in $L_\gamma) = 1$ because all L_γ are wl-compact.

2. A note on almost compact locales

2.1. Lemma. If L is a locale and $Q \subseteq L$ is an ideal maximal with respect to the property $Q \subseteq S_L$ then

(i) $x \in Q \Rightarrow x^{**} \in Q$,

(ii) Q is prime in $Id(L)$, i.e., $x \wedge y \in Q \Rightarrow x \in Q$ or $y \in Q$.

Proof. (i) If $x \in Q$, $x^{**} \notin Q$ then $y \in Q$ exists such that $0 = (x^{**} \vee y)^* = x^* \wedge \wedge y^* = (x \vee y)^*$, a contradiction with the fact that $x \vee y \in Q \subseteq S_L$.

(ii) If $x \wedge y \in Q$, $x \in L \setminus Q$, $y \in L \setminus Q$ then $x_1, y_1 \in Q$ exist such that $(x \vee x_1)^* = = 0 = (y \vee y_1)^*$. Now, we have $0 = (x^* \wedge x_1^*) \vee (y^* \wedge y_1^*) \geqq (x^* \vee y^*) \wedge \wedge (x_1^* \wedge y_1^*)$. If we put $z_1 = x_1 \vee y_1$ then $z_1 \in Q$, $z_1^* = x_1^* \wedge y_1^*$. Clearly, $x^* \vee \vee y^* \leqq z_1^{**} \in Q$, i.e., $x^* \vee y^* \in Q$. Now, we have that $a = (x \wedge y)^{**} \vee x^* \vee \vee y^* \in Q$ and $a^* = (x \wedge y)^* \wedge (x \wedge y)^{**} = 0$, a contradiction with with $a \in \in Q \subseteq S_L$.

2.2. Theorem. Let L_γ $(\gamma \in \Gamma)$ be locales. Then the product ΠL_γ is almost compact iff L_γ are almost compact for all $\gamma \in \Gamma$.

Proof. $\Rightarrow$: Let $i_\gamma: L_\gamma \to \Sigma L_\gamma$ be the canonical injections, $\gamma_0 \in \Gamma$ and $S_{\gamma_0} \subseteq L_{\gamma_0}$ be such that $\bigvee S_{\gamma_0} = 1$.

We put $S = \{i_{\gamma_0}(s): s \in S_{\gamma_0}\}$. Clearly, $S \subseteq \Sigma L_\gamma$, $\bigvee S = 1$ and by almost compactness there exists a finite set $F \subseteq S$ such that $\bigvee(F)^* = 0$. Now, we have that there exists a finite set $F_{\gamma_0} \subseteq S_{\gamma_0}$ such that $0 = [\bigvee(i_{\gamma_0}(s): s \in F_{\gamma_0})]^* = [i_{\gamma_0}(\bigvee(s: s \in F_{\gamma_0}))]^* = = i_{\gamma_0}([\bigvee(s: s \in F_{\gamma_0})]^*)$. Since i_{γ_0} is dense then there exists a finite dense subset $F_{\gamma_0} \subseteq S_{\gamma_0}$, i.e., L_{γ_0} is almost compact.

$\Leftarrow$: If L_γ $(\gamma \in \Gamma)$ are almost compact frames and if ΣL_γ is not almost compact then there exists a maximal ideal Q with regard to the property $Q \subseteq S_{\Sigma L_\gamma}$ such that $\bigvee Q = 1$. Let $Q_\gamma = \{x_\gamma \in L_\gamma: i_\gamma(x_\gamma) \in Q\}$. Since Q is an ideal, each Q_γ is an ideal, $Q_\gamma \subseteq S_{L_\gamma}$. We put $q_\gamma = \bigvee Q_\gamma$. Clearly, $q_\gamma \neq 1$ because L_γ is almost compact. If $X = \bigvee(i_\gamma(q_\gamma): \gamma \in \Gamma)$ then $X \neq 1$, $Q \subseteq \uparrow X$. Namely, if $i_{\gamma_1}(x_1) \wedge \ldots \wedge i_{\gamma_n}(x_n) \in Q$ then γ_j exists such that $i_{\gamma_j}(x_j) \in Q$ because Q is prime. Now, we have $i_{\gamma_j}(x_j) \leqq \leqq i_{\gamma_j}(q_{\gamma_j})$, i.e., $i_{\gamma_1}(x_1) \wedge \ldots \wedge i_{\gamma_n}(x_n) \in \downarrow X$. On the other hand, $1 = \bigvee Q \leqq \bigvee \downarrow X = = X$, a contradiction. Finally, ΣL_γ is almost compact.

2.3. Proposition. If L is an almost compact locale, $a \in L$, then the closed sublocale $\uparrow a$ is almost compact.

Proof. If $x_i \in \uparrow a$, $\bigvee x_i = 1$ then $(\bigvee_{j=1}^{n} x_{ij})^{**} = 1$ for some finite set of x_{ij}, $1 \leqq j \leqq n$. If $z \wedge \bigvee_{j=1}^{n} x_{ij} \leqq a$ then $a^* \leqq (z \wedge \bigvee_{j=1}^{n} x_{ij})^{***} = [z^{**} \wedge (\bigvee_{j=1}^{n} x_{ij})^{**}]^* = z^*$, i.e., $z \leqq \leqq z^{**} \leqq a^{**} = a$. Now, we have $(\bigvee_{j=1}^{n} x_{ij})^{\otimes\otimes} = 1$, where $\otimes$ denotes the pseudo-complement in $\uparrow a$.

2.4. Proposition. If L is a locale, $j_i\colon L \to L_{j_i}$, $i \in \{1, \ldots, n\}$ are nuclei on L such that the locales L_{j_i} are almost compact then the locale L_j is almost compact, where $j = \bigwedge_{i=1}^{n} j_i$.

Proof will be done for $n = 2$. Let $(j_1 \wedge j_2)(\bigvee(a_k\colon k \in I)) = 1$, $a_k \in L$. Since L_{j_1} and L_{j_2} are almost compact then a finite set $K \subseteq I$ exists such that $j_i(x) \wedge \wedge\, j_i(\bigvee(a_k\colon k \in K)) = j_i(0)$ implies $j_i(x) = j_i(0)$ for each $x \in L$, $i = 1, 2$.

If $(j_1 \wedge j_2)(x) \wedge (j_1 \wedge j_2)(\bigvee(a_k\colon k \in K)) = (j_1 \wedge j_2)(0)$ then $j_i(x) \wedge \wedge\, j_i(\bigvee(a_k\colon k \in K)) = j_i(0)$, i.e., $j_i(x) = j_i(0)$ for $i = 1, 2$. Now, we have that $(j_1 \wedge j_2)(x) = (j_1 \wedge j_2)(0)$ and $L_{j_1 \wedge j_2}$ is almost compact.

2.5. Lemma. ([5]). If L is a locale, $j \leqq k$ are nuclei of L, $a, b \in L$ then
(i) $k(a) \neq k(b) \Rightarrow j(a) \neq j(b)$,
(ii) $k(a) > k(0) \Rightarrow j(a) > j(0)$ hold.

Proof. $j(a) = j(b) \Rightarrow k(a) = k(j(a)) = k(j(b)) = k(b)$.
Now we introduce a generalization of [8] on locales.

2.6. Proposition. Let L be a locale, A be a chain of nuclei of L such that each nuclei $j \in A$ is not 1 and L_j is almost compact. Then the set $G = \{g \in L\colon j(g)$ is dense in L_j for some $j \in A\}$ has the finite intersection property.

Proof. Let $g_1, \ldots, g_n \in G$, $j_i(g_i)$ is dense in L_{j_i}, $1 \leqq i \leqq n$, $j_1 \leqq j_2 \leqq \ldots \leqq j_n$. Then $j_n(g_n) > j_n(0)$ and from lemma 2.5 we have $j_{n-1}(g_n) > j_{n-1}(0)$. Since $j_n(g_{n-1})$ is dense in $L_{j_{n-1}}$ we have $j_{n-1}(g_{n-1}) \wedge j_{n-1}(g_n) > j_{n-1}(0)$. Consequently, $j_{n-2}(g_{n-1} \wedge g_n) > j_{n-2}(0)$. Now, we have $j_{n-2}(g_{n-2} \wedge g_{n-1} \wedge g_n) > j_{n-2(0)}$. Finally, we obtain $j_1(g_1 \wedge \ldots \wedge g_n) > j_1(0)$, i.e., $g_1 \wedge \ldots \wedge g_n \neq 0$.

2.7. Lemma. If L is a Hausdorff locale, $1 \neq a \in L$ such that $\uparrow a$ is almost compact then for each dual atom $d \in D(L)$ such that $d \vee a = 1$ there exists $h \in L$ with $d \vee h^* = 1$, $a \vee h$ is dense in $\uparrow a$.

Proof. Clearly, $1 = a \vee d = a \vee \bigvee(x\colon x \lhd d)$, i.e., there exists $h \lhd d$ such that $a \vee h$ is dense in $\uparrow a$.

2.8. Lemma. If L is a dually atomic almost compact Hausdorff locale and $A \subseteq L$ is a chain such that $a \in A$ implies $1 \neq a$, $\uparrow a$ is almost compact, then $\bigvee A \neq 1$.

Proof. From 2.6 we know that $G = \{g \in L\colon a \vee g$ is dense in $\uparrow a$ for some $a \in A\}$ has the finite intersection property, i.e., $\bigvee(g^*\colon g \in G) \neq 1$. Now, there exists a dual atom $d \in D(L)$ such that $d \geqq g^*$ for all $g \in G$.

Let $1 = \bigvee A$. Then $a \in A$ exists with $a \vee d = 1$, i.e., $h \in L$ exists such that $d \vee h^* = 1$, $a \vee h$ is dense in $\uparrow a$. Evidently, $h \in G$, i.e., $d \geqq h^*$, a contradiction.

Recall that a locale L is *compact* iff for each chain $\{a_i\}_{i \in I}$, $a_i \neq 1$ for each $i \in I$, is $\bigvee_{i \in I} a_i \neq 1$.

2.9. Theorem. Let L be a Hausdorff locale. Then L is compact iff $\uparrow a$ is almost compact for each $a \in L$.

Proof. $\Rightarrow$: It is evident.

$\Leftarrow$: Clearly, L is almost compact and dually atomic. Namely, $L = \uparrow 0$ and $\uparrow a$ is almost compact for each $1 \neq a \in L$, i.e., there exists an element d such that $a \leqq d \in D(\uparrow a) \subseteq D(L)$ (see [10], 2.13). The rest follows from 2.8.

Recall that a topological space T is *locally almost compact* if for each $x \in T$ there exists a neighbourhood $U(x)$ of x such that $U(x)$ is almost compact. Equivalently, T is locally almost compact iff for each $x \in T$ there exists an open set U such that $x \in U$, $\overline{U}$ is almost compact.

Let L be a locale. We put $F_a = \{x \in L: \uparrow x^{**}$ is almost compact$\}$. Clearly, $D(L) \subseteq F_a$ and each dense element lies in F_a.

2.10. Proposition. Let T be a topological space. Then T is locally almost compact iff $\bigvee(x^*: x \in F_a) = 1$.

Proof is similar as for wl-compact spaces.

Definition. We say that a locale L is locally almost compact if $\bigvee(x^*: x \in F_a) = 1$.

Clearly, each wl-compact locale is locally almost compact and each almost compact locale is locally almost compact.

2.11. Lemma. Let L be a locale, $l \in L_r$. Then $\uparrow l$ is almost compact iff for each $S \subseteq L$ such that $\bigvee S = 1$ there exists $S' \subseteq S$, S' finite such that $(l \vee \bigvee S')^* = 0$.

Proof. $\Rightarrow$: If $S \subseteq L$, $\bigvee S = 1$ then there is $S' \subseteq S$, S' finite such that $(l \vee \bigvee S')$ is dense in $\uparrow l$, i.e., $y \wedge (l \vee \bigvee S') \leqq l$ implies $y \leqq l$. If $y \wedge (l \vee \bigvee S') = 0$ then $y \leqq (l \vee \bigvee S') = l^* \wedge \bigvee(S')^*$. Now, we have $y = y \wedge l \leqq l \wedge l^* \wedge (\bigvee S')^* = 0$.

$\Leftarrow$: If $S \subseteq L$, $\bigvee S = 1$ then there exists $S' \subseteq S$, S' finite such that $(l \vee \bigvee S')^* = 0$. If $y \wedge (l \vee \bigvee S') \leqq l$ then $l^* \leqq (y^* \vee (l \wedge \bigvee S')^*)^{**} = y^*$, i,e, $y \leqq y^{**} \leqq l^{**} = l$.

2.12. Proposition. Let L be a locale which is not almost compact. Then L is locally almost compact iff F_a is an α-filter.

Proof follows from 2.11.

2.13. Lemma. Let L be a regular locale, $l \in L_r$. Then $l \in F_c$ iff $l \in F_a$.

Proof. $F_c \subseteq F_a$. If $l \in F_a$ then $\uparrow l$ is almost compact and regular, i.e., $\uparrow l$ is compact (see [10], 2.7). Now, we have that $l \in F_c$.

2.14. Proposition. If L is a regular locally almost compact locale then L is wl-compact.

Proof. Evidently, $1 = \bigvee(x^*: x \in F_a) = \bigvee(x^*: x^{**} \in F_a) = \bigvee(x^*: x^{**} \in F_c)$.

2.15. Proposition. If L is a locally almost compact locale then L has at least one semiprime element. Moreover, for each $1 \neq x \in L_r$, $x \in F_a$ there exists $p \in S(L)$ such that $x \leqq p$.

Proof. The Proposition can be proved similarly as 1.4.

3. The one-point extensions

Definition. (i) Let K be a locale and L be a dense sublocale in K. Then we say that K is an extension of L.

(ii) Let L be a locale, $F \subseteq L$ be a filter on L. The sublocale $L_F \subseteq L + 2$, generated by the set $\{(l, 0): l \in L\} \cup \{(a, 1): a \in F\}$ is called a one-point extension of L.

This construction is a special case of the "Artin glueing" construction for locales (see [12]).

Evidently, L is a dense sublocale of L_F. We shall denote $\varepsilon_a = \bigvee(\varepsilon: (a, \varepsilon) \in L_F)$ for each $a \in L$.

3.1. Lemma. If L is a locale then $(a, \varepsilon)^* = (a^*, \varepsilon_{a^*})$ holds in L_F.

Proof. We have $(a, \varepsilon) \wedge (a^*, \varepsilon_{a^*}) = (0, 0)$ because $0 \notin F$. If $(a, \varepsilon)^* = (b, \beta)$ then $b \leqq a^*$ and $\beta \leqq \varepsilon_b \leqq \varepsilon_{a^*}$.

Now, we give an explicite description of the sets $P(L_F)$ and $D(L_F)$.

3.2. Proposition. Let L be a locale, $F \subseteq L$ be a filter and $(a, \varepsilon) \in L_F$. Then the following propositions hold:

1. $(a, \varepsilon) \in P(L_F)$ iff $a = 1$, $\varepsilon = 0$ or $a \in P(L)$, $\varepsilon = \varepsilon_a$.
2. $(a, \varepsilon) \in D(L_F)$ iff $a = 1$, $\varepsilon = 0$ or $a \in D(L)$, $\varepsilon = 1$.

Proof. 1. $\Rightarrow$: If $(a, \varepsilon) \in P(L_F)$ then $a \in P(L) \cup \{1\}$. Namely, if $a \neq 1$, $a \notin P(L)$ then $x, y \in L$ exist such that $x \wedge y \leqq a$, $x \nleqq a$, $y \nleqq a$. Clearly, $(x, 0) \wedge (y, 0) \leqq$ $\leqq (a, \varepsilon)$, $(x, 0) \nleqq (a, \varepsilon)$, $(y, 0) \nleqq (a, \varepsilon)$, a contradiction.

If $a = 1$ then $\varepsilon = 0$. If $a \neq 1$, $a \in P(L)$ then $(1, 0) \wedge (a, \varepsilon_a) \leqq (a, \varepsilon)$, i.e., $\varepsilon_a \leqq$ $\leqq \varepsilon \leqq \varepsilon_a$.

$\Leftarrow$: Evidently, $(1, 0) \in D(L_F) \subseteq P(L_F)$. Consider (a, ε_a) for some $a \in P(L)$. If $(x, \beta) \wedge (y, \gamma) \leqq (b, \varepsilon_a)$ then $x \leqq a$ or $y \leqq a$, i.e., $\beta \leqq \varepsilon_a$ or $\gamma \leqq \varepsilon_a$. Now, we have $(a, \varepsilon_a) \in P(L_F)$.

2. The proof is similar.

3.3. Corollary. Let L be a locale, $F \subseteq L$ be a filter of L.Then L_F is a T_1-locale iff L is a T_1-locale and $D(L) \subseteq F$.

Proof. $\Leftarrow$: Clearly, L is a T_1-frame. If $d \in D(L)$ then $(d, \varepsilon_d) \in P(L_F) = D(L_F)$, i.e., $\varepsilon_d = 1$. We have $d \in F$.

$\Rightarrow$: Let $(a, \varepsilon) \in P(L_F)$. Clearly, $(1, 0) \in D(L_F)$ and if $a \neq 1$, $a \in P(L)$, $\varepsilon = \varepsilon_a$ then $a \in D(L) \subseteq F$, i.e., $(a, \varepsilon) \in D(L_F)$.

3.4. Corollary. Let L be a locale. Then L_F is dually atomic iff for each $1 \neq f \in F$ there exists $d \in D(L)$ such that $f \leqq d$.

Proof. $\Rightarrow$: If $1 \neq f \in F$ then $(f, 1) \in L_F$ and $(d, 1) \in D(L_F)$ exists such that $(f, 1) \leqq$ $\leqq (d, 1)$, i.e., $f \leqq d$, $d \in D(L)$.

$\Leftarrow$: Let $(a, \varepsilon) \neq (1, 1)$, $(a, \varepsilon) \in L_F$. If $\varepsilon = 0$ then $(a, \varepsilon) \leqq (1, 0) \in D(L_F)$. If $\varepsilon = 1$, $1 \neq a \in F$ then $d \in D(L)$ exists such that $a \leqq d$, i.e., $(a, \varepsilon) \leqq (d, 1) \in D(L_F)$.

3.5. Proposition. Let L be a locale, F be a filter of L and $(a, \varepsilon) \in L_F$ then the following propositions hold:

1. $a \in S(L) \Rightarrow (a, \varepsilon_a) \in S(L_F)$.
2. $(a, \varepsilon) \in S(L_F) \Rightarrow a \in S(L) \cup \{1\}$.
3. $(a, \varepsilon) \in S(L_F)$, F is an α-filter of $L \Rightarrow a \in S(L)$, $\varepsilon = 1$ or $a = 1$, $\varepsilon = 0$.

Proof. 1., 2. are evident.

3. Let $(a, \varepsilon) \in S(L_F)$. If $a = 1$ then $\varepsilon = 0$. If $a \neq 1$, $a \in S(L)$ then $x \in F$ exists such that $x^* \nleqq a$. We have $(x, 1) \wedge (x^*, 0) \leqq (0, 0)$, i.e., $(x, 1) \leqq (a, \varepsilon)$ and $\varepsilon = 1$.

3.6. Corollary. Let F be an α-filter on a locale L. Then L_F is an S-locale iff L is an S-locale.

Proof. $\Rightarrow$: L is a homomorphic image L_F, i.e., L is an S-frame. $\Leftarrow$: If $(a, \varepsilon) \in S(L_F)$ and $a \neq 1$ then $a \in S(L) = D(L)$, $\varepsilon = 1$, i.e., $(a, \varepsilon) \in D(L_F)$.

3.7. Proposition. L_F is spatial iff L is spatial.

Proof. $\Rightarrow$: If $1 \neq a \in L$ then $(a, 0) = (1, 0) \wedge \bigwedge\{(p, \varepsilon_p) \geqq (a, 0) : p \in P(L)\}$, i.e., $a = \bigwedge\{p \geqq a : p \in P(L)\}$.

$\Leftarrow$: If $(a, \varepsilon) \neq (1, 1)$, $(a, \varepsilon) \in L_F$ then $(a, \varepsilon) = \bigwedge\{(p, \varepsilon_p) \geqq (a, \varepsilon) : (p, \varepsilon_p) \in P(L_F)\}$ because $a = \bigwedge\{p \geqq a : p \in P(L)\}$.

3.8. Proposition. L_F is conjunctive iff for arbitrary two elements $a, b \in L$ such that $1 \neq a \nleqq b$ there exists $c \in F$ such that $a \vee c = 1, b \vee c \neq 1$ and $F \setminus \{1\}$ is cofinal in $L \setminus \{1\}$.

Proof. $\Rightarrow$: If $1 \neq a \nleqq b$, $a, b \in L$ then $(1, 1) \neq (a, 0) \nleqq (b, 0)$, i.e., $(c, \varepsilon) \in L_F$ exists such that $(a, 0) \vee (c, \varepsilon) = (1, 1)$, $(1, 1) \neq (b, 0) \vee (c, \varepsilon)$. We have $\varepsilon = 1$, $a \vee c = 1$, $b \vee c \neq 1$ and $c \in F$.

If $1 \neq b \in L$ then $(1, 1) \neq (1, 0) \nleqq (b, 0)$, i.e., $(c, \varepsilon) \in L_F$ exists with $(1, 0) \vee \vee (c, \varepsilon) = (1, 1)$, $(1, 1) \neq (b, 0) \vee (c, \varepsilon)$ and we have $\varepsilon = 1$, $b \leqq c \vee b \neq 1$, $c \vee \vee b \in F$.

$\Leftarrow$: If (a, ε), $(b, \beta) \in L_F$, $(1, 1) \neq (a, \varepsilon) \nleqq (b, \beta)$ then we have the following cases:

a) If $1 \neq a \nleqq b$ then $c \in F$ exists such that $a \vee c = 1$, $b \vee c \neq 1$, i.e., $(a, \varepsilon) \vee \vee (c, 1) = (1, 1), (b, \beta) \vee (c, 1) \neq (1, 1)$.

b) If $1 = a \nleqq b$ then $\varepsilon = 0$, $b \neq 1$ and $1 \neq c \in F$ exists such that $b \nleqq c$. We have $(1, 0) \vee (c, 1) = (1, 1), (b, \beta) \vee (c, 1) = (c, 1) \neq (1, 1)$.

c) If $1 \neq a \leqq b$ then $\varepsilon = 1$, $\beta = 0$ and we have $(a, \varepsilon) \vee (1, 0) = (1, 1), (b, \beta) \vee \vee (1, 0) = (1, 0) \neq (1, 1)$.

Finally, L_F is conjunctive.

3.9. Lemma. If L is a locale, $F \subseteq L$ is a filter of L, $x \in L_r$ then $x \in F \Leftrightarrow (x^*, 0) \lhd \lhd (1, 0)$.

Proof. $\Rightarrow$: If $x \in F$ then $(x, 1) \vee (1, 0) = (1, 1)$, $(x^*, 0) \leqq (1, 0)$, i.e., $(x^*, 0) \lhd \lhd (1, 0)$.

$\Leftarrow$: If $(x^*, 0) \lhd (1, 0)$ then $(x, \varepsilon_x) \vee (1, 0) = (1, 1)$, i.e., $\varepsilon_x = 1$. We have $x \in F$.

3.10. Corollary. F is an α-filter iff $(1, 0) = \bigvee(z \in L_F\colon z \lhd (1, 0))$.
Proof follows from 3.9.

3.11. Theorem. If L is a locale and F is a filter of L then the following propositions are equivalent:

1. L_F is a Hausdorff locale.
2. L is a Hausdorff locale and F is an α-filter.
3. (i) $a = \bigvee(x \,\square\, a\colon x^* \in F)$ for each $a \in L$,
   (ii) For each $1 \neq a \in F$ there exists $x \in F$ such that $x \,\square\, a$.

Proof. $1 \Rightarrow 2$: Clearly, L is a Hausdorff frame and $(1, 0)$ is a dual atom in L_F. Since $(1, 0) = \bigvee(z\colon z \lhd (1, 0))$ we have that F is an α-filter.

$2 \Rightarrow 3$: (i) If $a \in L$ then $a = \bigvee(x \in L: x \,\square\, a) = \bigvee(y \wedge x: y^* \in F,\ x \,\square\, a) = \bigvee(z \in L: z \,\square\, a, z^* \in F)$.

(ii) If $1 \neq a \in F$ then $x \nleqq a$ exists with $x^* \in F$. If we put $z = a \wedge x^*$ then $z \leqq a$, $z^* \nleqq a$ because $a^* \vee x^{**} \nleqq a$, i.e., $z \in F$, $z \,\square\, a$.

$3 \Rightarrow 1$: Let $(1, 1) \neq (a, \varepsilon) \in L_F$. If $\varepsilon = 0$ then $(a, 0) = \bigvee((x, 0): x \,\square\, a, x^* \in F) = \bigvee((x, 0): (x^*, 1) \nleqq (a, 0))$. If $\varepsilon = 1$ then $z \in F$ exists with $z \,\square\, a$. Clearly $(a, 1) = (a, 0) \vee (z, 1) = \bigvee((x, \beta): (x, \beta) \,\square\, (a, 1))$, i.e., L_F is a Hausdorff frame.

3.12. Theorem. If L is a locale, F is a filter of L then the following are equivalent:

1. L_F is regular.
2. (i) $a = \bigvee(x \lhd a: x^* \in F)$ for each $a \in L$.
   (ii) For each $a \in F$ there exists $x \in F$ such that $x \lhd a$.

Proof. $1 \Rightarrow 2$: (i) If $a \in L$ then $(a, 0) = \bigvee((x, \varepsilon) : (x, \varepsilon) \lhd (a, 0)) = \bigvee((x, \varepsilon) : (x^*, \varepsilon_{x^*}) \vee (a, 0) = (1, 1)) = \bigvee((x, \varepsilon) : x \lhd a,\ x^* \in F)$. Now, we have $a = \bigvee(x: x \lhd a, x^* \in F)$.

(ii) If $a \in F$ then $(a, 1) = \bigvee((x, \varepsilon) : (x^*, \varepsilon_{x^*}) \vee (a, 1) = (1, 1))$. Clearly, $(x, 1) \leqq (a, 1)$ exists such that $x^* \vee a = 1$, i.e., $x \in F$ exists with $x \lhd a$.

$2 \Rightarrow 1$: Let $(a, \varepsilon) \in L_F$. If $\varepsilon = 0$ then $(a, 0) = \bigvee((x, 0): x \lhd a, x^* \in F) = \bigvee((x, 0) : (x, 0) \lhd (a, 0))$. If $\varepsilon = 1$ then $x \in F$ exists with $x \lhd a$. We have $(a, 1) = (a, 0) \vee (x, 1) = \bigvee((y, \varepsilon): (y, \varepsilon) \lhd (a, 1))$.

4. The one-point compactifications

4.1. Proposition. If L is a non-compact locale then the locale L_{F_c} is compact.

Proof. If $\bigvee((x_i, \varepsilon_i): i \in I) = (1, 1)$ then there exists $i_0 \in I$ with $\varepsilon_{i_0} = 1$, i.e., $x_{i_0} \in F_C$. Clearly, a finite set $K \subseteq I$ exists such that $\bigvee(x_i: i \in K) \vee x_{i_0} = 1$, i.e. $\bigvee((x_i, \varepsilon_i): i \in K) \vee (x_{i_0}, 1) = (1, 1)$.

Definition. Let L be a non-compact locale. We say that L_{F_C} is the one-point compactification of L.

Evidently, if L is spatial then L_{F_C} is the Alexandroff extension of L.

4.2. Proposition. Let L be a non-compact locale. Then L_{F_C} is a T_1-locale iff L is a T_1-locale.

Proof follows from 3.3 because $D(L) \subseteq F_C$.

The following is a locale analogy of the Alexandroff compactification for Hausdorff spaces.

4.3. Proposition. Let L be a non-compact locale. Then L_{F_C} is a Hausdorff locale iff L is a wl-compact Hansdorff locale.

Proof follows from 3.11.

4.4. Corollary. A wl-compact Hausdorff locale is a T_2'-locale.

Proof. Clearly, L_{F_C} is a compact Hausdorff frame, i.e., L_{F_C} is a T_2'-frame (see [10], 1.4) because L_{F_C} is dually atomic. Since L is a homomorphic image of L_{F_C} we have that L is a T_2'-frame.

4.5. Proposition. Let L be a non-compact locale. Then L_{F_C} is regular iff L is wl-compact and regular.

Proof. $\Rightarrow$: It follows from 4.3 and from the fact that homomorphic images of regular frames are regular.

$\Leftarrow$: It follows from 1.9, 1.12 and 3.1.

4.6. Corollary. A wl-compact regular locale L is spatial. Moreover, L is completely regular.

Proof. If L is non-compact then L_{F_C} is spatial and completely regular, i.e., L is spatial and completely regular.

4.7. Proposition. If L is a locale which is not almost compact then *thr* locale L_{F_a} is almost compact.

Proof. If $\bigvee((x_i, \varepsilon_i): i \in I) = (1, 1)$ then $i_0 \in I$ exists with $\varepsilon_{i_0} = 1$, i.e., $x^*_{i_0} \in F_a$. Further, a finite set $K \subseteq I$ exists such that $[\bigvee(x_i: i \in K)]^* \wedge x_{i_0} = 0$, i.e., $[\bigvee((x_i, \varepsilon_i): i \in K) \vee (x_{i_0}, 1)]^* = \bigwedge(x^*_i, \varepsilon_{x_i*}) \wedge (x^*_{i_0}, 0) = (0, 0)$.

Definition. Let L be a locale which is not almost compact. We say that L_{F_a} is the one-point almost compactification of L.

4.8. Proposition. Let L be a locale which is not almost compact. Then it holds:

1. L_{F_a} is a T_1-locale iff L is a T_1-locale.
2. L_{F_a} is a Hausdorff locale iff L is a Hausdorff locale which is locally almost compact.

Proof. 1. It follows from 3.3 because $D(L) \subseteq F_a$. 2. It follows from 3.11.

The proposition 4.8.2 is well known for spaces (see [8]).

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